New Additional Mathematics

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Marshall Cavendish Education

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New Additional Mathematics is specially written for students preparing for the GCE 'O' Level examinations.

Written by experienced practising teachers, *Ho Soo Thong* and *Khor Nyak Hiong*, this book provides comprehensive coverage of the revised Additional Mathematics syllabus.

Special features

- Clear and concise explanation of concepts
- Relevant examples
- IT-based activities for exploration and extension
- Great variety of questions
- Questions that develop higher-order thinking skills
- Overview of each chapter
- Revision exercises and assessment papers
- Answers and index for easy reference

The well-organised content and reader-friendly presentation makes this an essential textbook that will help students achieve mastery of vital mathematical principles.

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Preface

New Additional Mathematics is written for students preparing for the GCE 'O' Level Additional Mathematics examination paper.

This book closely follows the revised syllabus and includes five new chapters on Sets, Matrices, Permutations and Combinations, Kinematics and Relative Velocity. Some Challenging questions (indicated by *) and Critical Thinking Problems (indicated by ) are placed at appropriate places for more able students. Suggestions on the use of IT and open tools to promote dynamic learning have been added where relevant (indicated by ). These open tools include spreadsheets and graph plotters.

All basic concepts are explained and developed clearly through comprehensive illustrations and worked examples. Important concepts applied for problem solving in some examples are highlighted by short notes. Simple questions for direct application of basic concepts and formulae are given at the beginning of each graded exercise-students are encouraged to attempt these.

The main text contains clear presentation of basic concepts with simple illustrations, the derivation and application of useful formulae, important facts and the process of problem solving through examples.

At the end of each chapter, all basic concepts, formulae and problem-solving methods are summarised in the Important Notes. This is followed by harder examples and more questions in the Miscellaneous Exercise.

To assist students in their revision and examination preparation, five sets of graded revision exercises are placed at various appropriate stages of learning, in addition to two assessment papers at the end of the book.

We would like to thank Mdm Tan Meow Lang for her constructive suggestions on the presentation and approach, Dr Leong Yu Kiang for his discussion on certain basic concepts and terminology in the first edition and those who offered suggestions in this new edition.

We are grateful to the Cambridge University Local Examination Syndicate for their permission to reproduce some of the past examination questions. Finally, we would like to thank our publisher and all those involved in the production of this book.

Ho Soo Thong
David Khor Nyak Hiong

Contents

1 Sets

1.1 Introduction to Sets

Introduction

In the word 'mathematics', the **collection** of distinct letters a, c, e, h, i, m, s, t is called a **set**. Each of the letters is called a **member** or an **element** of the set. We denote the set by the capital letter M and list its members, separated by commas and enclosed within braces as shown below.

$$M = \{a, c, e, h, i, m, s, t\}$$

The letter a is a member of M and we write $a \in M$, where '$\in$' denotes '**belongs to**', '**is a member of**' or '**is an element of**'. On the other hand, the letter b is not a member of M, so we write $b \notin M$.

The set M is said to be well-defined. Why? This is because we can determine if any letter is a member of M.

Similarly, D denotes the set of days in a week and it is well-defined. We write

D = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}.

A set is a collection of well-defined elements (objects).

Other examples of well-defined sets are:

(a) M denotes the set of months in a year, and so
M = {January, February, March, April, May, June, July, August, September, October, November, December}.

(b) F denotes the set of first ten positive integers, and so
$F = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

(c) C denotes the set of countries in South-East Asia, and so
C = {Myanmar, Cambodia, Laos, Indonesia, Malaysia, Singapore, Thailand, Vietnam, Philippines}.

The set of all public housing estates in Singapore as at 1 Jan 2000, E, is

$$E = \{\text{Ang Mo Kio, Bishan, Bedok, ..., Woodlands, Yishun}\}.$$

Instead of listing all the elements, we use '...' to indicate the presence of the other elements.

Finite and Infinite Sets

The set of all positive integers less than 50 is $A = \{1, 2, 3, ..., 49\}$. The number of elements in A is 49 and we write $n(A) = 49$. The set A is called a **finite** set. Why?

On the other hand, the set of all positive even integers consists of an infinite number of elements and is called an **infinite** set. Note that it is impossible to list all the elements of the set, thus, it is written as $E = \{2, 4, 6, ...\}$ and we also use '...' to indicate the remaining numbers. Can we write down the value of $n(E)$?

Example 1

List all the elements, if possible, of each of the following sets:
(a) A, the set of positive integers less than 12.
(b) B, the set of prime numbers less than 15.
(c) C, the set of positive integers which are also multiples of 3.
(d) D, the set of integers x defined by $-3 < x < 6$.
Write down the number of elements for each finite set.

Solution:

(a) $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$
(b) $B = \{2, 3, 5, 7, 11, 13\}$
(c) $C = \{3, 6, 9, ...\}$
(d) $D = \{-2, -1, 0, 1, 2, 3, 4, 5\}$
We have $n(A) = 11$, $n(B) = 6$, C is an infinite set and $n(D) = 8$.

Set-builder Notation

Using x as a member of the set A in Example 1, we write

$$A = \{x : x \text{ is a positive integer and } x < 12\}$$

which is called a **set-builder notation**.
Similarly,

$$B = \{x : x \text{ is a prime number and } x < 15\}.$$
$$C = \{x : x \text{ is a positive integer and } x \text{ is a multiple of } 3\}.$$
$$D = \{x : x \text{ is an integer and } -3 < x < 6\}.$$

The following commonly used notation for sets on the real line will be used subsequently:

$\mathbb{R}$ is the set of real numbers
$\mathbb{R}^+$ is the set of positive real numbers, $\{x : x > 0, x \in \mathbb{R}\}$.
$\mathbb{N}$ is the set of natural numbers, $\{1, 2, ...\}$.
$\mathbb{Z}$ is the set of integers, $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$.
$\mathbb{Z}^+$ is the set of positive integers $\{1, 2, 3, ...\}$.

The set of 10 smallest prime numbers is {2, 3, 5, 7, 11, 13, 17, 19, 23, 29}.

With the above notation, some of the sets in Example 1 can be simply written as

$$A = \{n : n < 12, n \in \mathbb{Z}^+\}$$
$$B = \{p : p < 15, p \text{ is a prime number}\}$$
$$C = \{3n : n \in \mathbb{Z}^+\}$$
$$D = \{x : -3 < x < 6, x \in \mathbb{Z}\}$$

Note that in the set notation of A, n is a 'dummy variable'; we can use m instead of n. So

$$\{n : n < 12, n \in \mathbb{Z}^+\} = \{m : m < 12, m \in \mathbb{Z}^+\}$$

Similarly, in the set notation of B, p is a dummy variable.

The set of positive real numbers less than 2 is written as

$$\{x : x < 2, x \in \mathbb{R}^+\} \quad \text{or} \quad \{x : 0 < x < 2, x \in \mathbb{R}^+\}$$

Example 2 List the 5 smallest numbers of the following sets:

(a) $C = \{6n : n \in \mathbb{Z}^+\}$,

(b) $D = \{x : x^2 \geqslant 20, x \in \mathbb{N}\}$

Solution:

(a) $C = \{6 \times 1, 6 \times 2, 6 \times 3, 6 \times 4, 6 \times 5, ...\}$
$= \mathbf{\{6, 12, 18, 24, 30, ...\}}$

(b) $D = \mathbf{\{5, 6, 7, 8, 9 ...\}}$

Example 3 Use the commonly used set notation to represent the following sets:

$A = \{x : x \text{ is an even positive integer and } x \text{ is less than } 7\}$ and
$B = \{x : x \text{ is a positive integer and } 4x < 19\}$.

(a) List the elements of the sets A and B.

(b) Find the integer x such that $x \in A$ and $x \notin B$.

(c) Find an integer x such that $x \notin A$ and $x \in B$.

Solution:

$A = \{2n : 2n < 7, n \in \mathbb{Z}^+\}$, $B = \{m : 4m < 19, m \in \mathbb{Z}^+\}$

(a) Since $2 \times 3 < 7$ and $2 \times 4 \nless 7$, $A = \{2, 4, 6\}$.
Since $4 \times 4 < 19$ and $4 \times 5 \nless 19$, $B = \{1, 2, 3, 4\}$.

(b) Next, we identify the common elements in the sets A and B as shown:
$A = \{2, 4, 6\}$
$B = \{1, 2, 3, 4\}$
Since $6 \in A$ and $6 \notin B$, $x = 6$.

(c) Since $1 \notin A$ and $1 \in B$, $x = 1$. **(another integer is 3)**

Empty or Null Set

Example 4 List the members of each of the following sets:

(a) P = The set of days in a week which starts with the letter A.

(b) Q = The set of positive integers which are both even and odd.

Solution:

(a) The set of days in a week is

W = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}. None of the days starts with A.

P = { } and is known as an **empty set** or a **null set**.

(b) Any positive integer is either even or odd.

Therefore, $Q = \{ \}$.

In the above, any set without a single element is written as { }. We also use the symbol 'ϕ' to denote an empty set and write $\phi = \{ \}$. Some examples are

$$A = \{x : x \geqslant 0, x^2 < 0\} = \phi$$

$$B = \{x : \tfrac{1}{2} < x < \tfrac{2}{3}, x \in \mathbb{Z}\} = \phi$$

$$C = \text{The set of months with 32 days} = \phi.$$

Exercise 1.1A

1. List the elements of the following sets:
   (a) The set of days in a week which starts with the letter 'S'.
   (b) The set of months in a year which starts with the letter 'A'.
   (c) The set of first five prime numbers.

2. For each of the following sets, represent the set by commonly used set notation and list all its elements.
   (a) A, the set of odd positive integers less than 12.
   (b) B, the set of first five natural numbers.
   (c) C, the set of multiples of 5 which are less than 19.
   (d) D, the set of multiples of 3 between 12 and 20.
   (e) E, the set of real numbers between -2 and 3.

3. List the elements of each of the following sets:
   (a) $A = \{x : x \text{ is a natural number and } 2x < 15\}$.
   (b) $B = \{x : x \text{ is a multiple of 3 and } 2x > 9\}$.
   (c) $C = \{x : x \text{ is a prime number and } x \text{ is a factor of } 42\}$.
   (d) $D = \{x : x \text{ is an even number and } x^2 < 8\}$.

4. List the elements of the following sets:
   (a) $A = \{x : 2 < x < 3, x \in \mathbb{Z}\}$.
   (b) $B = \{x : 3x = 5, x \in \mathbb{N}\}$.
   (c) $C = \{2n : n^3 = 8, n \in \mathbb{Z}\}$.
   (d) $D = \{7n : 7n < 30, n \in \mathbb{Z}^+\}$.

5. List the elements of the sets $F = \{x : x \text{ is a positive integer and } x \text{ is a factor of } 30\}$ and $G = \{x : x \text{ is an even integer and } x \text{ is a multiple of } 3\}$. Find an element x such that $x \in F$ and $x \in G$.

6. List the elements of the sets $C = \{x : x \text{ is a positive integer, } 3x < 5 \text{ or } 5 < x \leqslant 8\}$ and $T = \{x : x \text{ is a positive integer, } 4 < 3x \leqslant 24\}$. Find an element x such that $x \in C$ and $x \notin T$.

7. List the elements of the set $S = \{x : x \text{ is a perfect square and } 7 < x \leqslant 48\}$.

8. List the elements of the following sets:
   (a) $A = \{x : x \text{ is a factor of } 36, x \in \mathbb{N}\}$.
   (b) $B = \{x : x \text{ is an integer and } 9 < 4x \leq 33, x \in \mathbb{Z}\}$.
   (c) $C = \{x : x \in A \text{ and } x \notin B\}$.

9. List the elements of the sets $D = \{x : x \text{ is a multiple of 3 and } x < 15\}$ and $E = \{x : x \text{ is a multiple of 6 and } x < 15\}$. Find the set $C = \{x : x \in D \text{ and } x \in E\}$.

10. List the elements of the set $A = \{x : x \text{ is an odd positive integer and } 3x - 2 < 17\}$.

11. List the elements of the sets $S = \{x : x^2 < 15, x \in \mathbb{Z}^+\}$ and
    $T = \{x : x \text{ is a factor of } 45, x \in \mathbb{Z}^+\}$.
    (a) Find an element x such that $x \in S$ and $x \notin T$.
    (b) Find the set $U = \{x : x \in S \text{ or } x \in T\}$.

Equal Sets

Consider the following sets:

$$A = \{x : x \text{ is a positive integer and } 2x < 9\} = \{1, 2, 3, 4\}.$$
$$B = \{x : x \text{ is a positive integer and } x^2 \leq 20\} = \{1, 2, 3, 4\}.$$
$$C = \{x : x \text{ is a positive integer and } 5 < 3x < 16\} = \{2, 3, 4, 5\}.$$

The sets A and B have the same list of elements. They are said to be **equal**, so we write

$$A = B$$

Obviously, the sets A and C are not equal, so we write $A \neq C$.

Example 5 List the elements of the following sets:
(a) $P = \{x : x \text{ is a positive integer and } 5x \leq 16\}$.
(b) $Q = \{x : x \text{ is a positive integer and } x^2 < 20\}$.
(c) $R = \{x : x \text{ is a positive integer and } \sqrt{x} \leq 2\}$.
Show that $P \neq Q$ and $Q = R$.

Solution:
(a) $P = \{1, 2, 3\}$
(b) $Q = \{1, 2, 3, 4\}$
(c) $R = \{1, 2, 3, 4\}$
Obviously, $P \neq Q$ and $Q = R$.

Subsets

Consider again the sets $P = \{1, 2, 3\}$, $Q = \{1, 2, 3, 4\}$ and $R = \{1, 2, 3, 4\}$ from Example 5. Do you notice the relationship between P and R?
Every element of P is an element of R, that is, $x \in P \Rightarrow x \in R$.
The set P is a **subset** of R and we write $P \subseteq R$.
Similarly, every element of Q is an element of R, that is, $x \in Q \Rightarrow x \in R$.
So Q is also a **subset** of R and we write $Q \subseteq R$.

Although both P and Q are subsets of R, there is a difference between them. Note that $n(P) = 3$ and $n(R) = 4$. P is called a **proper subset** of R and we write $P \subset R$.

For any set A,

(i) $A \subseteq A$,

(ii) $\phi \subseteq A$. **(The empty set, ϕ, is a subset of any set.)**

> **If A is a subset of a finite set B, i.e. $A \subseteq B$, then $n(A) \leqslant n(B)$.**
> **If A is a proper subset of a finite set B, i.e. $A \subset B$, then $n(A) < n(B)$.**

We use the symbol $\not\subseteq$ to mean 'is not a subset of' and the symbol $\not\subset$ to mean 'is not a proper subset of'.

Universal Set

In Example 5, the sets

$$P = \{x : x \text{ is a positive integer and } 5x \leqslant 16\},$$
$$Q = \{x : x \text{ is a positive integer and } x^2 < 20\}$$

and
$$R = \{x : x \text{ is a positive integer and } \sqrt{x} \leqslant 2\}$$

consist of only positive integers. If we define

$$\varepsilon = \{x : x \text{ is a positive integer}\},$$

then the set P, Q and R are subsets of ε and ε is called the **universal set**.

> **A universal set is a set which consists of all elements under consideration.**

With $\varepsilon = \{x : x \text{ is a positive integer}\}$ as the universal set, we can write the set notation of P, Q and R as

$$P = \{x : 5x \leqslant 16\},$$
$$Q = \{x : x^2 < 20\}$$

and
$$R = \{x : \sqrt{x} \leqslant 2\}.$$

Complement of a Set

With $\varepsilon = \{x : x \text{ is a positive integer}\}$ and $P = \{1, 2, 3\}$, we define the set $P' = \{x : 5x > 16\}$ which consists of all the elements not in P, so $P' = \{4, 5, 6, \ldots\}$ and is called the **complement** of P.

Similarly, for the set $Q = \{1, 2, 3, 4\}$, the complement of Q is $Q' = \{5, 6, 7, \ldots\}$.

> **If ε is the universal set, the complement of P is the set**
> $$P' = \{x : x \notin P, x \in \varepsilon\}$$

Example 6 Given that $\varepsilon = \{x : x \text{ is an integer}, 0 < x \leqslant 10\}$,
$A = \{x : 2x > 7\}$
and $B = \{x : 3x < 20\}$,
list the elements of
(a) the sets A and A',
(b) the sets B and B'.
State whether each of the following is true or false:
$$A' \subseteq B,\ B' \not\subset A,\ A \not\subset B$$

Solution:

$\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

(a) $A = \{x : 2x > 7\} = \mathbf{\{4, 5, 6, 7, 8, 9, 10\}}$
$A' = \mathbf{\{1, 2, 3\}}$

(b) $B = \{x : 3x < 20\} = \mathbf{\{1, 2, 3, 4, 5, 6\}}$
$B' = \mathbf{\{7, 8, 9, 10\}}$

$A' \subseteq B$ is **true**, $B' \not\subset A$ is **false**, $A \not\subset B$ is **true**.

Exercise 1.1B

1. Given that $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, list the elements of the following sets.
   (a) $A = \{x : 5x > 37\}$ **(b)** $B = \{x : x + 5 < 12\}$
   (c) $C = \{x : 6 < 2x < 17\}$ **(d)** $D = \{x : x^2 < 37\}$

2. Given that $\varepsilon = \{x : 1 \leqslant x \leqslant 20, x \in \mathbb{Z}^+\}$, list the elements of the following sets.
   (a) $A = \{x : x \text{ is a multiple of } 2\}$
   (b) $B = \{x : x \text{ is a multiple of } 5\}$
   (c) $C = \{x : x \text{ is a multiple of } 10\}$
   State whether each of the following is true or false: $C \subset A$, $B \subset A$, $C \subset B$.

3. Given that $\varepsilon = \{5, 6, 7, 8, 9, 10, 11, 12\}$, list the elements of the following sets.
   (a) $A = \{x : x \text{ is a factor of } 60\}$ **(b)** $B = \{x : x \text{ is a prime number}\}$
   (c) $C = \{x : x \text{ is divisible by } 3\}$

4. Given that $\varepsilon = \{x : 1 \leqslant x \leqslant 20, x \in \mathbb{Z}\}$, list the elements of the following sets.
   (a) $A = \{x : x \text{ is a multiple of } 3\}$ **(b)** $B = \{x : x \text{ is a not a multiple of } 3\}$
   Show that $n(A) + n(B) = n(\varepsilon)$.

5. Given that $\varepsilon = \{8, 9, 10, 11, 12\}$, list the elements of the sets $A = \{2n : n \in \mathbb{Z}\}$ and $B = \{3n : n \in \mathbb{Z}\}$.
   (a) Find the element x such that $x \notin A$ and $x \notin B$.
   (b) Describe the sets A and B.

6. Given that $\varepsilon = \{x : 3 \leqslant x \leqslant 8, x \in \mathbb{Z}\}$, list the elements of the sets $A = \{x : 10 < x^2 \leqslant 25\}$ and $B = \{x : 8x - 9 > 30\}$.
   (a) Find an element x such that $x \in A$ and $x \notin B$.
   (b) Is $A \subseteq B$?

7. Given that $\varepsilon = \{a, b, c, d, e, f, g, h\}$, $A = \{a, c, e\}$ and $B = \{b, d, f, h\}$, list the elements of the sets A' and B'.
   Hence find an element x such that $x \notin A$ and $x \notin B$. Is A a proper subset of B'?

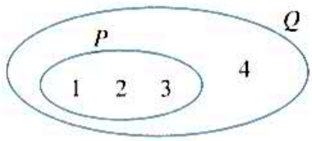
8. Given that $\varepsilon = \{x : x \text{ is an integer and } 1 \leqslant x \leqslant 20\}$,
$A = \{x : x \text{ is a multiple of } 6\}$
and $B = \{x : x \text{ is a multiple of } 2\}$.
List the elements of A and B. Is $A \subset B$?

*9. Given that $\varepsilon = \{x : 3 \leqslant x \leqslant 20, x \in \mathbb{Z}\}$, $A = \{x : x \text{ is odd}\}$ and $B = \{x : x \text{ is prime}\}$, list the elements of
(a) A and B.
(b) $C = \{x : x \in A \text{ and } x \in B\}$ and $D = \{x : x \in A \text{ or } x \in B\}$.
Describe the sets C and D.

1.2 Intersection and Union of Sets

Venn Diagram

The pictorial representation of the relationship $P \subseteq Q$ between the sets $P = \{1, 2, 3\}$ and $Q = \{1, 2, 3, 4\}$ is called a **Venn diagram**.



The following Venn diagram Fig. (a) shows the universal set $\varepsilon = \{x : x \text{ is an integer}, 0 < x \leqslant 10\}$, the set $A = \{x : 2x > 7\}$ and $A' = \{x : 2x \leqslant 7\}$, the complement of A.

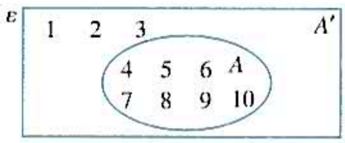


Fig. (a)

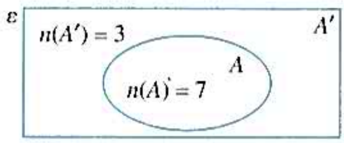


Fig. (b)

Instead of listing the numbers in each set, we can write down the number of elements of each set as shown in Fig. (b). In the following, when we quote $n(A)$, we assume that A is a finite set.

> **If ε is a universal set and A is any set, then**
> $$n(A) + n(A') = n(\varepsilon).$$

Example 7 It is given that $\varepsilon = \{x : 2 \leqslant x \leqslant 30, x \in \mathbb{Z}^+\}$
and
$$P = \{x : x \text{ is a factor } 30\}.$$
(a) List the elements of P.
(b) Describe the set P'.
(c) Find $n(P')$.

Solution:

(a) $P = \{2, 3, 5, 6, 10, 15, 30\}$

(b) $P' = \{x : x \text{ is not a factor of } 30\}$

(c) $n(P') = n(\varepsilon) - n(P)$
$= 29 - 7$
$= 22$

Intersection of Sets

Example 8

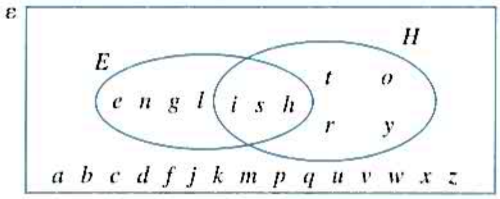
The universal set ε is the set of all letters in the English alphabet and the two subsets are $E = \{e, n, g, l, i, s, h\}$ and $H = \{h, i, s, t, o, r, y\}$.

(a) Illustrate, with a Venn diagram, the sets ε, E and H.

(b) List the elements of the set $\{x : x \in E \text{ and } x \in H\}$.

Solution:

(a)



(b) From the Venn diagram, $\{x : x \in E \text{ and } x \in H\} = \{i, s, h\}$.

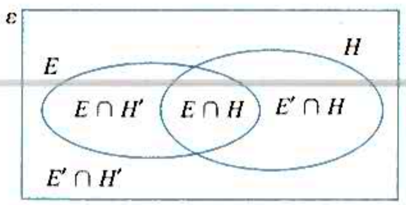
In Example 8, the set $\{x : x \in E \text{ and } x \in H\} = \{i, s, h\}$ is the set of common elements of the sets E and H. The set is known as the **intersection** of E and H and is denoted by $E \cap H$.

$$E \cap H = \{x : x \in E \text{ and } x \in H\}$$

Similarly, we have:

$$E \cap H' = \{x : x \in E \text{ and } x \in H'\} = \{e, n, g, l\}$$
$$E' \cap H = \{x : x \in E' \text{ and } x \in H\} = \{t, o, r, y\}$$
$$E' \cap H' = \{x : x \in E' \text{ and } x \in H'\}$$
$$= \{a, b, c, d, f, j, k, m, p, q, u, v, w, x, z\}$$

The above sets are shown in the following Venn diagram.

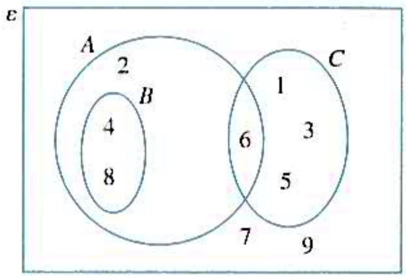


Example 9 It is given that $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$,
$A = \{2, 4, 6, 8\}$,
$B = \{4, 8\}$,
and $C = \{1, 3, 5, 6\}$.
By drawing a Venn diagram, list the elements of the sets
(a) $A \cap B$ and $A \cap B'$,
(b) $B \cap C$ and $B' \cap C'$.

Solution:

(a) Since $B \subseteq A$, $A \cap B = B = \{4, 8\}$
$A \cap B' = \{2, 6\}$

(b) $B \cap C = \{\ \}$
$B' \cap C' = \{2, 7, 9\}$



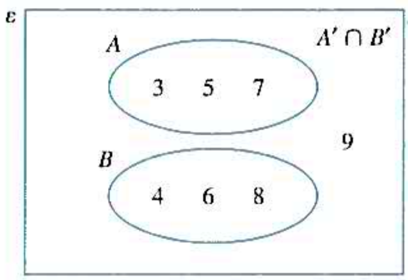
From Example 9, $B \cap C = \{\ \}$, therefore the sets B and C are said to be disjoint.

Two sets A and B are disjoint $\Leftrightarrow$ $A \cap B = \phi$.

Example 10 Given that $\varepsilon = \{3, 4, 5, 6, 7, 8, 9\}$, $A = \{x : x \text{ is prime}\}$ and $B = \{x : x \text{ is even}\}$, list the elements of the sets A and $A \cap B$. With the aid of a Venn diagram, show that
(a) $A' \cap B' = \{9\}$,
(b) $A \subseteq B'$ and $B \subseteq A'$.

Solution:

$A = \{3, 5, 7\}$
$B = \{4, 6, 8\}$
$A \cap B = \{\ \} = \phi$



(a) From the Venn diagram, we have
$A' \cap B' = \{9\}$.

(b) $A' = \{4, 6, 8, 9\}$ and $B' = \{3, 5, 7, 9\}$.
$\{3, 5, 7\} \subseteq \{3, 5, 7, 9\}$, i.e. $A \subseteq B'$
$\{4, 6, 8\} \subseteq \{4, 6, 8, 9\}$, i.e. $B \subseteq A'$

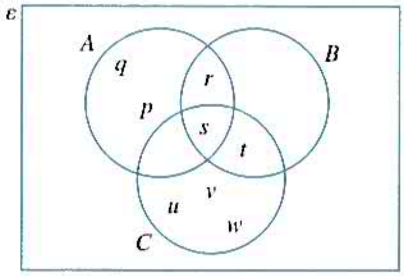
Example 11 $\varepsilon = \{p, q, r, s, t, u, v, w\}$, $A = \{p, q, r, s\}$, $B = \{r, s, t\}$ and $C = \{s, t, u, v, w\}$.

(a) List the elements of $A \cap B$, $B \cap C$ and $C \cap A$ and draw a Venn diagram.

(b) List the elements of $A \cap B \cap C$.

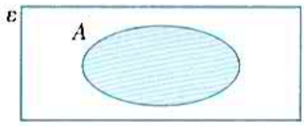
Solution:

(a) $A \cap B = \{r, s\}$
$B \cap C = \{s, t\}$
$C \cap A = \{s\}$



(b) $A \cap B \cap C = \{r, s\} \cap C$
$= \{r, s\} \cap \{s, t, u, v, w\}$
$= \{s\}$

In order to identify a set in a Venn diagram, we shade the region representing a set A as shown:



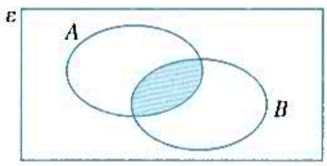
Example 12 Given that ε is the universal set and $A \cap B \neq \phi$, in separate Venn diagrams, shade the sets

(a) $A \cap B$, **(b)** $A' \cap B$,
(c) $A \cap B'$, **(d)** $A' \cap B'$.

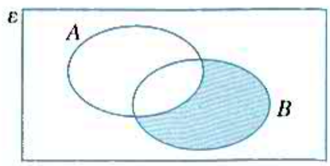
Show that $n(A \cap B) + n(A' \cap B) + n(A \cap B') + n(A' \cap B') = n(\varepsilon)$.

Solution:

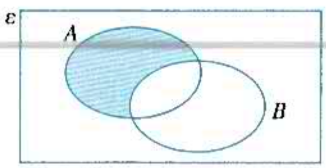
(a) $A \cap B$



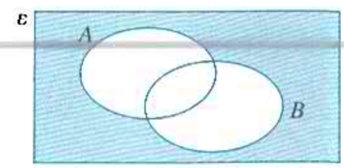
(b) $A' \cap B$



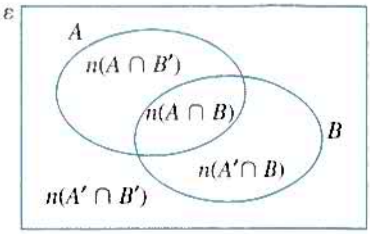
(c) $A \cap B'$



(d) $A' \cap B'$



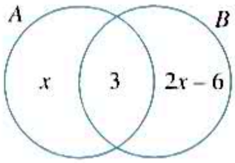
From the Venn diagram showing the number of elements in each subset of ε, we have
$n(A \cap B) + n(A' \cap B) + n(A \cap B') + n(A' \cap B') = n(\varepsilon)$.



For any two subsets A and B of a universal set ε,
$n(A \cap B) + n(A' \cap B) + n(A \cap B') + n(A' \cap B') = n(\varepsilon)$

Refer to Example 12 and study the case where A and B are disjoint.

Example 13 A and B are two sets and the number of elements in each set is shown in the Venn diagram.
Given that $n(A) = n(A' \cap B)$, calculate
(a) the value of x,
(b) $n(A)$ and $n(B)$.



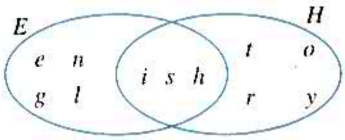
Solution:

(a) $n(A) = n(A' \cap B)$
$x + 3 = 2x - 6$
$x = \mathbf{9}$

(b) $n(A) = x + 3 = \mathbf{12}$
$n(B) = 3 + 2x - 6 = \mathbf{15}$

Union of Sets

In Example 8, we define the universal set ε of all letters in the English alphabet and two subsets $E = \{e, n, g, l, i, s, h\}$ and $H = \{h, i, s, t, o, r, y\}$.



The set $\{x : x \in E \text{ or } x \in H\} = \{e, g, h, i, l, n, o, r, s, t, y\}$ which includes elements of E or elements of H (or both) is known as the **union** of E and H and is denoted by $E \cup H$.

$E \cup H = \{x : x \in E \text{ or } x \in H\}$

Example 14 Given that

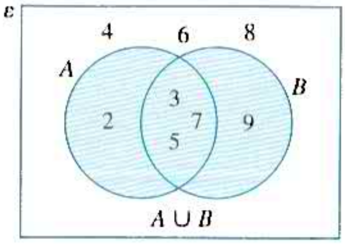
$\varepsilon = \{2, 3, 4, 5, 6, 7, 8, 9\}$,
$A = \{x : x \text{ is a prime number}\}$
and $B = \{x : x \text{ is an odd number}\}$.

(a) List the elements of the sets A, B and $A \cap B$.
(b) Draw a Venn diagram and shade the set $A \cup B$.
(c) List the elements of the sets $A \cup B$ and $(A \cup B)'$.

Solution:

(a) $A = \{2, 3, 5, 7\}$, $B = \{3, 5, 7, 9\}$ and $A \cap B = \{3, 5, 7\}$

(b)



(c) $A \cup B = \{x : x \in A \text{ or } x \in B\}$
$= \{2, 3, 5, 7, 9\}$
$(A \cup B)' = \{4, 6, 8\}$

Note: As shown in the above example, it is advisable to list the elements of the set $A \cap B$ before we draw the Venn diagram involving the sets A and B.

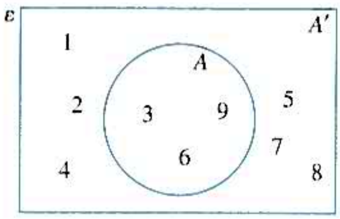
Example 15 It is given that $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A = \{x : x \text{ is a multiple of } 3\}$, show that

(a) $A \cup A' = \varepsilon$,
(b) $A \cap A' = \phi$.

Solution:

$A = \{3, 6, 9\}$ and $A' = \{1, 2, 4, 5, 7, 8\}$

(a) $A \cup A' = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} = \varepsilon$
(b) $A \cap A' = \{\ \} = \phi$



In general, for any set A and the universal set ε,
$A \cup A' = \varepsilon$ and $A \cap A' = \phi$.

Example 16 It is given that $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$,

$A = \{x : x \text{ is a multiple of } 2\}$,

$B = \{x : x \text{ is a multiple of } 3\}$

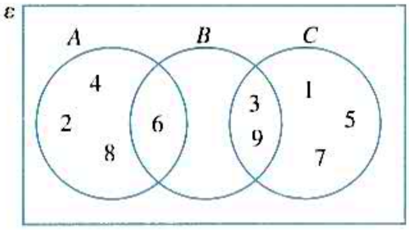
and $C = \{x : x \text{ is odd}\}$.

By drawing a Venn diagram, list the elements of the following sets:

(a) $A \cup B$ and $A \cup B \cup C$

(b) $B \cup C$ and $(B \cup C)'$

Solution: $A = \{2, 4, 6, 8\}$, $B = \{3, 6, 9\}$ and $C = \{1, 3, 5, 7, 9\}$



(a) $A \cup B =$ **$\{2, 3, 4, 6, 8, 9\}$**, $A \cup B \cup C = \varepsilon$

(b) $B \cup C = \{1, 3, 5, 6, 7, 9\}$

$(B \cup C)' =$ **$\{2, 4, 8\}$**

Note: When drawing the Venn diagram, we note that $A \cap B = \{6\}$, $B \cap C = \{3, 9\}$ and $A \cap C = \phi$.

Example 17 $\varepsilon = \{p, q, r, s, t, u, v, w\}$, $A = \{p, q, r, s\}$, $B = \{r, s, t\}$ and $C = \{s, t, u, v, w\}$.

(a) Find $n(A \cup B)$.

(b) List the elements of $(A \cup B)'$ and $A \cup B \cup C$.

Solution:

(a) $A \cup B = \{p, q, r, s, t\}$ and so $n(A \cup B) =$ **5**.

(b) $(A \cup B)' =$ **$\{u, v, w\}$**

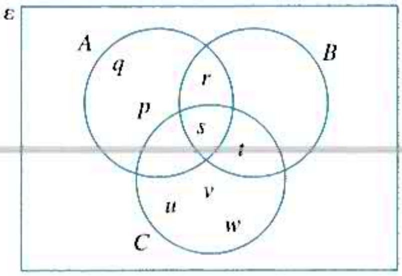
$A \cup B \cup C$

$= \{p, q, r, s, t\} \cup C$

$= \{p, q, r, s, t\} \cup \{s, t, u, v, w\}$

$= \{p, q, r, s, t, u, v, w\}$

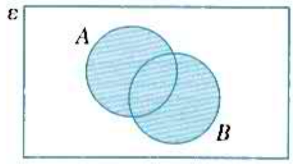
$= \varepsilon$



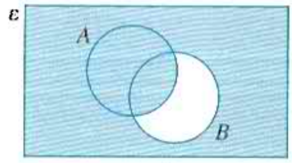
Example 18 Given that ε is the universal set and $A \cap B \neq \phi$, shade the sets
(a) $A \cup B$ **(b)** $A \cup B'$
(c) $A' \cup B$ **(d)** $(A \cup B)'$
in separate Venn diagrams.

Solution:

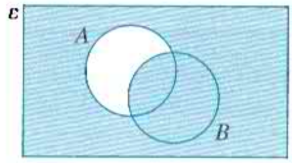
(a) $A \cup B$



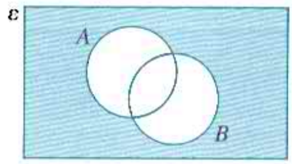
(b) $A \cup B'$



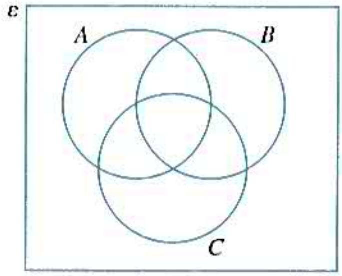
(c) $A' \cup B$



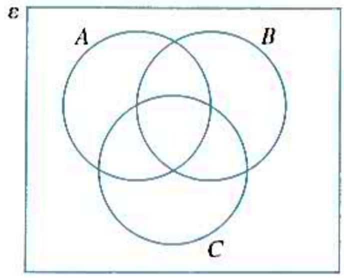
(d) $(A \cup B)'$



Example 19 In the Venn diagram (a), shade the set $A \cap (B \cup C)$.
In the Venn diagram (b), shade the set $A \cup (B \cap C)$.



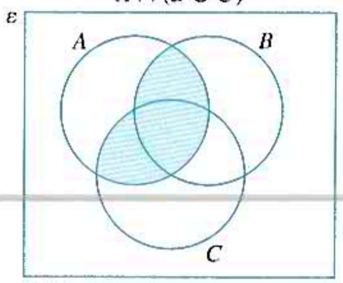
(a)



(b)

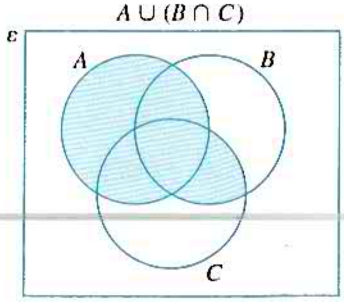
Solution:

$A \cap (B \cup C)$



(a)

$A \cup (B \cap C)$



(b)

Example 20

$\varepsilon = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{x : x \text{ is an even number}\}$ and $B = \{x : 7 < 3x < 25\}$.

(a) List the elements of A, B, $A \cap B$, $A \cup B$ and $A \cap B'$.

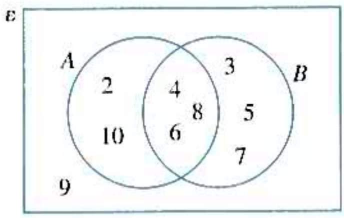
(b) Find an element x such that $x \in A$ and $x \notin B$.

(c) Find the element x such that $x \notin A$ and $x \notin B$.

Solution:

(a) $A = \{2, 4, 6, 8, 10\}$
$B = \{3, 4, 5, 6, 7, 8\}$
$A \cap B = \{4, 6, 8\}$
$A \cup B = \{2, 3, 4, 5, 6, 7, 8, 10\}$
$A \cap B' = \{2, 10\}$

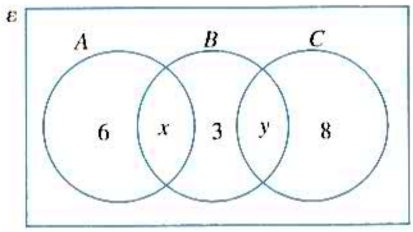
(b) $x \in A$ and $x \notin B$
$\Leftrightarrow \quad x \in A$ and $x \in B'$
$\Leftrightarrow \quad x \in A \cap B'$
$\therefore \quad x = 2$ or $x = 10$



(c) $x \notin A$ and $x \notin B$
$\Leftrightarrow \quad x \in A'$ and $x \in B'$
$\Leftrightarrow \quad x \in A' \cap B' = \{9\}$
$\therefore \quad x = 9$

Example 21

A, B and C are such that $\varepsilon = A \cup B \cup C$. The Venn diagram shows the number of elements in each subset of ε.



(a) Find x, given that $n(B) = n(C)$.

(b) Find y, given that $n(B \cap C) = n(A \cup B)'$.

(c) Find $n(\varepsilon)$.

Solution:

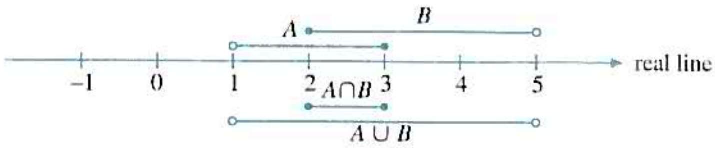
(a)
$$\begin{aligned} n(B) &= n(C) \\ x + 3 + y &= y + 8 \\ x &= 5 \end{aligned}$$

(b)
$$\begin{aligned} n(B \cap C) &= n(A \cup B)' \\ y &= 8 \end{aligned}$$

(c) $n(\varepsilon) = 6 + x + 3 + y + 8 = 30$

Sets of Real Numbers on the Real Line

The sets $A = \{x : 1 < x \leq 3, x \in \mathbb{R}\}$ and $B = \{x : 2 \leq x < 5, x \in \mathbb{R}\}$ can be represented graphically on the **real line** as shown below. With the aid of the graphical presentation, the sets $A \cap B$ and $A \cup B$ can be obtained as shown on the next page.

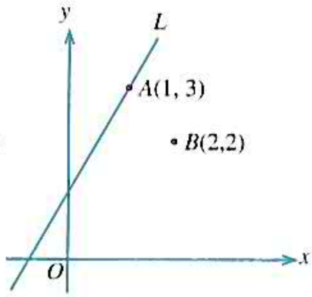


Then $A \cap B = \{x : 2 \leqslant x \leqslant 3, x \in \mathbb{R}\}$ and $A \cup B = \{x : 1 < x < 5, x \in \mathbb{R}\}$

Note that the solid dot indicates that the end number is a member of the set and an empty dot indicates that the end number is not a member of the set.

Sets of Points in a Plane

The set L denotes the set of points on the line $y = 2x + 1$ in the Cartesian plane. So, $L = \{(x, y) : y = 2x + 1, x \in \mathbb{R}\}$. Since $A(1, 3)$ is a point on the line, $(1, 3) \in L$. Since B $(2, 2)$ is not a point on the line, $(2, 4) \notin L$.

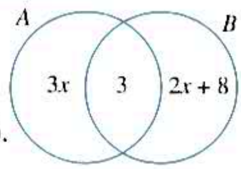


Exercise 1.2

1. A and B are two sets and the number of elements in each set is shown in the Venn diagram. Given that $n(A) = n(B)$, calculate

(a) the value of x,

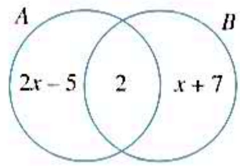
(b) $n(A \cup B)$ and $n(A \cap B')$.



2. A and B are two sets and the number of elements in each set is shown in the Venn diagram. Given that $n(A' \cap B) = n(A \cap B')$, calculate

(a) the value of x,

(b) $n(A)$ and $n(B)$.



3. $\varepsilon = \{x : x$ is a positive integer$\}$, $A = \{x : x \geqslant 5\}$ and $B = \{x : x < 12\}$. Find $n(A \cap B)$ and $n(A')$.

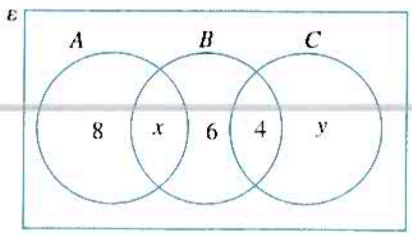
4. $\varepsilon = \{x : x$ is an even integer$\}$, $A = \{x : 3x \geqslant 25\}$ and $B = \{x : 5x < 12\}$. Find $n(A \cap B)$ and $n(A' \cap B')$.

5. A, B and C are such that $\varepsilon = A \cup B \cup C$. The Venn diagram represents the number of elements in each subset.

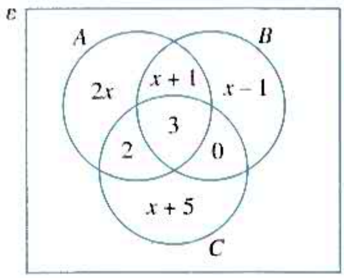
(a) Given that $n(A \cap B) = n(B \cap C)$, find the value of x.

(b) Given that $n(B \cap C') = n(A' \cap C)$, find the value of y.

(c) Find $n(\varepsilon)$.



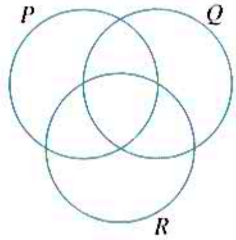
6. A, B and C are such that $\varepsilon = A \cup B \cup C$. The Venn diagram represents the number of elements in each subset.
   (a) Given that $n(\varepsilon) = 50$, find the value of x.
   (b) Find $n(B \cap C')$ and $n(A' \cap B)$.
   (c) Find $n(A \cap B \cap C')$.



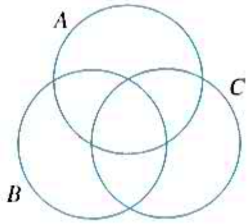
7. $\varepsilon = \{x : 2 \leqslant x \leqslant 10\}$, $P = \{x : x \text{ is a factor of } 24\}$ and $Q = \{x : x \text{ is a factor of } 15\}$.
   (a) List the elements of P, Q and $P' \cap Q'$.
   (b) Find $n(P \cap Q)$ and $n(P' \cap Q')$.

8. $\varepsilon = \{x : 1 \leqslant x \leqslant 18, x \in \mathbb{Z}\}$, $P = \{2n : n \in \mathbb{Z}^+\}$ and $Q = \{3n : n \in \mathbb{Z}^+\}$.
   (a) List the elements of P, Q and $P \cap Q$.
   (b) Find $n(P \cap Q)$ and $n(P' \cap Q')$.

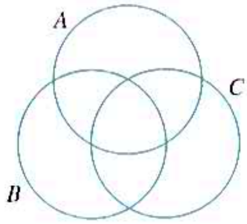
9. (a) In the Venn diagram, shade $(P \cap Q) \cap R'$.



   (b) In the Venn diagram, shade $A \cup (B \cap C)$.



   (c) In the Venn diagram, shade $(A \cap B') \cup C$.



10. Three sets A, B and C are such that $A \cap B \neq \phi$, $A \cap C = \phi$ and $C \subset B$. Draw a Venn diagram to illustrate the sets.

11. Given that $A = \{x : 2 < x \leqslant 5, x \in \mathbb{R}\}$ and $B = \{x : 1 \leqslant x < 3, x \in \mathbb{R}\}$, express in similar set notation each of the following sets: $A \cap B$, $A \cup B$, $A' \cap B'$ and $A' \cup B$.

12. It is given that $\varepsilon = \{x : x < 10, x \in \mathbb{R}^+\}$, $A = \{x : 1 < x \leqslant 4\}$ and $B = \{x : 3 \leqslant x < 6\}$. Represent the sets $A \cap B$, $A' \cap B$, $A \cap B'$ and $A' \cap B'$ on the real line and express each of the sets in similar set notation.

1.3 Applications

Now, we shall use Venn diagrams to depict the number of elements in each set for certain practical situations and apply the basic ideas on sets to solve practical problems. It is important to remember that only the number of elements in each set is written in the Venn diagram.

Example 22

(a) G and H are sets of students who study Geography and History respectively. Using the letters G, H, set notation and x to represent set members, write down an expression for the following statements:

(i) "x is a member of the set of students who study both Geography and History",

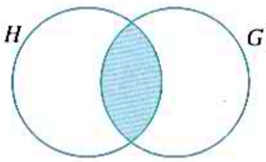
(ii) "x is a member of the set of students who study only History".

Illustrate, by shading on a Venn diagram, the region that x could lie in.

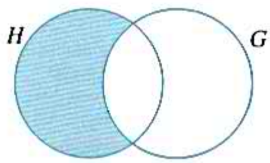
(b) There are 32 students in a class and each studies at least one of the subjects: Geography or History.
Of these, 22 study Geography and 15 study History. By drawing a Venn diagram, find the number of students who study both History and Geography.

Solution:

(a) **(i)** $x \in H$ and $x \in G$
i.e. $x \in H \cap G$



(ii) $x \in H$ and $x \notin G$
i.e. $x \in H \cap G'$

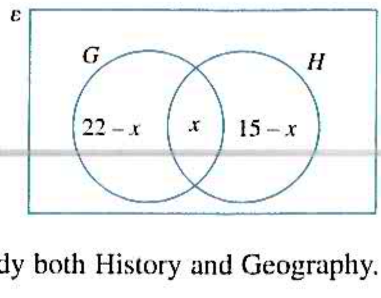


(b) Let H = {students who study History}
and G = {students who study Geography}.
Then $H \cap G$ = {students who study both History and Geography}.

Let $n(H \cap G) = x$.

Since each studies at least one of the subjects,

$$H \cup G = \varepsilon.$$
$$n(H \cup G) = n(\varepsilon)$$



i.e.

$$(22 - x) + x + (15 - x) = 32$$
$$37 - x = 32$$
$$x = 5$$

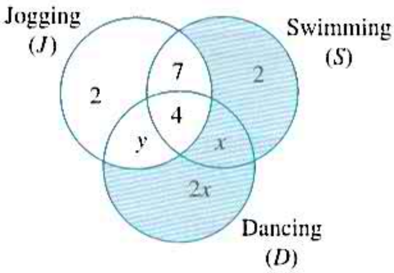
There are **5** students who study both History and Geography.

Example 23 Each of the 35 girls in a class takes part in at least one of the following three activities: Jogging, Swimming and Dancing
Of the 15 girls who choose Jogging,
4 also choose Swimming and Dancing,
2 choose Jogging only,
7 choose Swimming but not Dancing.
Of the 20 girls who do not choose Jogging,
x choose both Swimming and Dancing,
$2x$ choose only Dancing,
2 choose only Swimming.

(a) Draw a Venn diagram to illustrate this information.
(b) Find the value of x.
(c) Express in set notation {girls who choose Jogging and Dancing but not Swimming}.
(d) How many girls choose Jogging and Dancing but not Swimming?

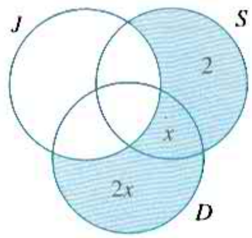
Solution:

(a) Let J = {girls who choose Jogging},
S = {girls who choose Swimming},
D = {girls who choose Dancing}.



(b) J' = {girls who do not choose Jogging}.

$$n(J') = 20$$
$$2x + x + 2 = 20$$
$$3x = 18$$
$$x = \mathbf{6}$$



(c) {Girls who choose Jogging and Dancing but not Swimming}
$= \boldsymbol{J \cap D \cap S'}$

(d) Let $n(J \cap D \cap S') = y$. Given that $n(J) = 15$.
$$\therefore \quad y + 4 + 7 + 2 = 15$$
$$y = 2$$

Only **2** girls choose Jogging and Dancing but not Swimming.

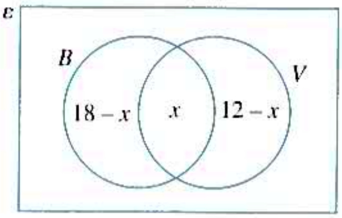
Example 24 Of the 24 students in a class, 18 like to play basketball and 12 like to play volleyball. It is given that

ε = {students in the class},
B = {students who like to play basketball} and
V = {students who like to play volleyball}.

Let $n(B \cap V) = x$ and draw a Venn diagram to illustrate this information.

(a) Describe the set $B \cup V$ and express $n(B \cup V)$ in terms of x.
(b) Find the smallest possible value of x.
(c) Find the largest possible value of x.

Solution:



(a) $B \cup V$ is the set of students who like to play basketball or volleyball.
$n(B \cup V) = (18 - x) + x + (12 - x) = \mathbf{30 - x}$

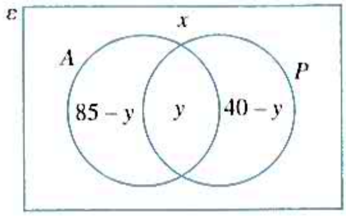
(b) $n(B \cap V)$ is smallest when $B \cup V = \varepsilon$
Then $n(B \cup V) = n(\varepsilon)$
$30 - x = 24$
$x = 6$
The smallest possible value of x is **6**.

(c) $n(B \cap V)$ is largest when $V \subseteq B$
Then $n(B \cap V) = n(V)$
$x = 12$
The largest possible value of x is **12**.

Example 25 In a secondary school, 150 secondary four students are in the Express Stream. Of these, 85 take Additional Mathematics and 40 take Physics.
ε is the set of secondary four students in the Express Stream,
A is the set of students taking Additional Mathematics, and
P is the set of students taking Physics.
Given that $n(A' \cap P') = x$ and $n(A \cap P) = y$, find, by drawing a Venn diagram,

(a) x in terms of y,
(b) the largest possible number of people taking neither Additional Mathematics nor Physics.

Solution: (a)



$$n(\varepsilon) = 150$$
$$(85 - y) + y + (40 - y) + x = 150$$
$$x = 25 + y$$

(b) x is largest $\Leftrightarrow$ y is largest
$\Leftrightarrow$ $P \subseteq A$
$\Leftrightarrow$ $y = 40$
$\Leftrightarrow$ $x = 65$

$\therefore$ the largest possible value of x is 65, i.e. the largest possible number of students taking neither Additional Mathematics nor Physics.

Exercise 1.3

1. The findings of a survey involving 100 undergraduates on the use of pagers and handphones are as follows: 86 use pagers, 52 use handphones and 10 do not use any of the products.
   (a) P and H are two sets of undergraduates who use pagers and handphones respectively. Express in set notation,
   {Undergraduates who use both products}.
   (b) Given that x undergraduates use both products, illustrate the results by drawing a Venn diagram and then find x.

2. In a certain class, 30 pupils took an examination paper in English and an examination paper in Chinese. The results are shown below:

Subject	Pass	Fail
English	22	8
Chinese	18	12

Given that x pupils passed both papers and 3 failed both papers, illustrate the results by drawing a Venn diagram and then find x.

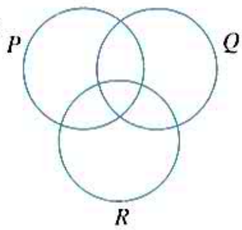
3. (a) F, P and C are sets of students who like fish, pork and chicken respectively. Using the letters F, P, C, set notation and x to represent set members, write down an expression for each of the following statements:
   (i) "x is a member of the set of students who like fish and pork or fish and chicken",
   (ii) "x is a member of the set of students who like fish and pork only".
   Illustrate, by shading on a Venn diagram, the region that x could lie in.

(b) There are 32 pupils in a class. Of these, 22 study History, 19 study Geography and 5 study neither History nor Geography. By drawing a Venn diagram, find the number of students who study both History and Geography.

4. In a supermarket of 36 employees, 24 speak Chinese, 20 speak Malay, 7 speak Tamil, 8 speak both Chinese and Malay, 6 speak both Malay and Tamil, 3 speak both Chinese and Tamil, and everyone speaks at least one of the three languages. By using a Venn diagram, find the number of employees who speak
(a) all the three languages, **(b)** Tamil only.

5. In a community club, a group of 40 women takes part in at least one of the three activities: Cooking, Singing and Dancing.
Of the 25 women who choose Cooking, 8 also choose Singing and Dancing, 3 choose Cooking only, 6 choose Dancing but not Singing.
Of the 15 women who do not choose Cooking, x choose both Singing and Dancing, $2x - 2$ choose only Dancing, 5 choose only Singing.
(a) Draw a Venn diagram to illustrate this information.
(b) Find the value of x.
(c) How many women choose Cooking and Singing but not Dancing?

6. **(a)** On the Venn diagram in the answer space shade $P \cup (Q \cap R)$.



(b) There are 30 boys in a class.
Of these 22 play football, 17 play cricket and 3 play neither football nor cricket.
It is given that

$$\varepsilon = \{\text{boys in the class}\},$$
$$F = \{\text{boys who play football}\},$$
$$C = \{\text{boys who play cricket}\}.$$

(i) Find $n(F \cap C)$.
(ii) Express in set notation
{Boys who play cricket but not football}. *(C)*

7. Each of a group of 20 students studies at least one of the three subjects Chemistry, Physics and Biology. All those who study Physics also study Chemistry.

3 students study all three subjects.
4 students study only Chemistry.
8 students study Physics.
14 students study Chemistry.

(a) Draw a Venn diagram to illustrate this information.
(b) How many students study only Biology?
(c) How many students study Chemistry and Biology but not Physics? *(C)*

8. Given that $n(\varepsilon) = 50$, $n(A) = 25$, $n(B) = 30$ and $n(A \cap B) = x$, find **(a)** the least possible value of x, **(b)** the greatest possible value of x.

9. Given that $n(\varepsilon) = 55$, $n(A) = 28$, $n(B) = x$ and $n(A \cap B) = 5$, express, in terms of x, $n(A \cup B)$ and $n(A' \cap B')$. Hence, find the greatest and the smallest possible values of x.

10. Of the 30 students in a class, 20 like Mathematics and 15 like Physics. It is given that

ε = {students in the class},
M = {students who like Mathematics}
and P = {students who like Physics}.

Express in set notation {students who like Mathematics but not Physics}.
Let $n(M \cap P) = x$.

(a) Express $n(M \cup P)$ in terms of x.
(b) Find the smallest possible value of x.

11. There are 25 children in a class. Of these, 12 are in the School Play and 18 are in the School Choir. It is given that

ε = {children in the class},
P = {children in the School Play}
and C = {children in the School Choir}.

(a) Find $n(P')$.
(b) Find the smallest possible value of $n(P \cap C)$.
(c) Express in set notation {children who are neither in the School Play nor the School Choir}. *(C)*

12. By drawing a Venn diagram, or otherwise, answer the following questions:

(a) Given that $n(\varepsilon) = 60$, $n(S) = 33$ and $n(F) = 36$, find the least possible value of $n(S \cap F)$.
(b) In a group of 60 people, 33 can speak Spanish and 36 can speak French. Find the greatest possible number of people who can speak Spanish, but not French. *(C)*

***13.** Of the 32 students in a class, 26 own desktop computers and 9 own notebook computers. It is given that

ε = {students in the class},
A = {students who own a desktop computer}
and B = {students who own a notebook computer}.

Let $n(A \cap B) = x$.

(a) Express $n(A' \cap B')$ in terms of x.
(b) Find the smallest and largest possible values of x.

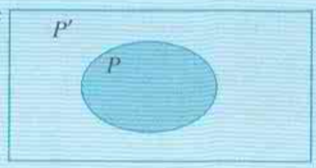
***14.** Given that $n(\varepsilon) = 50$, $n(P) = 28$, $n(Q) = 35$ and $n(P' \cap Q') = x$, express $n(P' \cap Q)$, $n(P \cap Q)$ and $n(P \cap Q')$ in terms of x. Find the greatest and the smallest possible values of x.

Important Notes

1. **(a)** A well-defined collection of objects is called a set and each object is called a member or an **element** of the set.
   (b) A set is denoted by a capital letter and is expressed by
   (i) listing its elements, e.g. $V = \{a, e, i, o, u\}$,
   (ii) a set-builder notation,
   e.g. $\{x : x \text{ is a prime number and } x < 30\}$.
   (c) For any finite set P, $n(P)$ denotes the number of elements in P.
   (d) A null or empty set is denoted by { } or ϕ.

2. (a) For any two sets P and Q:
   (i) $P = Q$ if they have the same elements
   (ii) $P \subseteq Q$ if $x \in P \Rightarrow x \in Q$ *(Subset)*
   (iii) $\phi \subseteq P$
   (iv) $P \cap Q = \{x : x \in P \text{ and } x \in Q\}$ *(Intersection of P and Q)*
   (v) $P \cap Q = \phi \Rightarrow P$ and Q are disjoint sets
   (vi) $P \cup Q = \{x : x \in P \text{ or } x \in Q\}$ *(Union of P and Q)*

(b) For any set P and a universal set ε which consists of all the elements under consideration:
   (i) $P \subseteq \varepsilon$ and $0 \leqslant n(P) \leqslant n(\varepsilon)$
   (ii) $P' = \{x : x \in \varepsilon \text{ and } x \notin P\}$, the complement of P
   (iii) $P \cap P' = \phi$
   (iv) $P \cup P' = \varepsilon$ and $n(P) + n(P') = n(\varepsilon)$



3. (a) For $P \subseteq \varepsilon$, $Q \subseteq \varepsilon$ and $P \cap Q \neq \phi$

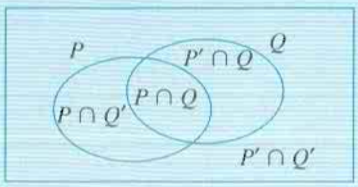


Fig. (a)

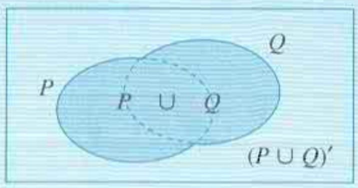


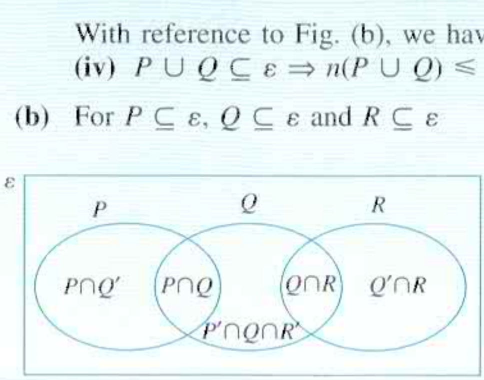
Fig. (b)

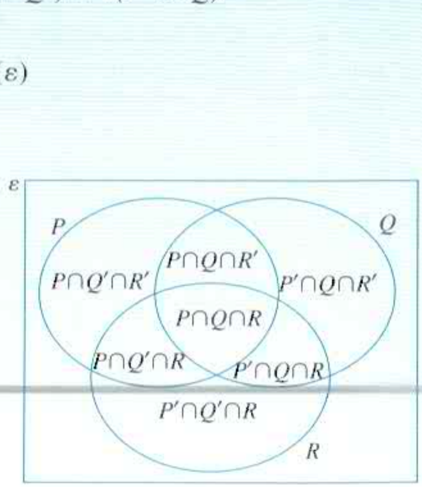
With reference to Fig. (a), we have
(i) $P \cap Q \subseteq P$ and $P \cap Q \subseteq Q$
$\Rightarrow$ $n(P \cap Q) \leqslant n(P)$ and $n(P \cap Q) \leqslant n(Q)$
(ii) $n(P \cap Q) + n(P' \cap Q) + n(P \cap Q') + n(P' \cap Q') = n(\varepsilon)$

With reference to Fig. (a) and Fig. (b),
(iii) $n(P \cap Q') + n(P \cap Q) + n(P' \cap Q) = n(P \cup Q)$
and $n(P' \cap Q') = n(P \cup Q)'$

With reference to Fig. (b), we have
(iv) $P \cup Q \subseteq \varepsilon \Rightarrow n(P \cup Q) \leqslant n(\varepsilon)$

(b) For $P \subseteq \varepsilon$, $Q \subseteq \varepsilon$ and $R \subseteq \varepsilon$





Miscellaneous Example

Example 26 It is given that

$$\varepsilon = \{x : x \text{ is a positive integer}\},$$
$$A = \{x : x \text{ is a multiple of } 2\},$$
$$B = \{x : x \text{ is a multiple of } 3\}$$
and $$C = \{x : x \text{ is a multiple of } 6\}.$$

Show that

(a) $C \subseteq A$ and $C \subseteq B$,

(b) $A \cap B = C$.

Solution:

(a) For any $x \in C$, $x = 6m = 2(3m) \in A$ where m $\in \mathbb{Z}$

$\Rightarrow C \subseteq A$.

Similarly, for any element $x \in C$, $x = 6m = 3(2m) \in B$

$\Rightarrow C \subseteq B$.

(b) $C \subseteq A$ and $C \subseteq B \Rightarrow C \subseteq A \cap B$ (1)

Next, for any $x \in A \cap B$

$\Rightarrow x \in A$ and $x \in B$

$\Rightarrow x = 2n$ and $x \in B$ *where* $n \in \mathbb{Z}$

$\Rightarrow x = 2n$ and $2n \in B$

$\Rightarrow x = 2n$ and $n = 3k$ *where* $k \in \mathbb{Z}$

$\Rightarrow x = 2(3k) = 6k$

$\Rightarrow x \in C$

Hence $A \cap B \subseteq C$ (2)

From (1) and (2): $A \cap B = C$

In the above example, we apply the following result for the equality of two sets:

$$P = Q \Leftrightarrow P \subseteq Q \text{ and } Q \subseteq P$$

Miscellaneous Exercise 1

1. List the elements of the following sets:

(a) $A = \{x : x \text{ is a positive integer and } x \text{ is multiple of } 3\}$

(b) $B = \{x : x \text{ is an integer and } 8 < 3x \leqslant 23\}$

(c) $C = \{x : x \text{ is a multiple of 3 and } x < 15\}$

(d) $A \cup (B \cap C)$ and $(A \cup B) \cap C$. Are these two sets equal?

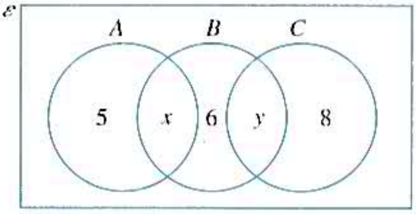
2. Given the universal set $\varepsilon = \{x : x \text{ is a positive integer and } x \leqslant 8\}$,

$$P = \{x : x \text{ is a prime}\}$$
and $$Q = \{x : x \text{ is a factor of } 90\},$$
find an element x such that $x \notin P$ and $x \notin Q$.

3. Given that $\varepsilon = \{x : 1 \leqslant x \leqslant 30, x \in \mathbb{Z}\}$,

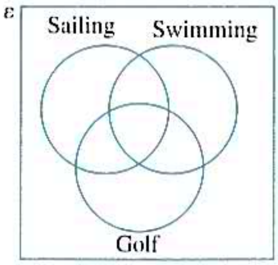
$$A = \{2n : n \in \mathbb{Z}\},$$
$$B = \{3n : n \in \mathbb{Z}\}$$
and $$C = \{6n : n \in \mathbb{Z}\}.$$
List the elements of $A \cap B$ and $(A \cup B) \cap C$.

4. A, B and C are three sets such that $\varepsilon = A \cup B \cup C$. The Venn diagram represents the number of elements in each subset.
Given that $n(A) = n(B' \cap C)$ and $n(A') = n(A \cup B')$, find the value of x and of y.



5. Given that $n(\varepsilon) = 23$, $n(A \cap B) = x$, $n(A) = y$, $n(B) = 2y$ and $n(A' \cap B') = 7$, find the least possible value of y.

6. People staying at a holiday hotel are able to take part in Sailing, Swimming and Golf.



4 people take part in all three activities.
17 people take part in Sailing and Swimming but not Golf.
21 people take part in Swimming and Golf but not Sailing.
12 people take part in Golf and Sailing but not Swimming.
42 people take part in Sailing only.
x people take part in Swimming only.
$(x - 2)$ people take part in Golf only.
16 people do not take part in any of these activities.

(a) Copy the Venn diagram and on your copy show the number of people in each subset.

(b) Given that 250 people are staying at the hotel, calculate
(i) x,
(ii) the number of people who do not take part in Swimming. *(C)*

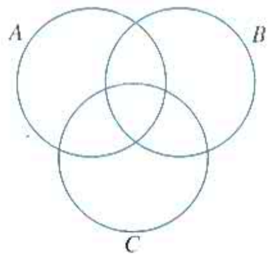
7. **(a)** In a survey carried out for a television company, the viewing choices of 100 families on a particular evening were recorded.
46 families said they had watched "The Syndicate", 62 families said they had watched "Casualty Ward" and 13 families said they had not watched either programme.
By drawing a Venn diagram, or otherwise, calculate
(i) the number of families who had watched both programmes,
(ii) the number of families who had watched "The Syndicate" only.

(b) It is given that

$\varepsilon = \{x : x \text{ is an integer, } 1 \leqslant x \leqslant 16\}$,
$A = \{x : x \text{ is a perfect square}\}$,
$B = \{x : x \text{ is a multiple of } 3\}$,
$C = \{x : x \text{ is a prime number}\}$.

(i) List the elements of A.
(ii) Find $n(A \cap B \cap C)$.
(iii) List the elements of $(A \cup B \cup C)'$. *(C)*

8. **(a)** On the Venn diagram in the answer space, shade the set $A \cup (B \cap C')$.



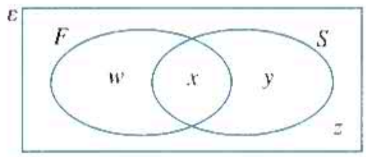
(b) There are 28 girls in a class. Of these, 17 sing in the choir and 15 play the piano. It is given that

ε = {girls in the class},
S = {girls who sing in the choir},
P = {girls who play the piano}.

(i) Find the smallest possible value of $n(S \cap P)$.

(ii) Express in set notation {Girls who neither sing in the choir nor play the piano}. *(C)*

9. Some people were interviewed to find out whether they spoke French, Spanish, French and Spanish, or neither French nor Spanish.
In the Venn diagram below



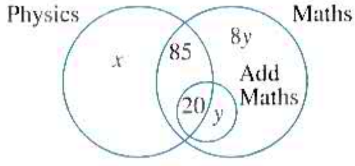
ε is the set of people who were interviewed, F is the set of people who spoke French, and S is the set of people who spoke Spanish.
The letter w, x, y and z represent the number of people in each of the subsets shown.
Given that $n(\varepsilon) = 250$, $n(F) = 80$ and $n(S) = 220$, find

(a) the maximum possible value of w,

(b) the maximum possible value of z,

(c) the maximum possible number of people who spoke both French and Spanish. *(C)*

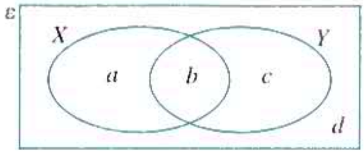
10. In a school, some of the subjects that it is possible to take are Mathematics, Additional Mathematics and Physics. The Venn diagram shows the combinations of these subjects that are possible and the numbers and letters represent the numbers of students in each subset.



(a) Given that the number of students taking Physics is 123, calculate the value of x.

(b) Given that one-sixth of those taking Mathematics also take Additional Mathematics, calculate the value of y and hence find the total number of students taking Mathematics. *(C)*

11.



In a school, 120 boys play cricket.
ε is the set of all the boys who play cricket, X is the set of batsmen and Y is the set of bowlers.
The letter a, b and c in the Venn diagram represent the number of boys in each subset of X and Y.
The letter d represents the number of boys who are neither batsmen nor bowlers.
Given that $n(\varepsilon) = 120$, $n(X) = 80$ and $n(Y) = 48$, find

(a) the value of b if $d = 0$,
(b) the value of d if $b = c$,
(c) the largest possible number of boys who are neither batsmen nor bowlers.

(C)

12. Given that n/m means 'n is a factor of m' and that

$$A = \{x : x \in \mathbb{Z}^+, 2/x\},$$
$$B = \{x : x \in \mathbb{Z}^+, 3/x\},$$
and
$$C = \{x : x \in \mathbb{Z}^+, 5/x\},$$

where $\mathbb{Z}^+$ is the set of positive numbers, list the five smallest members in each of the sets $A' \cap C$, $A' \cap B'$ and $B \cap C'$.
Find the smallest number which belongs to $(A' \cap B') \cup C$ and does not belong to $A' \cap (B' \cup C)$.

(C)

2 Simultaneous Equations

2.1 Simultaneous Linear Equations in Two Unknowns

As you already know, simultaneous linear equations may be solved either by **substitution** or by **elimination**.

Example 1 Solve the simultaneous equations:

$$2x + 3y = 7 \quad \text{.............. (1)}$$
$$3x - 4y = 2 \quad \text{.............. (2)}$$

Solution:

Method A (By substitution)

From (1): $x = \dfrac{7-3y}{2}$ .... (3)

Substitute (3) into (2):

$$3\left(\frac{7-3y}{2}\right) - 4y = 2 \quad \text{.............. (4)}$$

$2 \times (4)$: $3(7 - 3y) - 8y = 4$

$\Rightarrow \quad y = 1$

Substitute $y = 1$ into (3):

$x = 2$

Method B (By elimination)

To eliminate, say, y:

$4 \times (1)$: $8x + 12y = 28$ .... (3)

$3 \times (2)$: $9x - 12y = 6$ .... (4)

$(3) + (4)$: $17x = 34$

$x = 2$

Substitute $x = 2$ into (1) or (2), $y = 1$

Note: We may check our answer by verifying that $x = 2$, $y = 1$ satisfy both equations (1) and (2).

Exercise 2.1

Solve the simultaneous linear equations in questions 1–3.

1. $2x + y = 5$; $x + 2y = 7$

2. $2x - 3y = 12$; $3x + 4y = 1$

3. $4x + 3y = 8$; $2y - 3x = 11$

4. When given the simultaneous equations

$$5x + 2y = 9,$$
$$3x + 4y = 8,$$

a particular student gave the answer $x = \frac{3}{2}$, $y = \frac{3}{4}$.

Without actually solving the equations, check whether his answer is correct.

5. If $(x, y) = (2, 3)$ is a solution of the equations

$$ax + by = 1,$$
$$ay - bx = 8,$$

find the value of a and of b.

6. If $(x, y) = (p, 3)$ is a solution of the equations

$$3x + 2y = 6,$$
$$x - qy = 2,$$

find the value of p and of q.

7. Use a graph plotter to plot the simultaneous equations in Example 1.

(a) What are the shapes of their graphs?

(b) How is the solution of these simultaneous equations represented geometrically?

(c) Clear away the above graphs. Graph $y + 2x = 3$ and $2y + 4x = 5$. Do these simultaneous equations have a solution? Give your reason.

(d) What about the simultaneous equations $3y - 2x = 4$ and $4x - 6y = -8$? How many solutions are there?

2.2 Simultaneous Linear and Non-Linear Equations in Two Unknowns

A pair of simultaneous equations, one linear and the other non-linear, can generally be solved by substitution. Firstly, obtain an equation in one unknown by substitution and solve it for this unknown. Next, use the linear equation to find the other unknown. The following examples should make the process clear.

Example 2 Solve the simultaneous equations:

$$y^2 + (2x + 3)^2 = 10 \quad \text{.............................. (1)}$$
$$2x + y = 1 \quad \text{.............................. (2)}$$

Solution:

From (2):

$$2x = 1 - y \text{ .... } (3)$$

Substitute (3) into (1):

$$y^2 + [(1 - y) + 3]^2 = 10$$
$$y^2 + (4 - y)^2 = 10$$
$$2y^2 - 8y + 16 = 10$$
$$y^2 - 4y + 3 = 0$$
$$(y - 1)(y - 3) = 0$$
$$\Rightarrow \quad y = 1 \quad \text{or} \quad y = 3$$

Substitute these values into (3):

$$x = 0 \quad \text{or} \quad x = -1$$

$\therefore \quad x = 0, y = 1$ or $x = -1, y = 3$

Example 3

Solve the simultaneous equations:

$$y - x = 3$$
$$\frac{2}{x} - \frac{x}{y} = 1$$

Solution:

$$\frac{2}{x} - \frac{x}{y} = 1$$
$$\frac{2y - x^2}{xy} = 1$$
$$2y - x^2 = xy \text{ .... } (1)$$

Now

$$y - x = 3$$
$$y = x + 3 \text{ .... } (2)$$

Substitute (2) into (1):

$$2(x + 3) - x^2 = x(x + 3)$$
$$2x^2 + x - 6 = 0$$
$$(x + 2)(2x - 3) = 0$$

$$\Rightarrow \qquad x = -2 \quad \text{or} \quad x = \frac{3}{2}$$

Substitute these values into (2):

$$y = 1 \quad \text{or} \quad y = \frac{9}{2}$$

$\therefore \quad x = -2, y = 1$ or $x = \frac{3}{2}, y = \frac{9}{2}$

Fig. (a) shows the graphs whose equations are given in Example 2. Notice that the solutions of these simultaneous equations appear as (0, 1) and (–1, 3), the points of intersection of the two graphs.

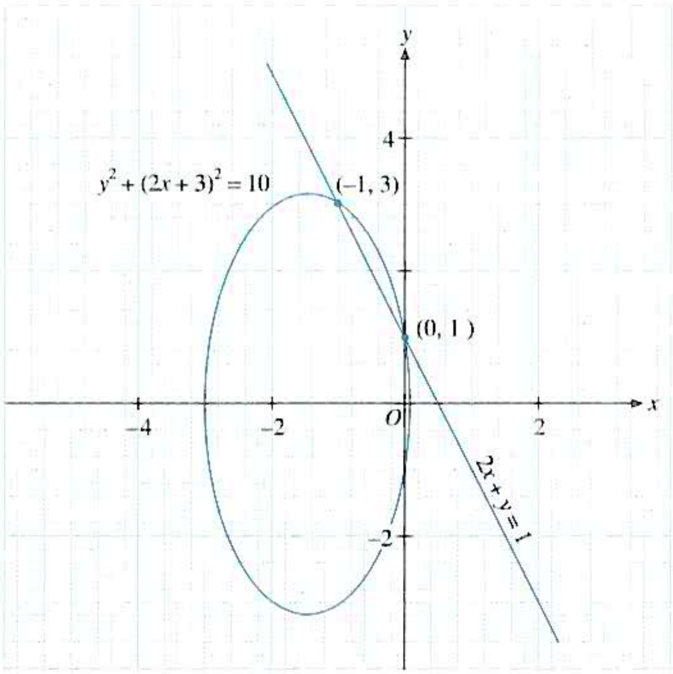


Fig. (a)

Similarly, Fig. (b) illustrates the solutions of the simultaneous equations in Example 3.

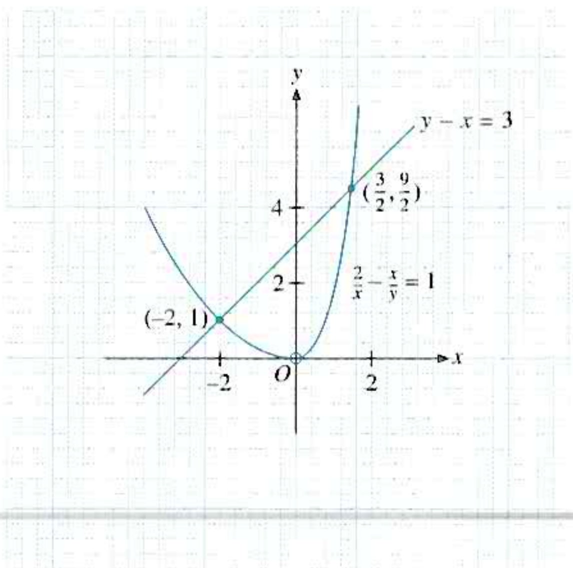


Fig. (b)

Exercise 2.2

Solve the pairs of simultaneous equations in questions 1–12.

1. $y^2 - 4x = 0$
$2x + y = 4$

2. $x + y = 2$
$2x^2 + xy + 1 = 0$

3. $3x + y = 1$
$x^2 + y^2 = 5$

4. $3s^2 + 2t^2 = 11$
$3s + 2t = 1$

5. $x + y = 5$
$(x + 1)^2 + (y + 1)^2 = 25$

6. $3x - 2y = 1$
$(x - 2)^2 + (2y + 3)^2 = 26$

7. $x^2 - 2xy + y^2 = 1$
$x - 2y = 2$

8. $2x + 3y = 5$
$x^2 + xy - 2y^2 = 10$

9. $3y - x = 3$
$\frac{2}{3y} - \frac{1}{x} = 2$

10. $3x + 2y = 10$
$\frac{3}{x} + \frac{2}{y} = 5$

11. $2x + y + 2 = 0$
$\frac{1}{x} + \frac{2}{y} = \frac{1}{2}$

12. $x - 5y = 8$
$x^2 + 9y^2 + 3x = 30 + xy$

13. Find the coordinates of the points of intersection of the line $2x + 3y + 2 = 0$ with the curve $2xy = -1$. *(C)*

14. The line $2x + 3y = 1$ intersects the curve $x(x + y) = 10$ at A and B. Calculate the coordinates of A and of B. *(C)*

15. The line $y = 2x + 3$ intersects the curve $xy + 20 = 5y$ at A and B. Calculate the coordinates of A and of B. *(C)*

16. On a graph plotter, plot the graphs whose equations are

$$21x + 6y = 10 \quad \text{and} \quad (7x - 20)^2 + (6y + 10)^2 = 200.$$

Estimate, to 1 decimal place, the solutions of these simultaneous equations from the coordinates of their points of intersection. Solve these equations algebraically and obtain their **exact** solutions.

Important Notes

1. Simultaneous linear equations can be solved either by substitution or by elimination.
2. Simultaneous linear and non-linear equations are generally solved by substitution as follows:
   Step 1: Obtain an equation in one unknown and solve this equation.
   Step 2: Substitute the results from step 1 into the linear equation to find the other unknown.
3. The points of intersection of two graphs are given by the solutions of their simultaneous equations.

Miscellaneous Example

Example 4 Solve the simultaneous equations:

$$5x + 3y = 3x^2 - y^2 = 2x + y + 1$$

Solution: We have $3x^2 - y^2 = 5x + 3y$ ....(1)

and $5x + 3y = 2x + y + 1$ ....(2)

From (2):

$$2y = 1 - 3x$$

$$y = \frac{1 - 3x}{2} \quad \text{..........(3)}$$

Substitute (3) into (1):

$$3x^2 - \left(\frac{1-3x}{2}\right)^2 = 5x + 3\left(\frac{1-3x}{2}\right)$$

$$3x^2 - \left(\frac{1 - 6x + 9x^2}{4}\right) = 5x + 3\left(\frac{1-3x}{2}\right) \quad \text{..........(4)}$$

$$4 \times (4): \; 12x^2 - (1 - 6x + 9x^2) = 20x + 6(1 - 3x)$$

$$3x^2 + 4x - 7 = 0$$

$$(x - 1)(3x + 7) = 0$$

$$\Rightarrow x = 1 \text{ or } -\frac{7}{3}$$

$$\text{and } y = -1 \text{ or } 4 \qquad \textit{From (3)}$$

$\therefore$ $x = \mathbf{1}$, $y = \mathbf{-1}$ or $x = -\frac{7}{3}$, $y = \mathbf{4}$

Miscellaneous Exercise 2

Solve the simultaneous equations in questions 1–3.

1. $x + 2y = 5$, $2x + y = 2xy$

2. $4x^2 - 9(y + 1)^2 = 36$, $2x + 3y = -5$

3. $\frac{2x}{y} + \frac{y}{x} = 3$, $3x - y = 2$

4. Solve for x and y, $\frac{x^2}{6} - \frac{y}{4} = 1$, $x + y = 5$. *(C)*

5. Find the coordinates of the points of intersection of the line $x + 2y = 10$ and the curve $2y^2 - 7y + x = 0$. *(C)*

6. Calculate the coordinates of the points of intersection of the curve $x^2 + y^2 = 8$ and the straight line $2x - y = 2$. *(C)*

7. If (2, 1) is a solution of the simultaneous equations

$$x^2 + xy + ay = b,$$
$$2ax + 3y = b,$$

find the value of a and of b. Find also the other solution.

***8.** If $(1, p)$ is a solution of the simultaneous equations

$$12x^2 - 5y^2 = 7,$$
$$2p^2x - 5y = 7,$$

find the value of p and the other solution.

9. Find the dimensions of the rectangle whose perimeter is 36 m and which is such that the square of the length of the diagonal is 170 m^2. *(C)*

3 Indices, Surds and Logarithms

3.1 Indices (Exponents) and Surds

When n is a positive integer, a^n is defined as:

$$a^n = \underbrace{a \times a \times a \times \ldots \times a}_{n \text{ factors}}$$

where a is called the **base**, and n, the **index** or **exponent** or **power**.

For example,

$$5^4 = 5 \times 5 \times 5 \times 5.$$

We shall restrict ourselves to positive bases (i.e. $a > 0$). Extending the definition to zero, negative and fractional indices, we have the following results:

For $a > 0$ and positive integers p and q:

$$a^0 = 1, \quad a^{-p} = \frac{1}{a^p}, \quad a^{\frac{1}{p}} = \sqrt[p]{a}, \quad a^{\frac{q}{p}} = (\sqrt[p]{a})^q$$

For example, $2^0 = 1$, $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$, $5^{\frac{1}{2}} = \sqrt{5}$ and $4^{\frac{5}{2}} = (\sqrt{4})^5 = 32$.

With these extended definitions, the following rules of indices hold for positive bases, a and b, and any rational indices, m and n.

$$\left.\begin{aligned} a^m \times a^n &= a^{m+n} \\ \frac{a^m}{a^n} &= a^{m-n} \end{aligned}\right\} \text{rules for same base}$$

$$(a^m)^n = a^{mn}$$

$$\left.\begin{aligned} a^n \times b^n &= (ab)^n \\ \frac{a^n}{b^n} &= \left(\frac{a}{b}\right)^n \end{aligned}\right\} \text{rules for same index}$$

Example 1 Simplify the following:

(a) $9^{\frac{1}{3}} \times 9^{\frac{1}{6}}$ (b) $\dfrac{4^{\frac{3}{4}}}{4^{\frac{9}{4}}}$

(c) $8^{\frac{1}{2}} \times 2$ (d) $12^3 \div 6^3$

Solution:

(a) $9^{\frac{1}{3}} \times 9^{\frac{1}{6}} = 9^{\frac{1}{3}+\frac{1}{6}}$
$= 9^{\frac{1}{2}}$
$= \sqrt{9}$
$= \mathbf{3}$

(b) $\dfrac{4^{\frac{3}{4}}}{4^{\frac{9}{4}}} = 4^{\frac{3}{4}-\frac{9}{4}}$
$= 4^{-\frac{3}{2}}$
$= (2^2)^{-\frac{3}{2}}$
$= 2^{-3}$
$= \mathbf{\frac{1}{8}}$

(c) $8^{\frac{1}{2}} \times 2^{\frac{1}{2}} = (8 \times 2)^{\frac{1}{2}}$
$= 16^{\frac{1}{2}}$
$= \sqrt{16}$
$= \mathbf{4}$

(d) $12^3 \div 6^3 = \left(\frac{12}{6}\right)^3$
$= 2^3$
$= \mathbf{8}$

Example 2 If $3^x = y$, express 3^{4-x} and 9^{x+1} in terms of y.

Solution:

$3^{4-x} = 3^4 \times 3^{-x}$
$= 81\left(\frac{1}{3^x}\right)$
$= \mathbf{\frac{81}{y}}$

$9^{x+1} = 9^x \times 9^1$
$= 9(3^2)^x$
$= 9(3^x)^2$
$= \mathbf{9y^2}$

In the rules for same power, namely $a^n \times b^n = (ab)^n$

$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n,$$

if $n = \frac{1}{2}$, we get $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

These two results are useful in simplifying expressions with the square root sign.

Example 3 Simplify the following:

(a) $\sqrt{12}$ (b) $\dfrac{\sqrt{1500}}{\sqrt{20}}$ (c) $(3 + \sqrt{2})(1 + \sqrt{2})$

Solution:

(a) $\sqrt{12} = \sqrt{4 \times 3}$
$= \sqrt{4} \times \sqrt{3}$
$= \mathbf{2\sqrt{3}}$

(b) $\dfrac{\sqrt{1500}}{\sqrt{20}} = \sqrt{\dfrac{1500}{20}}$
$= \sqrt{75}$
$= \sqrt{25 \times 3}$
$= \sqrt{25} \times \sqrt{3}$
$= \mathbf{5\sqrt{3}}$

(c)
$$\begin{aligned}(3 + \sqrt{2})(1 + \sqrt{2}) &= 3(1 + \sqrt{2}) + \sqrt{2}(1 + \sqrt{2}) \\ &= 3 + 3\sqrt{2} + \sqrt{2} + \sqrt{2}\sqrt{2} \\ &= 3 + 4\sqrt{2} + 2 \\ &= \mathbf{5 + 4\sqrt{2}}\end{aligned}$$

The answers in Example 3 involved the roots $\sqrt{3}$ and $\sqrt{2}$. Unlike $\sqrt{4}$, $\sqrt{81}$ and $\sqrt[3]{27}$ which have exact roots 2, 9 and 3 respectively, these roots cannot be evaluated exactly. They are called **surds**. They are irrational numbers as they cannot be expressed as fractions of the form $\frac{a}{b}$ where a and b are integers.

Observe that the product of two surds need not be a surd:

$$\sqrt{3}\sqrt{3} = 3$$

and

$$\begin{aligned}(5 - 3\sqrt{2})(5 + 3\sqrt{2}) &= 5^2 - (3\sqrt{2})^2 \\ &= 25 - 9(2) \\ &= 7\end{aligned}$$

More generally, the product of $a\sqrt{h} + b\sqrt{k}$ and $a\sqrt{h} - b\sqrt{k}$ is a rational number and we call these surds **conjugate surds**. Conjugate surds may be used to **rationalise the denominator**, that is, to change the denominator, which contains surd, to a rational number.

Example 4 By rationalising the denominators, simplify

(a) $\dfrac{3}{\sqrt{2}}$, (b) $\dfrac{8}{3 - \sqrt{5}}$ (c) $\dfrac{4\sqrt{3} - 3\sqrt{2}}{2\sqrt{3} - 3\sqrt{2}}$.

Solution:

(a)
$$\begin{aligned}\frac{3}{\sqrt{2}} &= \frac{3}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) \\ &= \mathbf{\frac{3}{2}\sqrt{2}}\end{aligned}$$

(b)
$$\begin{aligned}\frac{8}{3 - \sqrt{5}} &= \frac{8}{3 - \sqrt{5}}\left(\frac{3 + \sqrt{5}}{3 + \sqrt{5}}\right) \\ &= \frac{8(3 + \sqrt{5})}{3^2 - (\sqrt{5})^2} \\ &= \frac{8(3 + \sqrt{5})}{9 - 5} \\ &= \mathbf{6 + 2\sqrt{5}}\end{aligned}$$

(c)
$$\begin{aligned}\frac{4\sqrt{3} - 3\sqrt{2}}{2\sqrt{3} - 3\sqrt{2}} &= \frac{4\sqrt{3} - 3\sqrt{2}}{2\sqrt{3} - 3\sqrt{2}}\left(\frac{2\sqrt{3} + 3\sqrt{2}}{2\sqrt{3} + 3\sqrt{2}}\right) \\ &= \frac{8(3) + 12\sqrt{6} - 6\sqrt{6} - 9(2)}{(2\sqrt{3})^2 - (3\sqrt{2})^2} \\ &= \frac{6 + 6\sqrt{6}}{4(3) - 9(2)} \\ &= \mathbf{-1 - \sqrt{6}}\end{aligned}$$

Exercise 3.1

1. Without using a calculator, evaluate the following:

   (a) $(2^3 \times 3^2)^0$ (b) $8^{\frac{1}{2}} \times 8^{\frac{1}{6}}$ (c) $(2\sqrt{3})^4$

   (d) $\dfrac{9^{\frac{1}{3}} \times 3^{\frac{1}{3}}}{6}$ (e) $25^{\frac{1}{4}} \times 5^{\frac{1}{3}} \times 5^{\frac{1}{6}}$ (f) $\dfrac{3^{\frac{1}{3}} \times 3^0 \times 9^{\frac{1}{3}}}{27^{\frac{2}{3}}}$

2. Given $y = 2x^{\frac{3}{2}}$, find
   (a) y when $x = 9$, (b) x when $y = 16$.

3. If $2^x = y$, express the following in terms of y.
   (a) 2^{x+3} (b) 2^{2x-1} (c) $3(2^{2-x})$
   (d) $4(2^3)^{1-x}$ (e) $8(4^{x-2})$ (f) $8^x - 4^{-x}$

4. If $3^x = y$ and $2^x = z$, express the following in terms of y and/or z.
   (a) $2(9^{x+1})$ (b) 6^{x-1} (c) $9^{-x} \times 12^x$

5. Show that $(8x^2)^{8-r}\left(\dfrac{1}{2x}\right)^r = 2^{24-4r}(x^{16-3r})$.

6. Simplify the following:

   (a) $\sqrt{18}$ (b) $\sqrt{6} \times \sqrt{8}$ (c) $\dfrac{\sqrt{60}}{\sqrt{5}}$

   (d) $2\sqrt{3} + 5\sqrt{3} - 3\sqrt{3}$ (e) $4\sqrt{2} - \sqrt{8} + \sqrt{50}$ (f) $(2 + \sqrt{5})(2 - \sqrt{5})$
   (g) $(1 + \sqrt{3})(2 - \sqrt{3})$ (h) $(1 - 2\sqrt{7})^2$

7. By rationalising the denominators, simplify the following:

   (a) $\dfrac{6}{\sqrt{3}}$ (b) $\dfrac{3}{\sqrt{2} + 1}$ (c) $\dfrac{9}{2\sqrt{3} - 3}$

   (d) $\dfrac{12\sqrt{5}}{2\sqrt{5} - 4}$ (e) $\dfrac{8}{\left(\sqrt{6} + 2\right)^2}$ (f) $\dfrac{3\sqrt{2} - 4}{3\sqrt{2} + 4}$

   (g) $\dfrac{\sqrt{3} + 2\sqrt{2}}{\sqrt{3} - 2\sqrt{2}}$ (h) $\dfrac{3\sqrt{5} - 2\sqrt{3}}{3\sqrt{3} - \sqrt{5}}$

3.2 Exponential Equations

An equation that contains a variable in an index (or exponent) is called an **indicial** or **exponential equation**. The simplest form is $a^x = b$. If b can be expressed as a^n, then:

$$a^x = a^n \Rightarrow x = n, \text{ where } a \neq -1, 0, 1$$

Example 5 Solve the following exponential equations:

(a) $4^x = \frac{1}{16}$ **(b)** $5^{x^2-1} = 1$ **(c)** $9^{2x+1} - 27^x = 0$

Solution:

(a)
$$\begin{aligned} 4^x &= \frac{1}{16} \\ &= \frac{1}{4^2} \\ 4^x &= 4^{-2} \\ \therefore\ x &= -2 \end{aligned}$$

(b)
$$\begin{aligned} 5^{x^2-1} &= 1 \\ 5^{x^2-1} &= 5^0 \\ \Rightarrow x^2 - 1 &= 0 \\ x^2 &= 1 \\ \therefore\ x &= \mathbf{1 \text{ or } -1} \end{aligned}$$

(c)
$$\begin{aligned} 9^{2x+1} - 27^x &= 0 \\ 9^{2x+1} &= 27^x \\ (3^2)^{2x+1} &= (3^3)^x \\ \Rightarrow 2(2x+1) &= 3x \\ \therefore\ x &= -2 \end{aligned}$$

Example 6 Given that $y = ax^b + 10$, and that $y = 26$ when $x = 2$ and $y = 64$ when $x = 3$, find the value of a and of b.

Solution:

$y = ax^b + 10$

$y = 26$ when $x = 2$ $\Rightarrow$ $26 = a(2^b) + 10$

$a(2^b) = 16$ .... (1)

$y = 64$ when $x = 3$ $\Rightarrow$ $64 = a(3^b) + 10$

$a(3^b) = 54$ .... (2)

(2) ÷ (1):
$$\begin{aligned} \frac{3^b}{2^b} &= \frac{54}{16} \\ \left(\frac{3}{2}\right)^b &= \frac{27}{8} \\ \left(\frac{3}{2}\right)^b &= \left(\frac{3}{2}\right)^3 \\ \therefore\ b &= 3 \end{aligned}$$

Substitute $b = 3$ into (1): $8a = 16$

$\therefore\ a = 2$

Example 7 Solve the following simultaneous equations:

$$4^x(2^y) = 16$$
$$\frac{3^y}{27^x} = \frac{1}{3}$$

Solution:

$$4^x(2^y) = 16$$
$$(2^2)^x(2^y) = 2^4 \quad \textit{(change to the same base)}$$
$$2^{2x+y} = 2^4$$
$$\Rightarrow 2x + y = 4 \text{ .... (1)}$$
$$\frac{3^y}{27^x} = \frac{1}{3}$$
$$\frac{3^y}{\left(3^3\right)^x} = 3^{-1}$$
$$3^{y-3x} = 3^{-1}$$
$$\Rightarrow y - 3x = -1 \text{ .... (2)}$$

By solving (1) and (2), we get $x = \mathbf{1}$ and $y = \mathbf{2}$.

Example 8

By using the substitution $y = 2^x$, find the value of x such that $4^{x+1} = 2 - 7(2^x)$.

Solution:

Since $4^{x+1} = 4(4^x) = 4(2^2)^x = 4(2^x)^2$, the equation becomes

$$4(2^x)^2 = 2 - 7(2^x)$$

Substitute $2^x = y$ into the equation:

$$4y^2 = 2 - 7y$$
$$4y^2 + 7y - 2 = 0$$
$$(4y - 1)(y + 2) = 0$$
$$y = \frac{1}{4} \quad \text{or} \quad y = -2$$
$$2^x = \frac{1}{4} \quad \text{or} \quad 2^x = -2 \quad \textit{(no solution)}$$
$$2^x = 2^{-2}$$
$$\therefore \; x = \mathbf{-2}$$

Note: $a > 0 \Rightarrow a^x > 0$ and hence there is no solution for which $2^x < 0$.

Exercise 3.2

1. Solve the following equations:

(a) $3^{2x} = 27$

(b) $4^x = 32$

(c) $\left(\sqrt{2}\right)^{3x} = \frac{1}{8}$

(d) $\left(\frac{1}{9}\right)^{x+2} = 3$

(e) $4^x(5^{2x}) = 10$

(f) $5^x - 25 = 0$

(g) $7^{x^2-4} - 1 = 0$

(h) $8^x = 4^{x+1}$

(i) $9^x = \left(\sqrt{3}\right)^{x+2}$

(j) $\frac{4^x}{2^{x-1}} = 8^{2-x}$

(k) $2^{x^2} = 4^{2(x-1)}$

(l) $7^{x^2} = 49^{6-2x}$

(m) $4^{x^2-6} - 16^{x+1} = 0$

(n) $3^{x^2} = \frac{9^{x+1}}{27^x}$

2. Given that $y = ax^n - 23$, and that $y = 4$ when $x = 3$ and $y = 220$ when $x = 9$, find the value of a and of n.

3. Solve the following simultaneous equations:

   (a) $5^x(25^{2y}) = 1$ and $3^{5x}(9^y) = \frac{1}{9}$ (b) $2(4^x) = 32^y$ and $\frac{125^x}{25^y} = 625$

   (c) $\frac{3^x}{9^y} = 27$ and $4^{2x}(2^{6y}) = \frac{1}{4}$

4. Show that $\sqrt[n]{2 \times 4^m} = 2^{\frac{2m+1}{n}}$. Hence find the value of m and of n which satisfy the equations $\sqrt[n]{2 \times 4^m} = 8$ and $\frac{27^m}{9^{n+1}} = 81$ simultaneously.

5. By using the substitution $y = 2^x$, find the value of x such that $3(2^{x-1}) = 2^x + 4$.

6. By using the substitution $y = 3^x$, find the values of x such that

   (a) $3^x = 4 - 3(3^{-x})$, (b) $3^{2x} - 3^{x+2} = 3^x - 9$.

7. By using appropriate substitution or otherwise, solve the following equations:

   (a) $5^{2x} - 6(5^x) + 5 = 0$ (b) $2^{2x} - 10(2^x) + 16 = 0$

   (c) $2(16^x) - 5(4^x) + 2 = 0$ (d) $7^{x+1} - 2 = 2(7^x) + 3$

   (e) $9^{x+1} + 1 = 10(3^x)$ (f) $6(3^{x-1}) = 3^4 - 3^x$

8. Show that the equation $2^{2x+1} = 3(2^x) + 2$ is satisfied by only one value of x.

9. Solve the following equations:

   (a) $8\left(\frac{1}{2}\right)^x + 2 = 2^x$ (b) $4^x = 2^{\frac{8}{x}}$

   (c) $3^{x+1}(9^{2-x}) = \frac{1}{3}$ (d) $4^{x+\frac{1}{2}} - 2^{x+3} = 8 + 7(2^x)$

   (e) $\frac{5^{x+1} - 20}{10 + 5^x} = 3$ (f) $2^x(5^x) = \frac{1}{100}(10^{x-1})^4$

10. If $\frac{r^2}{4}(3x)^r\left(\frac{2}{9x^2}\right)^{6-r}$ can be simplified to $\frac{k}{x^3}$, find the values of the constants r and k.

*11. Solve the simultaneous equations $64(4^y) = 16^x$ and $3^y = 4(3^{x-2}) - 1$.

*12. By using an appropriate substitution, find the value of x for which $x^{\frac{3}{2}} - 8x^{-\frac{3}{2}} = 7$.

3.3 Logarithms

We know that $10 = 10^1$ and $100 = 10^2$; so if $40 = 10^x$, then $10^1 < 10^x < 10^2$ and hence $1 < x < 2$. The value of x may be found using a scientific calculator as follows:

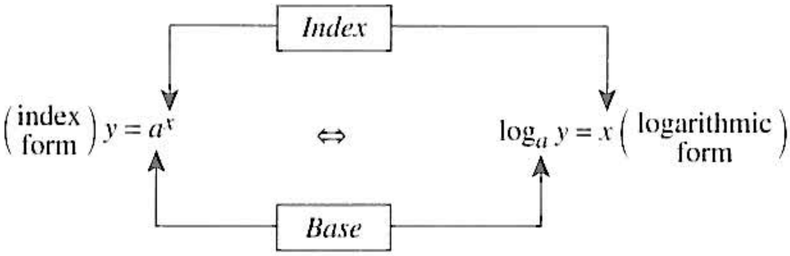
$$40 = 10^x \Rightarrow x = \log_{10} 40 \approx 1.602\ 06$$

For any positive number a, except 1, $y = a^x \Leftrightarrow \log_a y = x$

For example, $9 = 3^2 \Leftrightarrow 2 = \log_3 9$.
Observe that the **logarithm** $\log_3 9 (= 2)$ is the power, 2, in $9 = 3^2$.
Thus $\log_3 9$ = the power to which we raise 3 to get 9.

Note: We read $\log_a y$ as 'the logarithm of y to base a' or simply 'log, base a, of y'.

The relationship between the two forms is illustrated as follows:



Observe that the bases in both forms are the same and that **the logarithm is the power**.

Example 9 Convert the following to logarithmic form:

(a) $4^2 = 16$ (b) $2^5 = 32$
(c) $10^3 = 1000$ (d) $2^{-1} = \frac{1}{2}$

Solution:

(a) $4^2 = 16 \Leftrightarrow 2 = \log_4 16$ ***(The logarithm is the power.)***
(b) $2^5 = 32 \Leftrightarrow 5 = \log_2 32$
(c) $10^3 = 1000 \Leftrightarrow 3 = \log_{10} 1000$
(d) $2^{-1} = \frac{1}{2} \Leftrightarrow -1 = \log_2 \frac{1}{2}$

Example 10 Convert the following to index form:

(a) $3 = \log_2 8$ (b) $2 = \log_5 25$ (c) $\frac{1}{2} = \log_2 \sqrt{2}$

Solution:

(a) $3 = \log_2 8 \Leftrightarrow 2^3 = 8$
(b) $2 = \log_5 25 \Leftrightarrow 5^2 = 25$
(c) $\frac{1}{2} = \log_2 \sqrt{2} \Leftrightarrow 2^{\frac{1}{2}} = \sqrt{2}$

Observe that $x = \log_a y \Rightarrow y = a^x$. So $a > 0 \Rightarrow y = a^x > 0$.

If $a = 1$, then $1^2 = 1 \Rightarrow \log_1 1 = 2$.
But $1^3 = 1 \Rightarrow \log_1 1 = 3$.
Hence, $\log_1 1$ does not have a definite value and we say that $\log_a y$ is not defined when the base $a = 1$.

For $\log_a y$ to be defined:
(a) $y > 0$
(b) $a > 0, a \neq 1$

For example, $\log_1 2$, $\log_{-3} 4$ and $\log_2 (-1)$ are not defined.

Equations of the form $x = \log_a y$ can be solved (for any of the three variables y, a or x) by first writing them in index form. Care must be taken to check the answer(s) to see whether the logarithm is defined.

For example, $\log_x 4 = 2 \Rightarrow 4 = x^2$
$x = 2$ or -2

Since x is the base, $x > 0$ and $x \neq 1$; so $x = -2$ is rejected and the only solution is $x = 2$.

Example 11 Solve the following equations:
(a) $\log_3 x = 2$ (b) $x = \log_{\frac{1}{2}} 16$ (c) $\log_x (4x - 3) = 2$

Solution:
(a) $\log_3 x = 2 \Leftrightarrow 3^2 = x$
$\therefore x = 9$

(b) $x = \log_{\frac{1}{2}} 16 \Leftrightarrow \left(\frac{1}{2}\right)^x = 16$
$(2^{-1})^x = 2^4$
$-x = 4$
$\therefore x = -4$

(c) $\log_x (4x - 3) = 2 \Leftrightarrow x^2 = 4x - 3$
$x^2 - 4x + 3 = 0$
$(x - 1)(x - 3) = 0$
$\therefore x = 1$ or 3
For the logarithm to be defined, $x = 3$.

Check: Since x is the base, $x = 1$ is rejected as $\log_x (4x - 3)$ is not defined for this value.

Example 12 Prove that, where the logarithms are defined,
(a) $\log_a a = 1$ (b) $\log_a 1 = 0$
(c) $\log_a x^r = r \log_a x$, where $r \in \mathbb{R}$

Solution:
(a) Converting to logarithmic form,
$a = a^1 \Rightarrow \log_a a = 1$

(b) Similarly, $1 = a^0 \Rightarrow \log_a 1 = 0$

(c) Let $m = \log_a x$. Then $x = a^m$,
$x^r = (a^m)^r = a^{rm}$
$\Rightarrow \log_a x^r = rm$
$\therefore \log_a x^r = r \log_a x$

Note: You should remember the results in (a) to (c).
With these results, we have $\log_5 1 = 0$
and $\log_2 8 = \log_2 2^3 = 3 \log_2 2 = 3 \times 1 = 3$.

Exercise 3.3

1. Convert the following to logarithmic form:
   (a) $2^4 = 16$ (b) $3^{-2} = \frac{1}{9}$ (c) $100 = 10^2$
   (d) $a^3 = y$ (e) $2^x = p$ (f) $x^4 = 2 - k$

2. Convert the following to index form:
   (a) $3 = \log_5 125$ (b) $-2 = \log_2 \left(\frac{1}{4}\right)$ (c) $\log_4 64 = 3$
   (d) $\log_x 3 = 4$ (e) $\log_3 y = n$ (f) $p + 1 = \log_2 (4y)$

3. Check whether the logarithm $\log_x (5 - 2x)$ is defined for each of the following:
   (a) $x = 2$ (b) $x = 0.5$ (c) $x = 3$
   (d) $x = 2.5$ (e) $x = 1$ (f) $x = \sqrt{2}$

4. Solve the following equations:
   (a) $\log_2 x = 3$ (b) $\log_x 9 = 2$
   (c) $x = \log_4 8$ (d) $\log_3 (x - 2) = 1$
   (e) $\log_2 (2x + 1) = -3$ (f) $\log_9 \sqrt{27} = x$
   (g) $\log_x (6x - 8) = 2$ (h) $\log_x 8 = \frac{3}{2}$

5. Evaluate the following:
   (a) $\log_4 4 - \log_2 2^3$ (b) $\log_2 1 + 2 \log_5 5^2$ (c) $(3 - \log_3 3)^3$
   (d) $\left(\dfrac{3 \log_x x + 2}{4 - 2 \log_5 1}\right)^2$ (e) $\log_2 (6 - \log_7 7^{-2})$ (f) $\log_2 4 + \log_3 81$

6. Given that $\log_4 x = 2$ and $\log_2 y = 3$, evaluate $\frac{x}{y}$.

7. Given that $\log_3 x = a$ and $\log_{\sqrt{3}} y = b$, express xy^2 as a power of 3.

8. Solve the following simultaneous equations:
   (a) $\log_x 16 = 4$ and $\log_2 y = x$ (b) $\log_y x = 2$ and $xy = 8$

9. If $\log_2 (\log_3 x) = \log_5 5$, find x.

*10. Given that $\log_4 y = a$ and $\log_8 (2y) = b$, show that $2a = 3b - 1$.

3.4 Common and Natural Logarithms

Common Logarithms

Logarithms to base 10 are called **common logarithms**. We often write '$\log_{10}$' as 'lg'. For example $\log_{10} 5$ and $\log_{10} (x + 1)$ are abbreviated as lg 5 and lg $(x + 1)$ respectively. Common logarithms can be evaluated using a scientific calculator.

Recall that by the definition of logarithm, $\lg Y = X \Leftrightarrow Y = 10^X$.

Natural Logarithms

Besides base 10, another important base is e.

In the given table, observe that $\left(1+\frac{1}{n}\right)^n$ approaches a certain value as n becomes very large (i.e. as n tends to infinity, ∞). This limiting value is denoted by e and e ≈ 2.718 28 (to 5 decimal places).

n	$\left(1+\frac{1}{n}\right)^n$
10	2.593 742 5
10^2	2.704 813 8
10^3	2.716 923 9
10^4	2.718 145 9
⋮	⋮
10^7	2.718 281 7
10^8	2.718 281 8
↓	↓
∞	e

Logarithms to base e are called **natural** (or **Naperian**) **logarithms**. '$\log_e$' is often abbreviated as 'ln', so $\log_e 6$ and $\log_e x$ are written as $\ln 6$ and $\ln x$ respectively. Natural logarithms may also be evaluated using a scientific calculator.

By definition, $\mathbf{\ln Y = X \Leftrightarrow Y = e^X}$.

Using a calculator, we can use common and natural logarithms to solve equations of the form $a^x = b$, especially when b cannot be expressed as a^n.

We shall be using the result in Example 12(c): $\log_a x^r = r \log_a x$.

Example 13 Solve the following equations:
(a) $6^{x+2} = 21$ **(b)** $e^{3x} = 9$

Solution:

(a)
$$\begin{aligned} 6^{x+2} &= 21 \\ \lg 6^{x+2} &= \lg 21 \\ (x+2)\lg 6 &= \lg 21 \\ x + 2 &= \frac{\lg 21}{\lg 6} \\ x &= \frac{\lg 21}{\lg 6} - 2 \\ &\approx \mathbf{-0.301} \end{aligned}$$

(b)
$$\begin{aligned} e^{3x} &= 9 \\ \ln e^{3x} &= \ln 9 \\ 3x \ln e &= \ln 9 \\ 3x &= \ln 9 \\ x &= \frac{1}{3}\ln 9 \\ &\approx \mathbf{0.732} \end{aligned}$$

convert to log form directly

Example 14 Express $3^x(2^{2x}) = 7(5^x)$ in the form $a^x = b$. Hence find x.

Solution:

Since $3^x(2^{2x}) = 3^x(2^2)^x = (3 \times 4)^x = 12^x$

the equation becomes:
$$\begin{aligned} 12^x &= 7(5^x) \\ \left(\frac{12}{5}\right)^x &= 7 \end{aligned}$$

Taking lg of each side:
$$\begin{aligned} x \lg \frac{12}{5} &= \lg 7 \\ x &= \frac{\lg 7}{\lg \frac{12}{5}} \\ &\approx \mathbf{2.22} \end{aligned}$$

Exercise 3.4

Solve the equations in questions 1 to 15.

1. $5^x = 9$
2. $(1.6)^x = 21$
3. $2(3^x) = 5$
4. $4 - 7^{2x} = 1$
5. $e^x = 7$
6. $e^{3x} = 14$
7. $4e^{2x} = 21$
8. $e^{4x} - 125 = 0$
9. $3^{x+1} = 12$
10. $4^{2x-3} = 20$
11. $e^{1+x} = 19$
12. $e^{\frac{1}{2}x} = 0.7$
13. $(4.1)^x = \pi$
14. $6^{\frac{2}{x}} = 4$
15. $e^{x^2} = 312$
16. Solve the following equations giving your answers correct to 3 significant figures.
    (a) $2^x3^x = 10$ (b) $4^x = 9(5^x)$ (c) $2^{x+1} = 3^x$
17. Given $3^{x+1} \cdot 2^{x-2} = 21$, show that $6^x = 28$. Hence or otherwise, find x.
18. Find x, correct to 3 significant figures, in each of the following:
    (a) $5^{x-1} \cdot 3^{x+2} = 10$ (b) $2^{2x} \cdot 5^{x+1} = 7$
    (c) $4(3^{2x}) = e^x$ (d) $3^x \cdot 10^{2x} = 4 \cdot 20^{x-2}$
19. Using the substitution $y = e^x$, solve the following equations:
    (a) $2e^{2x} - 3e^x = 2$ (b) $e^x = 7 - 12e^{-x}$ (c) $e^{3x} + 2e^x = 3e^{2x}$
20. Using suitable substitutions, solve for x.
    (a) $9^x - 4 = 3^{x+1}$ (b) $2e^x = 7\sqrt{e^x} - 3$
21. Solve the simultaneous equations $4^{x+3} = 32(2^{x+y})$, $9^x + 3^y = 10$.
22. Given $y = 5e^{0.2x}$ find
    (a) y when $x = 3$, (b) x when $y = 12$.
23. For each of the following, find y in terms of x.
    (a) $10^y = x + 1$ (b) $e^{1-y} = 3x$ (c) $\ln (y + 1) = x$
    (d) $2 \lg y = x - 2$ (e) $e^{2y} = x - 4$ (f) $\ln (x + y) - 4x = 0$
24. Solve for x.
    (a) $\lg x = 0.61$ (b) $(\ln x)^2 = 3$ (c) $\ln x = \lg 2$
    (d) $\lg 3x = 9$ (e) $\ln 2 \cdot \ln 4x = 3$ (f) $\lg (x - 2) = (\lg 3)^2$
    (g) $\ln 4x = \lg 3 \cdot \lg 5$ (h) $\lg (2x + 1) = \log_3 3$ (i) $\lg (x - 1) = \ln (e^2 - 1)$

3.5 Laws of Logarithms

We shall now familiarise ourselves with the basic laws of logarithms. The proofs of some of these laws are given in Example 23.

(1) Change-of-Base Law

If a, b and c are positive numbers and $a \neq 1$, $c \neq 1$, then

$$\log_a b = \frac{\log_c b}{\log_c a}.$$

Proof: Let $x = \log_a b$, then $a^x = b$.
Taking logarithms, base c, of both sides.

$$\log_c a^x = \log_c b$$
$$x \log_c a = \log_c b$$
$$x = \frac{\log_c b}{\log_c a}$$
$$\therefore \quad \log_a b = \frac{\log_c b}{\log_c a}$$

In particular, when $c = b$, $\log_a b = \dfrac{\log_b b}{\log_b a} = \dfrac{1}{\log_b a}$.

$$\mathbf{\log_a b = \frac{1}{\log_b a}}$$

Using the change-of-base law, an alternative solution to Example 13 found on page 46 is as follows:

$$6^{x+2} = 21 \quad \Rightarrow \quad x + 2 = \log_6 21$$
$$= \frac{\lg 21}{\lg 6}$$
$$x \approx -0.301$$

Example 15 Evaluate $\log_7 5 \,.\, \log_5 9 \,.\, \log_9 7$.

Solution:

$$\log_7 5 \,.\, \log_5 9 \,.\, \log_9 7 = \frac{\lg 5}{\lg 7} \cdot \frac{\lg 9}{\lg 5} \cdot \frac{\lg 7}{\lg 9}$$
$$= \mathbf{1}$$

(2) Product Law

The **logarithm of a product** is the **sum** of the logarithms of the factors.

If a, x, y are positive numbers and $a \neq 1$, then
$$\mathbf{\log_a xy = \log_a x + \log_a y.}$$

Hence
$$\log_2 (3 \times 5) = \log_2 3 + \log_2 5$$
$$\log_3 7x = \log_3 7 + \log_3 x$$
$$\log_4 x(x + 3) = \log_4 x + \log_4 (x + 3)$$

Note: The expression $\log_4 (x + 3)$ is not equal to $\log_4 x + \log_4 3$.

Example 16 Simplify the following:

(a) $\log_6 3 + \log_6 2$ (b) $\log_2 40 + \log_2 0.1 + \log_2 0.25$

Solution:

(a) $\log_6 3 + \log_6 2 = \log_6 (3 \times 2) = \log_6 6 = 1$

(b) $$\begin{aligned}\log_2 40 + \log_2 0.1 + \log_2 0.25 &= \log_2 (40 \times 0.1 \times 0.25)\\ &= \log_2 1\\ &= 0\end{aligned}$$

(3) Quotient Law

The **logarithm of a quotient** is the logarithm of the numerator **minus** the logarithm of the denominator:

If a, x, y are positive numbers and $a \neq 1$, then

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y.$$

Hence $\log_3 \left(\frac{7}{2}\right) = \log_3 7 - \log_3 2$

$$\log_2 \left(\frac{x+1}{x}\right) = \log_2 (x+1) - \log_2 x$$

Note: The expression $\frac{\log_3 7}{\log_3 2}$ is not equal to $\log_3 \left(\frac{7}{2}\right)$.

Example 17 Simplify the following:

(a) $\log_4 8 - \log_4 2$ (b) $\log_2 x^4 - \log_2 x^3$

Solution:

(a) $$\begin{aligned}\log_4 8 - \log_4 2 &= \log_4 \left(\frac{8}{2}\right)\\ &= \log_4 4\\ &= 1\end{aligned}$$

(b) $$\begin{aligned}\log_2 x^4 - \log_2 x^3 &= \log_2 \left(\frac{x^4}{x^3}\right)\\ &= \log_2 x\end{aligned}$$

(4) Power Law

As seen in Example 12(c):

If a and x are positive numbers and $a \neq 1$, then
$\log_a x^r = r \log_a x$ for any real number r.

Hence, $\log_3 2^4 = 4 \log_3 2$

$\log_2 x^{-3} = -3 \log_2 x$

$\log_5 \sqrt{x} = \log_5 x^{\frac{1}{2}}$

$= \frac{1}{2} \log_5 x$

Note: $\log_a x^r$ means $\log_a (x^r)$. It is not the same as $(\log_a x)^r$, so $(\log_a x)^r \neq r \log_a x$.

Example 18 Evaluate the following:

(a) $\log_2 2\sqrt{2}$ **(b)** $\dfrac{\log_a 8}{\log_a 4}$

Solution:

(a)
$$\begin{aligned} \log_2 2\sqrt{2} &= \log_2 (2 \times 2^{\frac{1}{2}}) \\ &= \log_2 2^{\frac{3}{2}} \\ &= \frac{3}{2} \log_2 2 \\ &= \mathbf{\frac{3}{2}} \end{aligned}$$

(b)
$$\begin{aligned} \frac{\log_a 8}{\log_a 4} &= \frac{\log_a 2^3}{\log_a 2^2} \\ &= \frac{3 \log_a 2}{2 \log_a 2} \\ &= \mathbf{\frac{3}{2}} \end{aligned}$$

Note: For (a), we may let $x = \log_2 2\sqrt{2}$.

Then
$$\begin{aligned} 2^x &= 2\sqrt{2} \\ \Rightarrow \quad 2^x &= 2 \times 2^{\frac{1}{2}} \\ &= 2^{\frac{3}{2}} \\ \therefore \quad x &= \mathbf{\frac{3}{2}} \end{aligned}$$

Example 19 Given that $\log_a 2 = 0.301$ and $\log_a 3 = 0.477$, find

(a) $\log_a \frac{3}{4}$, **(b)** $\dfrac{\log_a 2a}{\log_a 3}$.

Solution:

(a)
$$\begin{aligned} \log_a \frac{3}{4} &= \log_a 3 - \log_a 4 \\ &= \log_a 3 - \log_a 2^2 \\ &= \log_a 3 - 2 \log_a 2 \\ &= 0.477 - 2(0.301) \\ &= \mathbf{-0.125} \end{aligned}$$

(b)
$$\begin{aligned} \frac{\log_a 2a}{\log_a 3} &= \frac{\log_a 2 + \log_a a}{\log_a 3} \\ &= \frac{0.301 + 1}{0.477} \\ &= \mathbf{2.727} \end{aligned}$$

Example 20 Given that $\lg x = m$ and $\lg y = n$, express $\lg \left(10\sqrt{\frac{x}{y}}\right)$ in terms of m and n.

Solution:

$$\begin{aligned}\lg \left(10\sqrt{\frac{x}{y}}\right) &= \lg 10 + \lg\sqrt{\frac{x}{y}} \\ &= 1 + \lg \left(\frac{x}{y}\right)^{\frac{1}{2}} \\ &= 1 + \frac{1}{2}\lg \left(\frac{x}{y}\right) \\ &= 1 + \frac{1}{2}(\lg x - \lg y) \\ &= \mathbf{1 + \frac{1}{2}(m - n)}\end{aligned}$$

Example 21 Given that $2 \lg xy = 2 + \lg (1 + x) + \lg y$ where x and y are both positive, express y in terms of x.

Solution:

$$\begin{aligned}2 \lg xy &= 2 + \lg (1 + x) + \lg y \\ \lg (xy)^2 &= 2 + \lg (1 + x)y \\ \lg x^2y^2 - \lg (1 + x)y &= 2 \\ \lg \frac{x^2y^2}{(1 + x)y} &= 2 \\ \frac{x^2y}{(1 + x)} &= 10^2 \\ y &= \mathbf{\frac{100(1 + x)}{x^2}}\end{aligned}$$

Example 22 Express $3 + \log_2 5$ as a single logarithm.

Solution:

Since $\log_2 2 = 1$, $\quad 3 + \log_2 5 = 3 \log_2 2 + \log_2 5$

$$\begin{aligned}&= \log_2 2^3 + \log_2 5 \\ &= \log_2 (8 \times 5) \\ &= \mathbf{\log_2 40}\end{aligned}$$

For any number n, $n = n \log_a a = \log_a a^n$.

Example 23 If a, x, y are positive numbers and $a \neq 1$, prove that

(a) $\log_a xy = \log_a x + \log_a y$,

(b) $\log_a \frac{x}{y} = \log_a x - \log_a y$,

Solution:

Let $m = \log_a x$ and $n = \log_a y$. Then $x = a^m$ and $y = a^n$.

(a) $\quad xy = a^m \times a^n = a^{m+n}$

$\Rightarrow \quad \log_a xy = m + n$

$\therefore \quad \log_a xy = \log_a x + \log_a y$

(b) $\frac{x}{y} = \frac{a^m}{a^n} = a^{m-n}$

$\Rightarrow \log_a \frac{x}{y} = m - n$

$\therefore \log_a \frac{x}{y} = \log_a x - \log_a y$

Exercise 3.5

1. Evaluate the following by converting them to common or natural logarithms using the change-of-base law:
   (a) $\log_5 7$ (b) $\log_3 11$ (c) $\log_4 (5.3)$
   (d) $\log_{\frac{1}{2}} 9$ (e) $\log_6 \pi$ (f) $\log_{2.5} (6.7)$

2. Evaluate the following:
   (a) $\log_3 5 \,.\, \log_5 27$ (b) $\frac{\log_5 4 \,.\, \log_2 10}{\log_{25} \sqrt{10}}$

3. If a, b and c are positive numbers other than 1, show that
   $$\log_b a \,.\, \log_c b \,.\, \log_a c = 1.$$

4. Evaluate the following logarithms without using a calculator:
   (a) $\log_2 4$ (b) $\lg \left(\frac{1}{10}\right)$ (c) $\log_2 8$
   (d) $\log_3 27$ (e) $\log_2 \left(\frac{1}{4}\right)$ (f) $\log_9 \sqrt{3}$

5. Evaluate the following expressions without using a calculator:
   (a) $\log_8 2 + \log_8 4$ (b) $\log_3 36 - \log_3 12$
   (c) $\log_2 60 - \log_2 15$ (d) $\log_3 4 + \log_3 2 - \log_3 72$
   (e) $\log_6 54 - 2 \log_6 3$ (f) $\log_5 4 + 2 \log_5 3 - 2 \log_5 6$

6. Simplify and express each of the following as a single logarithm:
   (a) $\log_a 8 - 2 \log_a 4$
   (b) $2 \log_x 5 - 3 \log_x 2 + \log_x 4$
   (c) $\lg \left(\frac{8}{75}\right) - 2 \lg \left(\frac{3}{5}\right) + 4 \lg \left(\frac{3}{2}\right)$
   (d) $2 \lg (x + 2) + \lg (x + 1) - \lg (x^2 + 3x + 2)$

7. Evaluate the following:
   (a) $\log_a \sqrt{a}$ (b) $\log_a \frac{1}{a^3}$ (c) $\frac{\log_a 9}{2 \log_a 27}$ (d) $\log_{\sqrt{a}} a^2$

8. Given that $\log_a 3 = 0.477$ and $\log_a 5 = 0.699$, evaluate the following:
   (a) $\log_a 15$ (b) $\log_a 3\sqrt{5}$ (c) $\log_a 0.6$
   (d) $\frac{\log_a 25}{\log_a 3a}$ (e) $\log_a (5a^2)$ (f) $\log_a \left(\frac{9}{5a}\right)$

9. Given that $\log_4 3 = a$ and $\log_4 5 = b$, express the following in terms of a and b:
   (a) $\log_4 45$ (b) $\log_4 20$ (c) $\log_4 75$
   (d) $\log_4 (0.6)$ (e) $\log_4 (0.75)$ (f) $\log_4 (1.8)$

10. Given that $\lg x = p$ and $\lg y = q$, express the following in terms of p and q:

(a) $\lg (xy^2)$ (b) $\lg \left(\frac{10x}{y}\right)$ (c) $\lg \sqrt{10x^3 y}$

(d) $\lg \left(\frac{100\sqrt{x}}{y^2}\right)$ (e) xy (f) $\lg (y^x)$

11. Find y in terms of x when

(a) $\lg y = 1 + 3 \lg x$, (b) $\lg (y + 1) = 2 - \frac{1}{2} \lg x$,

(c) $2 \log_3 y - 4 = 3 \log_3 (x + 2)$, (d) $3 + \log_2 (x + y) = \log_2 (x - 2y)$.

12. If $\log_2 (y + 1) = 2 \log_2 x + c$ and $y = 3$ when $x = 2$, find y in terms of x.

13. Express each of the following as a single logarithm:

(a) $2 + \log_3 5$ (b) $3 - 2 \lg 5$ (c) $3 \log_a 2 - 4 + \log_a a^3$

14. Express the following in the form $\ln x = ax + b$ and find a and b.

(a) $x^3 = e^{6x-1}$ (b) $xe^{-x} = 2.46$ (c) $(xe^x)^2 = 30e^{-x}$

3.6 Logarithmic Equations

An equation that contains a logarithm of a variable quantity is called a **logarithmic equation**. Logarithmic equations can generally be solved using the following property.

> **For two logarithms of the same base,**
> $\log_a M = \log_a N \Leftrightarrow M = N.$

For example, $\log_3 (x + 1) = \log_3 4 \Rightarrow x + 1 = 4$ and so $x = 3$.

Example 24 Solve the equation $\log_2 (x - 1) + \log_2 (x - 4) = \log_2 (2x - 6)$.

Solution:

$$\begin{aligned}
\log_2 (x - 1) + \log_2 (x - 4) &= \log_2 (2x - 6) \\
\log_2 (x - 1)(x - 4) &= \log_2 (2x - 6) \\
\Rightarrow (x - 1)(x - 4) &= 2x - 6 \\
x^2 - 5x + 4 &= 2x - 6 \\
x^2 - 7x + 10 &= 0 \\
(x - 2)(x - 5) &= 0 \\
x &= 2 \text{ or } 5
\end{aligned}$$

We need to check whether each logarithm is defined for these values of x.

When $x = 2$,
$$\begin{aligned}
\log_2 (x - 1) &= \log_2 1 \\
\log_2 (x - 4) &= \log_2 (-2), \text{ undefined} \\
\log_2 (2x - 6) &= \log_2 (-2), \text{ undefined}
\end{aligned}$$

So $x = 2$ is rejected.

When $x = 5$, $\log_2 (x - 1) = \log_2 4$
$\log_2 (x - 4) = \log_2 1$
$\log_2 (2x - 6) = \log_2 4$
All the logarithms are defined.
$\therefore$ for the logarithms to be defined, $x = 5$.

Example 25 Solve the equation $\log_3 (x - 1) + \log_3 (x + 3) - \log_3 (x + 1) = 1$.

Solution:

$$\log_3 (x - 1) + \log_3 (x + 3) - \log_3 (x + 1) = 1$$
$$\log_3 \frac{(x - 1)(x + 3)}{x + 1} = \log_3 3$$
$$\frac{(x - 1)(x + 3)}{x + 1} = 3$$
$$(x - 1)(x + 3) = 3(x + 1)$$
$$x^2 + 2x - 3 = 3x + 3$$
$$x^2 - x - 6 = 0$$
$$(x - 3)(x + 2) = 0$$
$$x = 3 \text{ or } -2$$

When $x = 3$, $\log_3 (x - 1) = \log_3 2$
$\log_3 (x + 3) = \log_3 6$
$\log_3 (x + 1) = \log_3 4$
All the logarithms are defined.
When $x = -2$, $\log_3 (x - 1) = \log_3 (-3)$, undefined
$\log_3 (x + 3) = \log_3 1$
$\log_3 (x + 1) = \log_3 (-1)$, undefined
So $x = -2$ is rejected.
$\therefore$ for the logarithms to be defined, $x = 3$.

Example 26 Solve the following equations:
(a) $\log_3 2 + \log_3 (x + 4) = 2 \log_3 x$
(b) $2 \log_p 8 - \log_p 4 = 2$

Solution:

(a) $\log_3 2 + \log_3 (x + 4) = 2 \log_3 x$
$$\log_3 2(x + 4) = \log_3 x^2$$
$$\Rightarrow \quad 2(x + 4) = x^2$$
$$x^2 - 2x - 8 = 0$$
$$(x - 4)(x + 2) = 0$$
$$x = 4 \text{ or } -2$$
For the logarithms to be defined, $x = 4$.

Note: **(1)** $x = -2$ is discarded as it causes $\log_3 x$ to be undefined.
(2) Had the original equation been
$$\log_3 2 + \log_3 (x + 4) = \log_3 x^2,$$
then both $x = 4$ and $x = -2$ would be acceptable.

(b) $2\log_p 8 - \log_p 4 = 2$

$$\log_p 8^2 - \log_p 4 = 2\log_p p$$
$$\log_p \frac{8^2}{4} = \log_p p^2$$
$$p^2 = 16$$
$$p = 4 \text{ or } -4$$

For the logarithms to be defined, $p = 4$.

Example 27 Solve the equation $\log_5 x = 4\log_x 5$.

Solution:

$$\log_5 x = 4\log_x 5 \quad \textit{(The bases are different.)}$$
$$\log_5 x = 4\frac{1}{\log_5 x}$$
$$\log_5 x \,.\, \log_5 x = 4$$
$$\therefore\ (\log_5 x)^2 = 4$$
$$\log_5 x = 2 \text{ or } -2$$
$$x = 5^2 \text{ or } 5^{-2}$$
$$\therefore\ x = 25 \text{ or } \frac{1}{25}$$

Exercise 3.6

For questions 1–10, solve for x.

1. $\log_2 (x-1) = \log_2 (4x-7)$
2. $\log_3 (x+2) + \log_3 (x-2) = \log_3 (2x-1)$
3. $\lg 18 + \lg\left(\frac{1}{3}x\right) - \lg (x+1) = 0$
4. $\log_3 x + \log_3 (x+2) = 1$
5. $2\lg 5 - \lg (x+2) = 1 - \lg (2x-1)$
6. $\log_2 (x-2) + \log_2 (8-x) - \log_2 (x-5) = 3$
7. $\log_2 (x-1)^2 = 2 + \log_2 (x+2)$
8. $\log_3 (x+2) + \log_3 (10-x) - 3 = 0$
9. $3\log_x 2 + \log_x 18 = 2$
10. $\log_p 2 + \log_p (x-1) = 0$
11. Solve the equation
    (a) $3\lg (x-1) = \lg 8$, (b) $\lg (20y) - \lg (y-8) = 2$. *(C)*
12. Solve the simultaneous equations $\ln (3x-y) = 2\ln 6 - \ln 9$ and $\dfrac{(e^x)^2}{e^y} = e$.
13. Solve the simultaneous equations $3^p = 9(27)^q$, $\log_2 7 - \log_2 (11q-2p) = 1$. *(C)*
14. By using the substitution $y = \log_3 x$ or otherwise, solve the equation
    (a) $\log_3 x + 2 = 3\log_x 3$, (b) $\log_3 x^3 = (\log_3 x)^3$.

15. Solve the following equations:

(a) $\log_3 x = 9 \log_x 3$ (b) $4 \log_4 x - 9 \log_x 4 = 0$

(c) $\log_2 x = \log_4 (x + 6)$ (d) $\log_5 (5 - 4x) = \log_{\sqrt{5}} (2 - x)$

*16. If $2 \log_a x = 1 + \log_a (7x - 10a)$, find x in terms of a.

*17. Find x for which $27 \times 3^{\lg x} = 9^{1 + \lg (x - 20)}$.

Important Notes

1. Indices

(a) *Definitions*

For $a > 0$ and positive integers p and q:

$$a^0 = 1,\ a^{-p} = \frac{1}{a^p},\ a^{\frac{1}{p}} = \sqrt[p]{a},\ a^{\frac{p}{q}} = \left(\sqrt[q]{a}\right)^p$$

(b) *Rules*

For $a > 0$, $b > 0$ and rational numbers m and n:

$a^m \times a^n = a^{m+n}$	$a^n \times b^n = (ab)^n$
$\frac{a^m}{a^n} = a^{m-n}$	$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$
$(a^m)^n = a^{mn}$	

2. Logarithms

(a) *Definition*

For $a > 0$ and $a \neq 1$,

$y = a^x \quad \Leftrightarrow \quad x = \log_a y$

(index form) (logarithmic form)

(b) For $\log_a y$ to be defined,

(i) $y > 0$ and **(ii)** $a > 0$, $a \neq 1$.

(c) Where the logarithms are defined,

$\log_a 1 = 0$ $\qquad \log_a xy = \log_a x + \log_a y$

$\log_a a = 1$ $\qquad \log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

$\log_a b = \frac{\log_c b}{\log_c a}$ $\qquad \log_a x^r = r \log_a x$

(d) When solving logarithmic equations, check the solution(s) with the **original** equation and discard any solution that causes a logarithm to be undefined.

3. Solution of $a^x = b$ where $a \neq -1, 0, 1$

If b can be easily written as a^n, then $a^x = a^n \Rightarrow x = n$.

Otherwise, take logarithms on both sides, i.e. $\lg a^x = \lg b$ and so $x = \frac{\lg b}{\lg a}$.

Example 28 Given that $\log_p (x^2y) = 8$ and $\log_p \left(\frac{y^2}{x}\right) = 6$, evaluate

(a) $\log_p (xy)$, **(b)** $\log_p \left(\frac{x}{y}\right)$.

If $\frac{y}{x} = 9$, find the value of p, of x and of y.

Solution:

$$\log_p (x^2y) = 8 \quad \Rightarrow \quad \log_p x^2 + \log_p y = 8$$
$$2 \log_p x + \log_p y = 8 \quad \text{.........................} \quad (1)$$

$$\log_p \left(\frac{y^2}{x}\right) = 6 \quad \Rightarrow \quad \log_p y^2 - \log_p x = 6$$
$$2 \log_p y - \log_p x = 6 \quad \text{.........................} \quad (2)$$

Eliminate $\log_p x$ by (1) + 2 × (2):

$$\log_p y + 4 \log_p y = 8 + 2(6)$$
$$5 \log_p y = 20$$
$$\log_p y = 4 \quad \text{.........................} \quad (3)$$

Substitute (3) into (1):

$$2 \log_p x + 4 = 8$$
$$\log_p x = 2 \quad \text{.........................} \quad (4)$$

(a) $\log_p (xy) = \log_p x + \log_p y$
$= 2 + 4$
$= \mathbf{6}$

(b) $\log_p \left(\frac{y}{x}\right) = \log_p y - \log_p x$
$= 4 - 2$
$= \mathbf{2}$

If $\left(\frac{y}{x}\right) = 9$, using the result of **(b)**, we get:

$$\log_p 9 = 2$$
$$\Rightarrow \quad p^2 = 9$$
$$p = 3 \text{ or } -3$$

For the logarithms to be defined, $p = \mathbf{3}$.

Substitute $p = 3$ into (3) and (4):

$\log_3 y = 4 \quad \Rightarrow \quad y = 3^4 = \mathbf{81}$

and $\log_3 x = 2 \quad \Rightarrow \quad x = 3^2 = \mathbf{9}$

Miscellaneous Exercise 3

1. Given that $y = 3(4)^{x+2}$, find, without using a calculator, the value of
   (a) y when $x = -0.5$, **(b)** x when $y = 6$.

2. Given that $y = ax^b + 2$ and that $y = 5$ when $x = 2$ and $y = 29$ when $x = 4$, find the value of a and of b.

3. Given that $\log_3 2 = 0.631$, use the substitution $y = 3^x$ to solve the following equations:
(a) $3^x + 10 = 2(3^{x+1})$ (b) $9^x + 2(3^x) = 3^{x+2} - 12$

4. Find x such that
(a) $2e^x = 3 - e^{x+1}$ (b) $e^{-x}(2e^{-x} + 1) = 15$

5. (a) Given that $\log_3 (x - 1) = 2$, evaluate $\lg x$.
(b) Solve the equation $\lg (3x + 2) + 6 \lg 2 = 2 + \lg (2x + 1)$.
(c) Find the value of x which satisfies the equation $e^{2x} - e^x - 6 = 0$. *(C)*

6. (a) Given that $2^x 4^y = 128$ and that $\ln (4x - y) = \ln 2 + \ln 5$, calculate the value of x and of y.
(b) Solve the equation
(i) $\lg (1 - 2x) - 2 \lg x = 1 - \lg (2 - 5x)$,
(ii) $3^{y+1} = 4^y$. *(C)*

7. (a) If $\log_2 k = 2 \log_2 6 + \log_2 10 - 3$, find k.
(b) Solve the simultaneous equations $8 \times 4^y = 2^{2x-1}$, $3^y\sqrt{3^x} = 81$.

8. Without using a calculator, solve the following equations:
(a) $(5^{x+1})^2 = 0.2\sqrt{5^x}$ (b) $\log_x 27 = 1.5$
(c) $\log_9 (3^{x+1}) = x^2$ (d) $\log_2 (\log_x 9) = 1$
(e) $\log_2 x \log_8 x = 12$ (f) $e^{4-x} = e^2 \cdot e^{x^2-4}$
(g) $\log_3 (x - 2) = 3 - \log_3 (x + 4)$ (h) $4^{3x} + \log_2 \left(\frac{1}{8}\right) = 5$
(i) $\log_2 x^2 - \log_2 (2x + 5) = 2$ (j) $\log_2 x = 4 \log_x 2$

9. Solve the following equations:
(a) $3^{x+1} = 8$ (b) $e^{\frac{3}{x}} = 4$ (c) $\log_x 5 = 3$
(d) $\lg (\ln x) = 0.1$ (e) $5^x = e^{2x+1}$ (f) $\ln (e^{2x} - 5) = 2$

10. Given that $\log_2 x = a$ and $\log_8 y = b$, express x^2y and $\frac{x}{y}$ as powers of 2. Given further that $x^2y = 32$ and $\frac{x}{y} = 0.5$, find the value of a and of b.

11. Given that $\log_3 2 = 0.631$, evaluate $\log_3 6$, $\log_3 (2.25)$ and $\log_{\sqrt{3}} 6$.

12. Given that $\log_2 a = p$, express $\log_2 (4a^3)$ and $\log_8 \sqrt{a}$ in terms of p.

13. Given that $\log_2 x = p$ and $\log_4 y = q$, express the following in terms of p and q:
(a) $\log_2 xy$ (b) $\log_4 \frac{x}{y}$
(c) $\log_x 4y$ (d) x^2y

14. Given that $\ln 2 = a$ and $\ln 5 = b$, express $\ln \sqrt[3]{10e}$ in terms of a and b. Find also the number x such that $\ln x = \frac{b - 2a}{2}$.

15. Given that $\log_b (xy^2) = m$ and $\log_b (x^3y) = n$, express $\log_b \frac{y}{x}$ and $\log_b \sqrt{xy}$ in terms of m and n.

16. (a) Solve the equation $2 \lg 5 + \lg (x + 1) = 1 + \lg (2x + 7)$.
(b) A liquid cools from its original temperature of 90°C to a temperature T°C in x minutes. Given that $T = 90(0.98)^x$, find the value of
(i) T when $x = 10$, (ii) x when $T = 27$.
(c) The curves $y = e^{x+1}$ and $y = e^{4-2x}$ meet at P. Find the coordinates of P. *(C)*

17. **(a)** The mass, m grams, of a radioactive substance, present at time t days after first being observed, is given by the formula $m = 24e^{-0.02t}$. Find
   (i) the value of m where $t = 30$,
   (ii) the value of t when the mass is half of its value at $t = 0$.
   (b) Solve the equation $\lg(20 + 5x) - \lg(10 - x) = 1$.
   (c) Given that $x = \lg a$ is a solution of the equation $10^{2x+1} - 7(10^x) = 26$, find the value of a. *(C)*

18. **(a)** Use a spreadsheet program such as Microsoft Excel to complete the following table:

x	$y = 3 + e^{-2x}$
–20	
–15	
–10	
–5	
0	
5	
10	
15	
20	

   What is the value that y approaches as x becomes very large? Does y approach any value when x is very small?
   (b) Repeat **(a)** with $y = 5 + 2e^{-x}$ and $y = 4 + e^{-x}$.
   (c) If $y = 3 + 2e^{x}$, does y approach any value as x becomes very large? Does y approach any value as x becomes very small?
   (d) State the value that $y = 5 - e^{3x}$ approaches as x becomes very small.
   (e) If $y = a + be^{cx}$, how do the values of a, b and c affect the value of y as x becomes
   (i) very large? **(ii)** very small?

19. **(a)** An object is heated in an oven until it reaches a temperature of X degrees Celsius. It is then allowed to cool. Its temperature, θ degrees Celsius, when it has been cooling for time t minutes, is given by the equation $\theta = 18 + 62e^{-\frac{t}{8}}$. Find
   (i) the value of X,
   (ii) the value of θ when $t = 16$,
   (iii) the value of t when $\theta = 48$.
   State the value which θ approaches as t becomes very large.
   (b) Solve the equation
   (i) $\lg x + \lg[5(x + 1)] = 2$, **(ii)** $3^{y+1} = 0.45$. *(C)*

20. The equation $2^{2x+p} - 2^{x+p} = 9(2^x) - 2$ has a solution $x = 1$. Find
(a) the value of p, (b) the other solution of the equation.

21. (a) Solve the following equations:
(i) $\log_{x-2}(2x^2 - 10x + 13) = 1$ (ii) $2 \log_y 5 - \log_y 10 + \log_y 40 = 4$
(b) Given that $\lg(xy) - 2 = 3 \lg y - \lg x + \lg 4$, express y in terms of x.

*22. (a) Find the positive values of x for which $9x^{\frac{2}{3}} + 4x^{-\frac{2}{3}} = 37$.
(b) If $\lg 2 = m$, express $\log_8 5$ in terms of m.

*23. (a) Solve the equation $\lg(3^x - 2^{4-x}) = 2 + \frac{1}{3} \lg 8 - \frac{1}{4} x \lg 16$.
(b) Without using a calculator, evaluate $(\lg 5)^2 + \lg 2 \lg 50$.

24. (a) Given that $p^n = 16p$, express $\log_2 p$ in terms of n.
(b) Without using a calculator, solve $4^x - 3^{x+\frac{1}{2}} = 3^{x-\frac{1}{2}} - 2^{2x-1}$.

*25. (a) Given that $\ln y = 2 \ln(x - 1) + c$ and that $y = 20$ when $x = 3$, find the value of x when $y = 45$.
(b) Given that $a > b > 1$ and $2 \log_a b + 4 \log_b a = 9$, find b in terms of a.

*26. (a) Solve for x in terms of a given that $\log_a 5 + 2 = \log_a(x + a) + \log_a(x - 3a)$.
(b) Find the exact value of x if $(3x)^{\lg 3} = (4x)^{\lg 4}$.

27. (a) The result '$\lg xy = \lg x + \lg y$' is not always true! Give a pair of values of x and y such that the result will not hold. (*Hint:* Notice that, under **2(c)** in the 'Important Notes', we state the result with a certain qualification!)
Similarly, give values of x, y and r such that the following results fail:
(i) $\lg\left(\frac{x}{y}\right) = \lg x - \lg y$ (ii) $\lg x^r = r \lg x$
(b) Are the following results true for *any* real value of a, m and n? If not, give a counter example (i.e. give values of a, m and n which cause the result to fail).
(i) $m^a = n^a \Rightarrow m = n$ (ii) $a^m = a^n \Rightarrow m = n$

28. Use a graph plotter to plot the graph of
(a) $y = 10^x$, (b) $y = e^x$, (c) $y = 2^x$,
(d) $y = \lg x$, (e) $y = \ln x$, (f) $y = \log_2 x$.
Note the shapes of the graphs in **(a)** to **(c)** and **(d)** to **(f)**. Do you notice any relationship between the graphs of **(a)** and **(d)**, **(b)** and **(e)**?
(We shall examine the graphs of exponential and logarithmic functions and their relationship more closely in Chapter 19.)

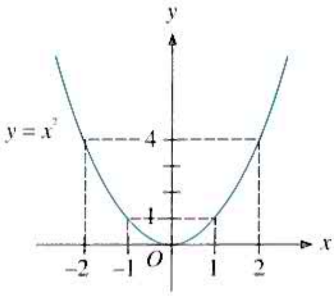
4 Quadratic Expressions and Equations

4.1 Maximum/Minimum Value of a Quadratic Expression

A quadratic expression such as $2x^2 - 8x + 11$ or $-4x^2 + 12x - 9$ has either a minimum or a maximum value. In this section, we shall learn how to find this extreme value for any quadratic expression.

First, let us consider the simplest case $y = x^2$.

x	-2	-1	0	1	2
y	4	1	0	1	4



Observe that the graph is symmetrical about the y-axis and that the least value, 0, of y occurs when $x = 0$. Hence:

$$\mathbf{y = x^2 \geqslant 0 \text{ for all real values of } x}$$

That is, **the square of a real number is non-negative.**

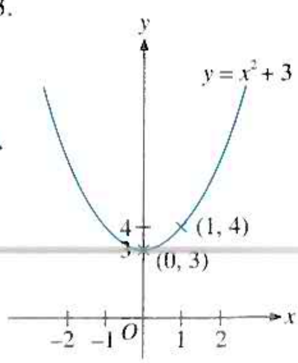
Example 1 State the minimum value of $x^2 + 3$ and the corresponding value of x. Hence sketch the graph of $y = x^2 + 3$.

Solution: For all real x, $\quad x^2 \geqslant 0$

$\Rightarrow \quad x^2 + 3 \geqslant 3$

$\therefore$ the minimum value is **3** when $x =$ **0**.

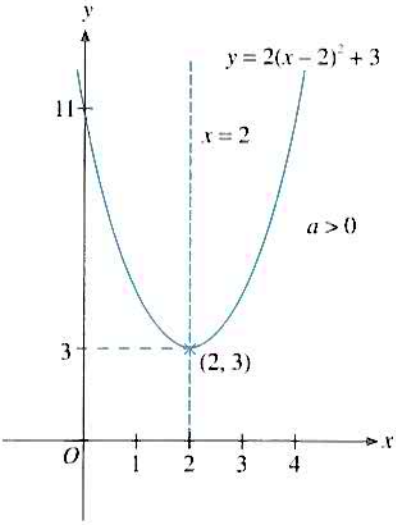
So, the graph of $y = x^2 + 3$ has minimum point $(0, 3)$ as shown in the figure.



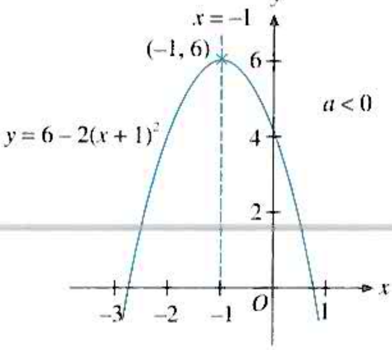
Note: Do you know why we need to give the coordinates of another point $(1, 4)$?

Let us now consider the **completed square form**, $a(x - h)^2 + k$. The expression $2(x - 2)^2 + 3$ is of the form $a(x - h)^2 + k$, where $a = 2$, $h = 2$ and $k = 3$. Since $2(x - 2)^2 + 3 = 2x^2 - 8x + 11$, $2(x - 2)^2 + 3$ is a quadratic expression.

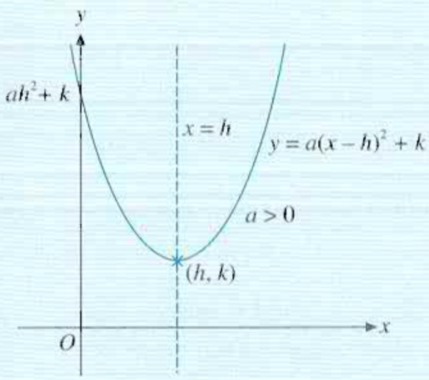
Consider the curve $y = 2(x - 2)^2 + 3$ as shown below.
Note that
(a) $a > 0$ $(a = 2)$
(b) $y - 3 = 2(x - 2)^2 \geqslant 0 \quad \Rightarrow \quad y \geqslant 3$ and so the minimum value of y is 3 and this minimum value occurs when the squared term $(x - 2)^2 = 0$, i.e. when $x = 2$. Hence the curve has a minimum turning point at (2, 3).
(c) the curve is a parabola symmetrical about the vertical line through the turning point, i.e. $x = 2$.

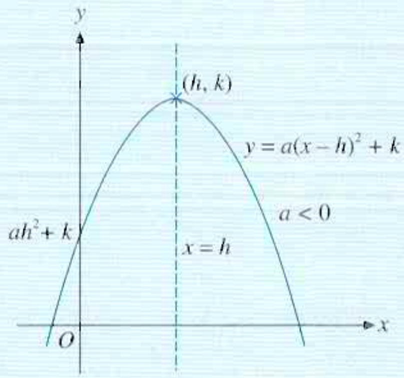


Next, consider the curve $y = 6 - 2(x + 1)^2$, where $a = -2$, $h = -1$ and $k = 6$, as shown below.
Note that
(a) $a < 0$ $(a = -2)$
(b) $y - 6 = -2(x + 1)^2 \leqslant 0 \quad \Rightarrow \quad y \leqslant 6$ and so the maximum value of y is 6. This maximum value occurs when the squared term $(x + 1)^2 = 0$, i.e. when $x = -1$. Hence the curve has a maximum turning point at $(-1, 6)$.
(c) the curve is symmetrical about the line $x = -1$.



In general, the curve $y = a(x - h)^2 + k$ is a parabola of one of the following types.





Note that the squared term $(x - h)^2 = 0 \Rightarrow x = h$ and $y = k$. Hence the turning point is at (h, k).

(a) When $a > 0$, (h, k) is a minimum point and k is the minimum value of y.

(b) When $a < 0$, (h, k) is a maximum point and k is the maximum value of y.

(c) The curve is symmetrical about the line $x = h$.

Example 2

State the minimum value of $2(x - 1)^2 + 5$ and the corresponding value of x. Hence sketch the curve $y = 2(x - 1)^2 + 5$, indicating the coordinates of the turning point and the y-intercept.

Solution:

For $2(x - 1)^2 + 5$, the minimum value is **5** and it occurs when $x =$ **1**.

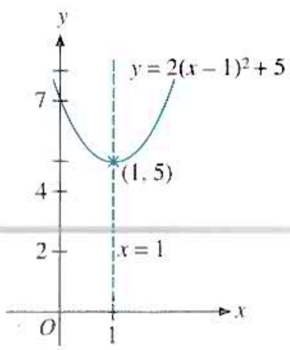
Since $a = 2 > 0$, the turning point **(1, 5)** is a minimum point.

The curve cuts the y-axis at $x = 0$, $y = 7$.

The curve is sketched as shown in the diagram.

Note: (1) The y-intercept (when $x = 0$) is used in sketching a curve.

(2) The parabola is symmetrical about the vertical line through the turning point, i.e. $x = 1$.



Recall that $\left(x+\frac{n}{2}\right)^2 = x^2 + nx + \left(\frac{n}{2}\right)^2$. Hence:

halve square

$$x^2 + nx = \left(x + \frac{n}{2}\right)^2 - \left(\frac{n}{2}\right)^2$$

We may use the above result to **complete the square** and express $ax^2 + bx + c$ in the form $a(x-h)^2 + k$.

For example,

halve square

$$\begin{aligned} 2x^2 + 12x + 8 = 2[x^2 + 6x] + 8 &= 2[(x+3)^2 - 3^2] + 8 \\ &= 2(x+3)^2 - 18 + 8 \\ &= 2(x+3)^2 - 10 \end{aligned}$$

Example 3 Express $y = -4x^2 + 12x - 9$ as $y = a(x-h)^2 + k$. Hence state the maximum value of y and the corresponding value of x. Sketch the curve $y = -4x^2 + 12x - 9$.

Solution:

halve square

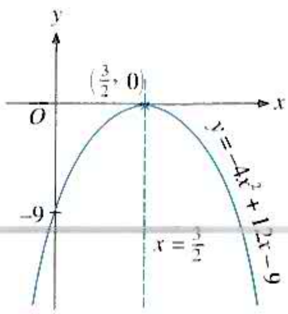
$$\begin{aligned} y = -4[x^2 - 3x] - 9 &= -4\left[\left(x - \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2\right] - 9 \\ &= -4\left(x - \frac{3}{2}\right)^2 + 4\left(\frac{9}{4}\right) - 9 \\ &= -4\left(x - \frac{3}{2}\right)^2 \end{aligned}$$

The maximum value of y is **0** and it occurs when $\left(x - \frac{3}{2}\right)^2 = 0$, i.e. when $x = \frac{3}{2}$.

$\therefore$ the curve has a maximum point at $\left(\frac{3}{2}, 0\right)$.

At the y-axis, $x = 0 \Rightarrow y = -9$.

The curve is sketched as shown in the diagram.



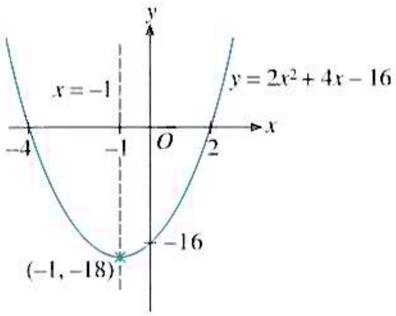
Example 4 By completing the square, sketch the graph of $y = 2x^2 + 4x - 16$, indicating the coordinates of the turning point and the x-intercepts.

Solution: $y = 2[x^2 + 2x] - 16 \quad = 2[(x + 1)^2 - 1^2] - 16 = 2(x + 1)^2 - 18$

Since $a = 2 > 0$, the curve has a minimum point at **(–1, –18)**.
At the y-axis, $x = 0 \quad \Rightarrow \quad y = -16$.
At the x-axis, $y = 0 \quad \Rightarrow \quad 2(x + 1)^2 - 18 = 0$

$$\begin{aligned}(x+1)^2 &= \frac{18}{2}\\ x + 1 &= \pm 3\\ x &= \mathbf{-4,\ 2}\end{aligned}$$

The curve is sketched as shown.



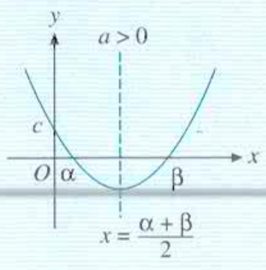
In Example 4, we could have factorised the quadratic expression as follows:

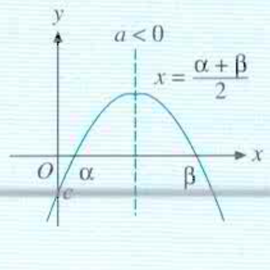
$$\begin{aligned}y &= 2x^2 + 4x - 16\\ &= 2(x^2 + 2x - 8)\\ &= 2(x + 4)(x - 2)\end{aligned}$$

This readily yields the x-intercepts –4 and 2, and the line of symmetry

$$x = \frac{(-4) + 2}{2} = -1.$$

Hence in the case when $ax^2 + bx + c$ can be easily factorised to $a(x - \alpha)(x - \beta)$, the graph of $y = a(x - \alpha)(x - \beta)$ can be sketched using the x-intercepts α and β as follows:





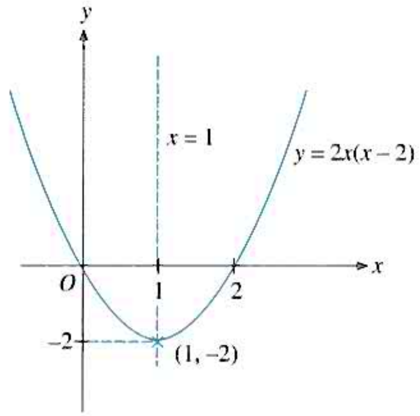
Example 5 Sketch, on separate diagrams, the curve
(a) $y = 2x(x - 2)$, **(b)** $y = (x + 2)(6 - x)$.

Solution: **(a)** For the curve $y = 2x(x - 2)$, $y = 0 \Rightarrow x = 0$ or $x = 2$.
So, the x-intercepts are 0 and 2.
The coefficient of x^2 is 2 (positive)

$\Rightarrow$ the curve has a minimum point at $x = \frac{0 + 2}{2} = 1$,
$y = -2$.

$\therefore$ the minimum point is $(1, -2)$.

The curve is sketched as follows:

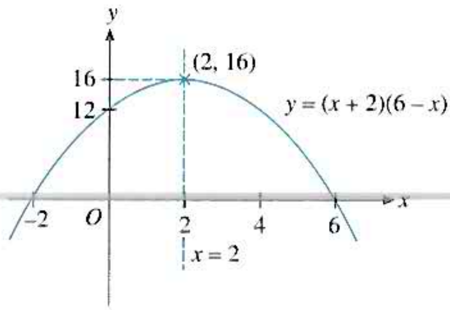


(b) Similarly for $y = (x + 2)(6 - x)$, $y = 0 \Rightarrow x = -2$ or $x = 6$.
So, the x-intercepts are -2 and 6.
The coefficient of x^2 is -1 (negative)

$\Rightarrow$ the curve has a maximum point at $x = \frac{(-2) + 6}{2} = 2$,
$y = 16$.

$\therefore$ the maximum point is $(2, 16)$.

The curve is sketched as follows:



Exercise 4.1

1. Use a graph plotter to explore how the graph of $y = a(x-h)^2 + k$ is affected by varying the value of
   (a) a, (b) h, (c) k.

2. In each of the following, state the minimum or maximum value of y and the corresponding value of x. Hence sketch the curve, indicating the y-intercept.
   (a) $y = 2(x-3)^2 + 1$ (b) $y = 4 - (x-2)^2$
   (c) $y = 4x^2 + 2$ (d) $y = (x+1)^2 - 5$
   (e) $y = 1 - (x+3)^2$ (f) $y = 9 - (2x-1)^2$

3. State the minimum value of $3(x+2)^2$ and the corresponding value of x. Sketch the curve $y = 3(x+2)^2$ for $-3 \leqslant x \leqslant 0$.

4. Express $3 + 9x - x^2$ in the form $q - (x+r)^2$, where q and r are constants. Hence
   (a) state its extreme value and the value of x when this occurs,
   (b) sketch the curve $y = 3 + 9x - x^2$.

5. Given that $4x^2 - 16x + 15 = a(x-p)^2 + q$ for all values of x, find the values of the constants a, p and q. Hence state the coordinates of the minimum point of the curve $y = 4x^2 - 16x + 15$.

6. Find the least value of $2x^2 - 4x + 3$ and the corresponding value of x.

7. Given that the greatest value of $k + 8x - 2x^2$ is 9, find the value of the constant k and sketch the curve $y = k + 8x - 2x^2$.

8. By finding the minimum value of $2x^2 - 3x + 2$, show that $2x^2 - 3x + 2$ is always positive.

9. Find the equation of the quadratic curve with a turning point at $(-2, 3)$ and which passes through $(-1, 5)$.

10. By completing the square, prove that $12x - 8 - 3x^2$ can never be greater than 4.

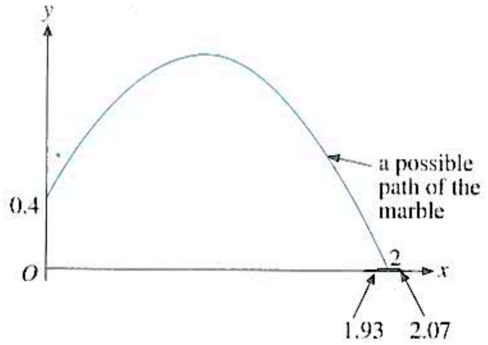
11. Find the x- and y-intercepts and hence sketch, on separate diagrams, the following curves. State also the coordinates of the maximum or minimum point.
    (a) $y = x(x+4)$ (b) $y = (x+4)(3-x)$
    (c) $y = (x+1)(x-7)$ (d) $y = 6x - x^2$
    (e) $y = 2x^2 - x - 3$ (f) $y = x^2 - 5x - 6$

12. (a) For each of the curves in question 11, without any further calculation, write down its equation in completed-square form.
    (b) Prove that $a(x-\alpha)(x-\beta)$ can always be expressed as $a(x-h)^2 + k$, stating h and k in terms of α and β.
    (c) Can $a(x-h)^2 + k$ be expressed as $a(x-\alpha)(x-\beta)$?

13. Sketch $y = (x-a)(x+3a)$ for $a > 0$. Hence sketch the graph of $y = x^2 + 2ax - 3a^2 + 1$ and state the coordinates of its turning point in terms of a.

14. A quadratic curve cuts the y-axis at -10 and the x-axis at 1 and k, where $k > 1$. Its maximum point occurs at $x = 3$. State the value of k and find the equation of the curve.

15. At a food cum fun fair, Lester plays a game in which he has to throw a marble into a hole of diameter 0.14 m and whose centre is at a distance of 2 m away. Taking into consideration his height and that the marble's path is a parabola, we can model the situation as shown in the diagram.



Use a graph plotter to find a possible equation for the parabola which starts at (0, 0.4) and passes through the hole on the x-axis. Can you find one such equation for Lester within 3 tries?

In the above model we had assumed that the diameter of the marble was inconsequential. What if Lester has to throw a ball of diameter 0.08 m instead? Would your earlier answer still hold good? If not, find a possible equation of the parabola in this case.

16. Someone challenged Jack to show that:

if $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ .... (1)

(a) Jack knew he had to complete the square but he could not get the desired result. The first four lines of his attempt are:

$$\begin{aligned} ax^2 + bx + c &= 0 \\ x^2 + \frac{b}{a}x &= -\frac{c}{a} \\ \left(x + \frac{b}{a}\right)^2 - \left(\frac{b}{a}\right)^2 &= -\frac{c}{a} \\ \left(x + \frac{b}{a}\right)^2 &= \left(\frac{b}{a}\right)^2 - \frac{c}{a} \end{aligned}$$

Can you spot his mistake? Correct it and complete the proof for Jack.

(b) Actually, the result in (1) was not stated correctly and a condition was missing. Had Jack known what it was, he would have a triumphant victory over his challenger! Can you figure out what was missing?

***17.** Given that $y = kx^2 - 4x + 3k$, express y in the form $a(x - p)^2 + q$, where a, p and q are in terms of k. Hence find the value of k if the maximum value of y is 4.

4.2 Roots of a Quadratic Equation

In Example 5, we saw that the x-intercepts of the curves are found by solving their respective quadratic equations

$$2x(x-2) = 0 \quad \text{and} \quad (x+2)(6-x) = 0.$$

For the curve $y = x^2 - 2x - 1$, the **roots** or **solutions** of the quadratic equation can be found using the **quadratic formula** (see Exercise 4.1 Question 16).

$$\boldsymbol{ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \; a \neq 0}$$

Notice that the formula involves the expression $\sqrt{b^2 - 4ac}$. This expression may or may not give a real value. In the next two examples, take note of the expression $\sqrt{b^2 - 4ac}$ and observe the relationship between the value of $b^2 - 4ac$ and the nature of the roots of the quadratic equation.

Example 6 Solve the equation $x^2 - 2x - 1 = 0$. Hence sketch the curve $y = x^2 - 2x - 1$, indicating the x-intercepts clearly.

Solution: When we compare $x^2 - 2x - 1 = 0$ with $ax^2 + bx + c = 0$, we have $a = 1$, $b = -2$ and $c = -1$.
So, $b^2 - 4ac = (-2)^2 - 4(1)(-1) = 8$.

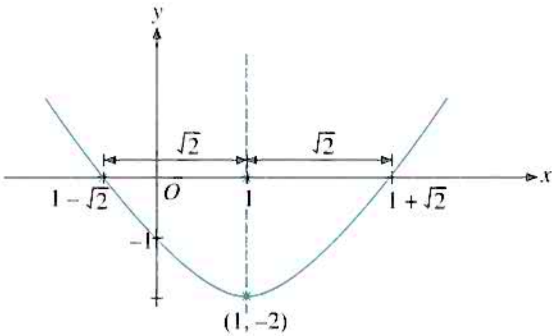
By the quadratic formula,

$$\begin{aligned} x^2 - 2x - 1 = 0 \quad \Rightarrow \quad x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{2 \pm \sqrt{8}}{2} \\ &= \mathbf{1 \pm \sqrt{2}} \end{aligned}$$

Hence the x-intercepts are $1 - \sqrt{2}$ and $1 + \sqrt{2}$.
$a = 1 > 0$ and so the curve has a minimum point at

$$\begin{aligned} x &= \frac{(1-\sqrt{2}) + (1+\sqrt{2})}{2} \\ &= 1 \\ \therefore \quad y &= -2 \end{aligned}$$

$\therefore$ the minimum point is $(1, -2)$ and the curve is sketched as follows:



Note: $b^2 - 4ac = 8 > 0$ and the equation $x^2 - 2x - 1 = 0$ has 2 distinct real roots.

Example 7

Find the roots of the following equations:

(a) $x^2 + 4x + 4 = 0$ **(b)** $2x^2 - x + 1 = 0$

Solution:

(a) Here $a = 1$, $b = 4$ and $c = 4$.

Then,
$$b^2 - 4ac = 4^2 - 4(1)(4) = 0$$
$$x = \frac{-4 \pm 0}{2}$$
$$\therefore \quad x = \frac{-4+0}{2} \text{ or } \frac{-4-0}{2}$$

The two roots are equal to **-2**.

(b) Here $a = 2$, $b = -1$ and $c = 1$. Then, $b^2 - 4ac = -7$.

But the value $\sqrt{b^2 - 4ac} = \sqrt{-7}$ is not a real number. In this case, the equation has **no real root**.

Note: Recall that $x^2 \geqslant 0$ for real x.

Hence $\left(\sqrt{-7}\right)^2 = -7 \ngeqslant 0 \quad \Rightarrow \quad \sqrt{-7}$ is not real.

Examples 6 and 7 show that there are three possible cases for the solution of the equation $ax^2 + bx + c = 0$, $(a \neq 0)$.

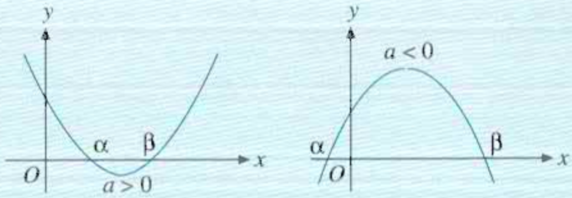
(a) $b^2 - 4ac > 0 \iff$ **the roots are real and distinct**
(b) $b^2 - 4ac = 0 \iff$ **the roots are real and equal**
(c) $b^2 - 4ac < 0 \iff$ **the roots are not real**

From (a) and (b), we conclude that

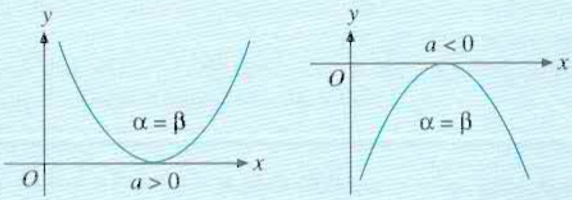
$$b^2 - 4ac \geqslant 0 \iff \text{the roots are real.}$$

The value $b^2 - 4ac$ is called the **discriminant** of the quadratic expression $ax^2 + bx + c$, $(a \neq 0)$, as it allows us to **discriminate** among the possible types of roots. It also tells us the position of the curve $y = ax^2 + bx + c$ relative to the x-axis as summarised in the following:

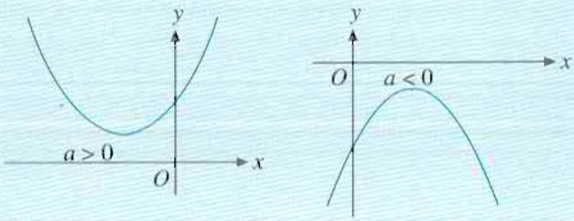
(a) $b^2 - 4ac > 0$:
Two distinct real roots
$\Rightarrow$ two x-intercepts



(b) $b^2 - 4ac = 0$:
Equal real roots
$\Rightarrow$ only one x-intercept and the x-axis is a tangent to the parabola



(c) $b^2 - 4ac < 0$:
No real roots
$\Rightarrow$ no x-intercept and $y = ax^2 + bx + c$ is either always positive or always negative



Example 8

Find the values of p for which $px^2 = 2x - p$, where $p \neq 0$, has equal real roots.

Solution:

$px^2 = 2x - p \quad \Rightarrow \quad px^2 - 2x + p = 0$

Here $a = p$, $b = -2$ and $c = p$.

For equal real roots, $b^2 - 4ac = 0$

$$\text{i.e.} \qquad (-2)^2 - 4(p)(p) = 0$$
$$p^2 = 1$$
$$\therefore p = \mathbf{1 \text{ or } -1}$$

Example 9

The quadratic equation $x^2 + 2kx + (k - 1)(k - 3) = 0$ has no real roots. Find the range of values of k.

Solution:

Here $a = 1$, $b = 2k$ and $c = (k - 1)(k - 3)$.

Since the roots are not real, $\quad b^2 - 4ac < 0$

$$\text{i.e.} \qquad (2k)^2 - 4(1)(k - 1)(k - 3) < 0$$
$$4k^2 - 4(k^2 - 4k + 3) < 0$$
$$16k < 12$$
$$\mathbf{k < \tfrac{3}{4}}$$

Example 10 Show that the roots of the equation $x^2 - 2x + 2 = p$ are real and distinct if $p > 1$.

Solution: $x^2 - 2x + 2 = p \quad \Rightarrow \quad x^2 - 2x + 2 - p = 0$

Here $a = 1$, $b = -2$ and $c = 2 - p$.

Now $b^2 - 4ac = (-2)^2 - 4(1)(2 - p)$

$= 4p - 4$

$= 4(p - 1)$

If $p > 1$, then $p - 1 > 0$ and so $b^2 - 4ac > 0$.

$\therefore$ the roots of the equation are real and distinct if $p > 1$.

Example 11 Show that the roots of the equation $x^2 + (1 - p)x - p = 0$ are real for all real values of p.

Solution: Here $a = 1$, $b = 1 - p$ and $c = -p$.

We need to show that $b^2 - 4ac \geqslant 0$.

Now, $b^2 - 4ac = (1 - p)^2 - 4(1)(-p)$

$= p^2 + 2p + 1$

$= (p + 1)^2$

Since p is real, $(p + 1)^2 \geqslant 0$ and so $b^2 - 4ac \geqslant 0$.

Hence the roots are real for all real values of p ($p \in \mathbb{R}$).

Intersection Problems Leading to Quadratic Equations

Consider the line and the curve whose respective equations are:

$$y = 2x + 1$$
$$y = x^2 + 6x + k$$

The following diagrams show the three possible cases with regard to their intersection:

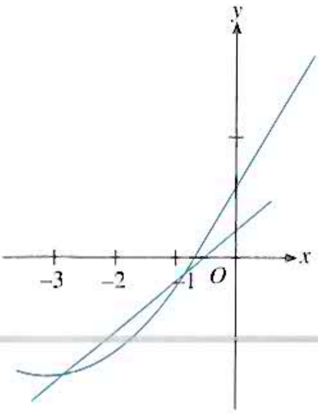


Fig. (a)

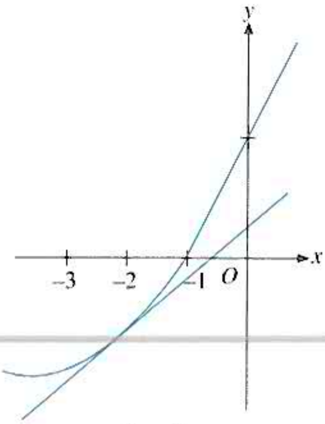


Fig. (b)

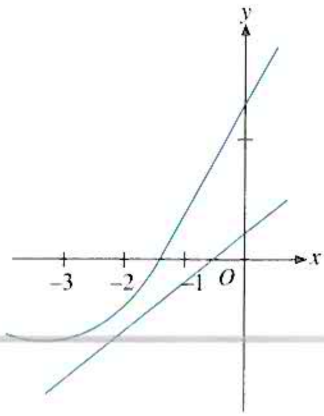


Fig. (c)

In order to know which values of k will correspond to which case, we solve the equations simultaneously:

$$x^2 + 6x + k = 2x + 1$$
$$x^2 + 4x + (k - 1) = 0$$

Notice that whether the line intersects the curve or not depends on the roots and hence the discriminant of the quadratic equation as follows:

Diagram	Discriminant	Nature of Roots	Points of Intersection
Fig. (a)	> 0	2 distinct real roots	2 points
Fig. (b)	$= 0$	2 equal real roots	1 point; line is a tangent to the curve
Fig. (c)	< 0	no real roots	none

Hence in Fig. (c), for the line not to intersect the curve,

$$b^2 - 4ac < 0$$
$$4^2 - 4(1)(k - 1) < 0$$
$$16 - 4k + 4 < 0$$
$$k > 5$$

Can you find k for the other two cases?

Example 12 If the line $y = mx - 8$ meets the curve $y = x^2 - 5x + m$, show that $m^2 + 6m - 7 \geqslant 0$.

Solution: By solving the two equations, we get:

$$x^2 - 5x + m = mx - 8$$
$$x^2 - (5 + m)x + m + 8 = 0$$

For the line to meet the curve, this quadratic equation must have real roots.

So,
$$b^2 - 4ac \geqslant 0$$
$$(5 + m)^2 - 4(1)(m + 8) \geqslant 0$$
$$25 + 10m + m^2 - 4m - 32 \geqslant 0$$
$$\therefore\ \mathbf{m^2 + 6m - 7 \geqslant 0}$$

How do we find m from $m^2 + 6m - 7 \geqslant 0$? This enquiry leads us into the next section on solving quadratic inequalities.

Exercise 4.2

1. Use the discriminant to determine the nature of the roots of the following quadratic equations. When the roots are real, find these roots:
   (a) $x^2 - 2x - 5 = 0$ **(b)** $4x^2 + 4x + 1 = 0$ **(c)** $3x^2 = 8x + 3$
   (d) $x(1 - 3x) = 2$ **(e)** $(2x - 1)(x + 1) = 2$ **(f)** $2x(x - 3) = 3 - 4x$
2. The equation $x^2 + k = 6x$ has two distinct real roots. Find the range of values of k.

3. If the equation $2x(p - x) = 3$ has real and equal roots, find the exact values of p.

4. Find the value(s) or range of values of p for which the equation
   (a) $px^2 - 6x + p = 0$ has equal real roots,
   (b) $2x^2 - 4x + 3 = p$ has real roots,
   (c) $3x^2 = 2x + p - 1$ has distinct real roots,
   (d) $p(x + 1)(x - 3) = x - 4p - 2$ has no real roots.

5. Find the range of values of m such that the roots of the equation $mx^2 + 2 = x(x + 3)$ are not real.

6. What is the least value k can take if the roots of the equation $x^2 - 2kx + k^2 = 3 + x$ are real?

7. Find the range of values of k for which the roots of the equation $x^2 + (2k + 1)x + k^2 - 3 = 0$ are not real. State the range of values of k for which the roots are real.

8. Find the range of values of k for which the equation $x^2 - 2kx + k^2 - 2k = 6$ has real roots. Find the roots in terms of k.

9. Show that the roots of the equation $2x^2 + p = 2(x - 1)$ are not real if $p > -\frac{3}{2}$.

10. Given that the equation $px^2 + 3px + p + q = 0$, where $p \neq 0$, has two equal real roots, find q in terms of p.

11. The equation $x(x - 2) + k^2 = k(2x - 1)$ is satisfied by two distinct real values of x. Find the range of values of k.

12. Show that $px^2 - 4x + p = 2px - 3x^2 - 1$ has real root(s) for all real values of p.

13. Show that the solutions of the equation $x^2 + kx = 3 - k$ are real for all real values of k.

14. Given that $x(x - 2) = t - 2$, find x in terms of t. Hence deduce the range of values of t for x to be real.

15. The curve $y = 3x^2 - 2x + c - 1$ cuts the x-axis at 2 points. Find the range of values of c.

16. Find the values of p for which the x-axis is a tangent to the curve $y = x^2 + px - p + 3$. For each of these values, find the coordinates of the point of tangency.

17. (a) Find the value of p for which the equation $(1 - 2p)x^2 + 8px - (2 + 8p) = 0$ has two equal roots.
    (b) Show that the line $x + y = q$ will intersect the curve $x^2 - 2x + 2y^2 = 3$ in two distinct points if $q^2 < 2q + 5$. *(C)*

18. Find the value of c such that the straight line whose equation is $y = 2x + c$ is tangential to the curve with equation $y = 3x^2 - 6x + 5$.

19. Find m if the line $y = 2x - 4m^2$ intersects the curve $y = x^2 - 4mx + 9$ in
    (a) 2 distinct points.
    (b) 1 point.
    (c) no point at all.

20. If the line $y + 2x = p$ does not intersect the curve $y^2 = x + p$, find the range of values of p.

21. For what range of values of c does the line $y = cx + 4$ cut the curve $5x^2 - xy = 2$ at 2 points?

22. Show that the line $y = 2mx - 1$ intersects the curve $y = mx^2 - x + m$ for all non-zero values of m.

23. If the curve $y = kx(x + 2)$ meets the line $y = x - k$, find the range of values of k. State the value of k for which the line is a tangent.

4.3 Solving Quadratic Inequalities

An inequality with a quadratic expression in one variable on one side and zero on the other, is called a **quadratic inequality** in one variable. For example,

$$(x - 1)(x - 4) < 0$$

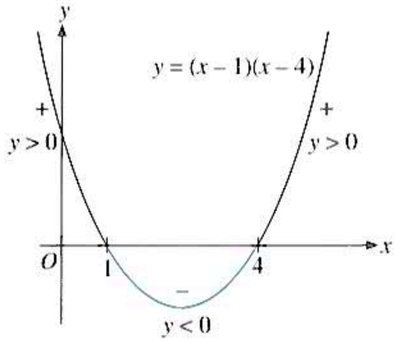
is a quadratic inequality in x. The range of values of x which satisfies this inequality can be found from the quadratic curve $y = (x - 1)(x - 4)$ as follows:

For $(x - 1)(x - 4) < 0$ (i.e. y is **negative**), we choose the interval for which the curve is **below** the x-axis.

$\therefore \quad (x - 1)(x - 4) < 0 \quad \Rightarrow \quad 1 < x < 4$

Similarly, if $(x - 1)(x - 4) > 0$ (i.e. y is **positive**), we choose the interval where the curve is **above** the x-axis.

$\therefore \quad (x - 1)(x - 4) > 0 \quad \Rightarrow \quad x < 1 \text{ or } x > 4$



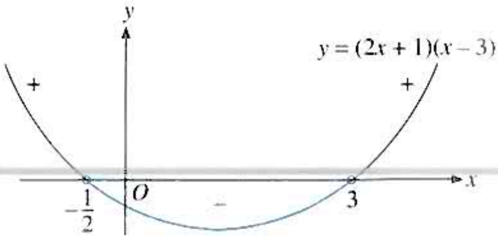
Example 13 Find the range of values of x for which $2x^2 < 5x + 3$.

Solution:

$$2x^2 < 5x + 3$$
$$2x^2 - 5x - 3 < 0$$
$$(2x + 1)(x - 3) < 0$$

From the graph, we see

that: $-\frac{1}{2} < x < 3$



Example 14 Find the set of values of m for which the equation

$$2x^2 - mx + 2 = 0 \text{ has real roots.}$$

Solution: Here $a = 2$, $b = -m$ and $c = 2$.

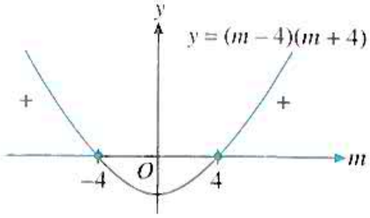
For real roots, $b^2 - 4ac \geqslant 0$

i.e. $(-m)^2 - 4(2)(2) \geqslant 0$

$$m^2 - 16 \geqslant 0$$

$$m^2 - 4^2 \geqslant 0$$

$$(m - 4)(m + 4) \geqslant 0$$



From the curve $y = (m - 4)(m + 4)$, we require $y \geqslant 0$ and so the solution set is $\{m: m \leqslant -4 \text{ or } m \geqslant 4, m \in \mathbb{R}\}$.

Note: Give your answer in set notation when the question requires you to find the **set** of values.

Exercise 4.3

1. Find the solution set of each of the following quadratic inequalities.
   (a) $x^2 > 9$
   (b) $x(x - 2) < 3$
   (c) $x^2 + 5x - 6 < 0$
   (d) $2x^2 - 7x + 3 \geqslant 0$
   (e) $x^2 > 4x + 12$
   (f) $4x(x + 1) \leqslant 3$
   (g) $(1 - x)^2 \geqslant 17 - 2x$
   (h) $(x + 2)(x + 3) \leqslant x + 6$
   (i) $(x - 1)(5x + 4) > 2(x - 1)$
   (j) $(x + 2)^2 < x(4 - x) + 40$
2. Find the set of values of x for which $2x^2 - 4x - 3$ is greater than x.
3. Sketch the curve $y = x^2 - 4x - 6$, indicating the exact values of the x-intercepts. Hence solve the inequality $x(x - 4) > 6$.
4. There is no real value of x for which $mx^2 + 8x + m = 6$. Find m.
5. The equation $2x^2 + 4x + 2 = p(x + 3)$ has two distinct real roots. Find the range of values of p.
6. If the equation $(p + 2)x^2 - 12x + 2(p - 1) = 0$ has real and distinct roots, find the set of values of p.
7. Find the set of values of m for the equation $2x^2 + 4x + 2 + m(x + 2) = 0$ to have real roots.
8. The roots of the equation $3kx^2 + (k - 5)x = 5x^2 + 2$ are not real. Find the set of values of k.
9. If the equation $px^2 - p + 10 = 2(p + 2)x$ has real solution(s), show that p cannot lie between 1 and 2.
10. Given that the line $y = c - 3x$ does not intersect the curve $xy = 3$, find the range of values of c. *(C)*
11. The curve $x^2 - xy + y^2 = 1$ cuts the line $2x - y = k$ in two distinct points. Find the set of values of k.

*12. The curve $y = (k - 6)x^2 - 8x + k$ cuts the x-axis at two points and has a minimum point. Find the range of values of k.

***13.** Given that $y = 2x^2 + px + 16$ and that $y < 0$ only when $2 < x < k$, find the value of p and of k.

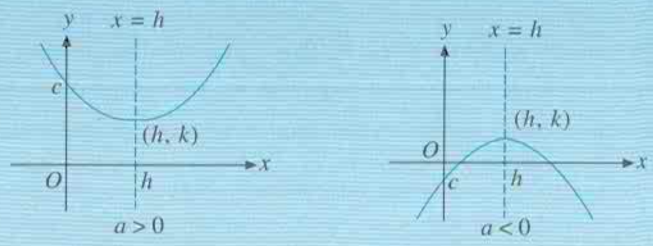
***14.** The equation $m^2x^2 - (m + n)x + n = 0$ has two equal real roots. Show that $n^2 + 2(m - 2m^2)n + m^2 = 0$. If n is real, find the range of values of m.

Important Notes

1. To sketch $y = ax^2 + bx + c, a \neq 0$

(a) ***Use the turning point***

Express $y = ax^2 + bx + c$ as $y = a(x - h)^2 + k$ by completing the square. Then the graphs are:

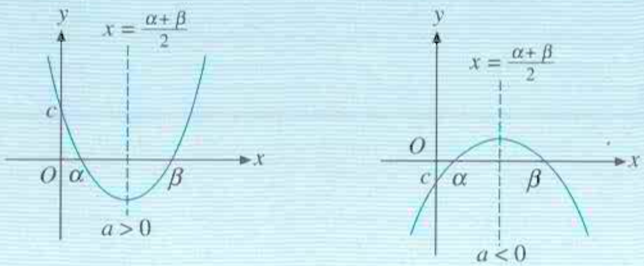


In either case, the turning point is (h, k) and the parabola is symmetrical about $x = h$.

The extreme value (minimum or maximum) is k. This value occurs when the squared term $(x - h)^2 = 0$, i.e. when $x = h$.

(b) ***Use the x-intercepts***

If $ax^2 + bx + c$ is readily factorised to $a(x - \alpha)(x - \beta)$, the graphs are:



2. Types of roots of $ax^2 + bx + c = 0, (a \neq 0)$

$b^2 - 4ac \geqslant 0$ (real roots)

- $b^2 - 4ac > 0$ (distinct real roots)
- $b^2 - 4ac = 0$ (equal, coincident or repeated real roots)

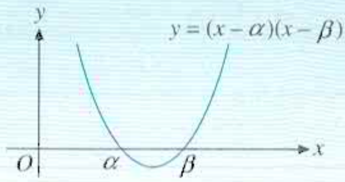
$b^2 - 4ac < 0$ (no real roots)

3. **Intersection of a line and a curve**
If the simultaneous equations of the line and the curve lead to a quadratic equation, then:

Discriminant	Outcome
> 0	line meets curve in 2 points } line meets curve
$= 0$	line is a tangent to the curve } line meets curve
< 0	line does not meet curve

4. A quadratic inequality may be solved using the corresponding **quadratic curve**.

$(x - \alpha)(x - \beta) < 0 \quad \Rightarrow \quad \alpha < x < \beta$
$(x - \alpha)(x - \beta) > 0 \quad \Rightarrow \quad x < \alpha$ or $x > \beta$



Miscellaneous Examples

Example 15 Find the set of values of m for which the line $y = mx - 1$ does not intersect the curve $y = x^2 - 2x + 3$. State also the values of m for which this line is a tangent to the curve.

Solution: We solve simultaneously the equations:

$$y = mx - 1 \quad \ldots\ldots \quad (1)$$
$$y = x^2 - 2x + 3 \quad \ldots\ldots \quad (2)$$

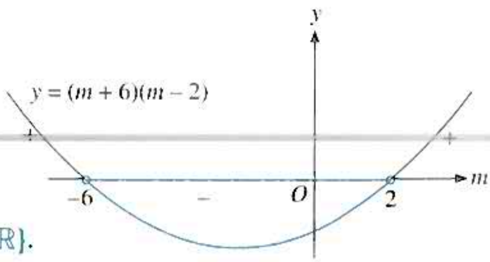
Equating (1) and (2) gives:

$$x^2 - 2x + 3 = mx - 1$$
$$x^2 - (2 + m)x + 4 = 0 \quad \ldots\ldots \quad (3)$$

Since there is no intersection, this equation has no real roots.

i.e. $\quad b^2 - 4ac < 0$

$$[-(2 + m)]^2 - 4(1)(4) < 0$$
$$4 + 4m + m^2 - 16 < 0$$
$$m^2 + 4m - 12 < 0$$
$$(m + 6)(m - 2) < 0$$
$$\Rightarrow \quad -6 < m < 2$$



$\therefore$ the solution set is
$\{m: -6 < m < 2, m \in \mathbb{R}\}$.

For the line to be a tangent, equation (3) must have equal real roots and so,

$$b^2 - 4ac = 0$$
$$\text{i.e. } (m + 6)(m - 2) = 0$$
$$m = \mathbf{-6 \text{ or } 2}$$

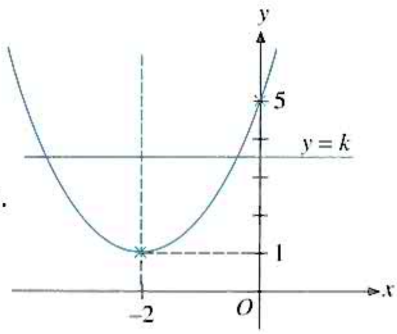
Example 16 Find the minimum value of $x^2 + 4x + 5$ and the corresponding value of x.
Sketch the curve $y = x^2 + 4x + 5$. Hence or otherwise, find the set of values of k for which $x^2 + 4x + 5 = k$ has two distinct real roots.

Solution:

$$x^2 + 4x + 5 = (x + 2)^2 - 2^2 + 5$$
$$= (x + 2)^2 + 1$$

$\therefore$ the minimum value is $\mathbf{1}$ when $x = \mathbf{-2}$.

The curve is sketched as shown.



The roots of $x^2 + 4x + 5 = k$ are given by the x-coordinates of the points of intersection of the graphs of $y = x^2 + 4x + 5$ and $y = k$. For the equation to have two distinct real roots, the line $y = k$ must be above the minimum point $(-2, 1)$ and so the solution set is $\{k: k > 1, k \in \mathbb{R}\}$.

Note: Otherwise, we have

$$x^2 + 4x + 5 = k$$
$$x^2 + 4x + (5 - k) = 0$$

For this equation to have 2 distinct real roots,

$$4^2 - 4(1)(5 - k) > 0$$
$$k > 1$$

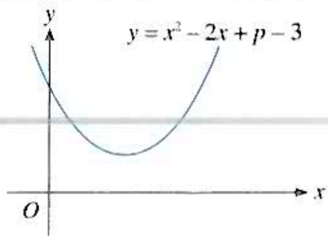
Example 17 Find the smallest integer p such that $x^2 - 2x + p$ is always greater than 3.

Solution:

For $x^2 - 2x + p > 3$, i.e. $x^2 - 2x + p - 3 > 0$ for all real values of x, the curve $y = x^2 - 2x + p - 3$ must lie above the x-axis.
So $a > 0$ and $b^2 - 4ac < 0$.
Now $a > 0$ is satisfied as $a = 1$.
While $b^2 - 4ac < 0$

$$\Rightarrow \quad (-2)^2 - 4(1)(p - 3) < 0$$
$$4 - 4p + 12 < 0$$
$$p > 4$$

$\therefore$ the smallest integer p is $\mathbf{5}$.



Miscellaneous Exercise 4

1. Sketch the following graphs, clearly labelling all intercepts.
   (a) $y = (x - 3)(5 - x)$ (b) $y = 3x^2 + 5$
   (c) $y = -2x^2 + 4x - 3$ (d) $y = x^2 - 5x - 6$

2. Find the range of values of p for which the equation
   (a) $x^2 + 3 = 2x + p$ has real roots,
   (b) $2x^2 + 2x\sqrt{3} + p = p(x^2 + 2)$ has distinct real roots.

3. Find the range of values of x for which
   (a) $(1 + x)(6 - x) \leqslant -8$, (b) $2x(x + 2) < (x + 1)(x + 3)$.

4. Given that $y = 5 + px - x^2 = 9 - (x + q)^2$, where $q > 0$, for all values of x,
   (a) calculate the values of the constants p and q,
   (b) state the maximum value of y and the value of x at which it occurs,
   (c) find the set of values of x for which y is negative.

5. (a) State the coordinates of the minimum point of the curve $y = 2(x - 1)^2 + 3$ and sketch the curve. Find the value of c for which the line $y = 2x + c$ is a tangent to the curve.
   (b) Find the largest negative integer p for the equation $4x^2 + 8 = px$ to have real roots.

6. (a) Find the values of c for which the equation $4x^2 - (c + 2)x + c = 1$ has equal roots.
   (b) Find the range of values of k for which the curve $y = \dfrac{16}{x}$ intersects the straight line $y = k - x$ at two distinct points. *(C)*

7. (a) Find the set of values of x for which $x(2x - 1) > 3$.
   (b) The line $y = 2k - x$ does not intersect the curve $y = (x - 1)^2 + k$. Find the range of values of k.

8. (a) Calculate the smallest positive integer k for which the equation $2x^2 + 2kx + 7 = 0$ has two distinct real roots.
   (b) Given that $3px^2 - 7qx + 3p = 0$ has equal roots and p and q are positive, find the ratio $p : q$ and solve the equation.

9. (a) Find the range of values of p for which the graph of $y = px^2 + 8x + p - 6$ crosses the x-axis. State also the values of p for which the x-axis is a tangent to the curve.
   (b) Express $y = \frac{1}{2}\{(x + 5)^2 + (x - 7)^2\}$ in the form $y = (x + q)^2 + r$. **Hence** find the least value of y and the corresponding value of x. *(C)*

10. (a) The quadratic equation $2px^2 + 4(p + q)x + 2(p + r) = 0$ has equal roots. Express p in terms of q and r.
    (b) Find the set of values of k for which the equation $(2 - k)x^2 + 3kx - 4k = 2$ has real root(s).

11. (a) Find the solution set of the inequality $x^2 + 2 \geqslant (2x - 1)(x + 2)$.
    (b) Show that the roots of the equation $x^2 - (2p - 1)x + p^2 + 1 = p$ are not real for all values of p.
    (c) Given that the line $y = mx + 3$ meets the curve $2xy - 2x = 1$, find the range of values of m.

12. (a) Find the range of values of k for which the graph of $y = -x^2 + 2(k - 3)x - 25$ lies entirely below the x-axis.
(b) Sketch the curve $y = (x - 3)(x + 1)$.
Hence, or otherwise, find the value of p if $(x - 3)(x + 1) = p$ has equal real roots and state the value of these roots.

13. If x is real and $(x + 1)^2 = k(x + 2)$, show that k cannot lie between -4 and 0.

14. (a) If p is real and $p \neq -1$, show that the equation $(p + 1)x^2 + (2p + 1)x + p = 0$ has two real roots. What can we conclude about the equation if $p = -1$?
(b) Solve for x if $x - 3 < x(x - 3) \leqslant 4$.

15. (a) Given that the curve whose equation is $y = p - (x - q)^2$ crosses the x-axis at the points $(-1, 0)$ and $(5, 0)$, find the maximum value of y.
(b) The line $(k - 2)y = 3x$ meets the curve $xy = 1 - x$ at two distinct points. Find the set of values of k. State also the values of k if the line is a tangent to the curve.

16. Find the minimum value of $2x^2 + 3x + 4$ and the corresponding value of x. Sketch the curve $y = 2x^2 + 3x + 4$. Hence or otherwise, find the range of values of k for which $2x^2 + 3x + 4 \geqslant k$ for all real values of x.

17. (a) Calculate the range of values of x for which $x^2 + 4x - 5 > 5x - 3$.
(b) Calculate the range of values of c for which $3x^2 - 9x + c > 2.25$ for all values of x. *(C)*

18. (a) Find the range of values of k for which $8 - 3x - x^2 \leqslant k$ for all real values of x.
(b) Find the range of values of m for which the line $y = mx + 1$ does not intersect the curve $y^2 = 8x$. *(C)*

*19. Find the range of values of k if $kx^2 + 8x > 6 - k$ for all real values of x.

20. Given that the equation $ax^2 + bx + c = 0$ has real roots, show that the roots of the following equations are also real.
(a) $cx^2 - bx + a = 0$
(b) $ax^2 + mbx + m^2c = 0$, where m is a constant

21. Sketch the curve $y = 2x^2 - 4x + 1$, indicating the coordinates of the turning point and the exact values of the x-intercepts. Hence find
(a) the set of values of x for which $2x^2 + 1 \leqslant 4x$,
(b) the range of values of p if $2x^2 - 4x + 1 + p = 0$ has no real roots.

22. Show that the line $y = 5x - 4$ is a tangent to the curve $y = x^2 + x$. Find also the condition for $y = mx - c$ to be a tangent to the curve.

23. A quadratic curve is symmetrical about the line $x = 3$ and passes through the points $(2, 13)$ and $(-1, -2)$. Find its equation and sketch this curve.

24. What values of m make the expression $(2m - 1)x^2 - 4(m - 2)x + (m - 1)$ a perfect square?

25. The curves $y = (x - a)^2 + b$ and $y = (x - c)^2 + d$ do not intersect. With the help of a graph plotter, compare their graphs and find what relationships exist between any or all of a, b, c and d.

26. For each of the following pairs of quadratic curves, where $a \neq 0$, are there any relationships between their turning points, their intercepts with the axes and their shapes?
With a graph plotter, investigate these relationships for
(a) $y = ax^2 + bx + c$ and $y = ax^2 - bx + c$ by varying the value of b,
(b) $y = ax^2 + bx + c$ and $y = -ax^2 + bx + c$ by varying the value of a,
(c) $y = ax^2 + bx + c$ and $y = -ax^2 + bx - c$ by varying the values of a and c.
In the above investigation, we have only induced results from **certain** values of a, b and c. How can we prove that our results will hold for **any** values of these constants?

5 Remainder and Factor Theorems

5.1 Polynomial Identities

We have come across algebraic expressions such as $3x^2 - 5x + 2$ and $3x - 5$.

They are called **polynomials**, a word derived from Greek and literally means 'many terms'. A polynomial in the variable x is therefore a sum of terms, each of the form ax^n, where a is a constant and the power n, a non-negative integer.

More examples of polynomials are $8x^3 - x^2 + 3x + 4$ and $5x^4 - \frac{1}{2}x + 1$.

Are $x^2 - 2x^{\frac{1}{2}} + 6$ and $x^2 + \frac{1}{x} - 2$ polynomials? No, they are not because the former contains $x^{\frac{1}{2}}$, a fractional power of x, whereas the latter contains $\frac{1}{x}$ $(= x^{-1})$, a negative power of x.

In the term ax^n, a is called the **coefficient** of x^n. For example, the polynomial $x^3 - 2x^2 + 3$ can be written as $1x^3 - 2x^2 + 0x + 3x^0$.

Hence, coefficient of $x^3 = 1$,
coefficient of $x^2 = -2$,
coefficient of $x = 0$,
coefficient of $x^0 = 3$. (This is the **constant term**.)

The **degree** of a polynomial in x is the highest power of x. For example, the degree of $x^3 + 2x$ is 3, $2x + 8$ is 1 and that of the constant 5 $(= 5x^0)$ is 0.

Consider the polynomials $x^2 + 2x$ and $4x + 3$.
Now, $x^2 + 2x = 4x + 3$ holds only for $x = -1$ and $x = 3$.
Thus, $x^2 + 2x = 4x + 3$ is an **equation** whose solutions are -1 and 3.

Consider the polynomials $x^2 - 4$ and $(x - 2)(x + 2)$. They are **identical**, that is, they are the same though expressed differently.

So $x^2 - 4 = (x - 2)(x + 2)$ is true for all values of x. .... (1)
The equality (1) is called an **identity**, and may be written as

$$x^2 - 4 \equiv (x - 2)(x + 2).$$

Many formulae are actually identities. For example,

$$(a-b)^2 \equiv a^2 - 2ab + b^2.$$

Example 1

Given that $x^3 - 2x^2 + 5 = ax(x-1)^2 + b(x-1) + c$ for all values of x, find the values of a, b and c.

Solution:

Since $x^3 - 2x^2 + 5 = ax(x-1)^2 + b(x-1) + c$ for **all** values of x, the equation holds for **any** value of x.

Let $x = 1$. Then $1 - 2 + 5 = c$

$$c = \mathbf{4}$$

Let $x = 0$. Then $5 = -b + c$

$$= -b + 4$$

$$b = \mathbf{-1}$$

Let $x = 2$. Then $8 - 8 + 5 = 2a + b + c$

$$5 = 2a + (-1) + 4$$

$$a = \mathbf{1}$$

Alternatively,

$$\begin{aligned} x^3 - 2x^2 + 5 &\equiv ax(x^2 - 2x + 1) + b(x-1) + c \\ &\equiv ax^3 - 2ax^2 + ax + bx - b + c \\ &\equiv ax^3 - 2ax^2 + (a+b)x + c - b \end{aligned}$$

Since the two polynomials are identical, the coefficients of every like powers of x must be equal.

Equating the coefficients of

x^3: $a = \mathbf{1}$

x: $a + b = 0$

$1 + b = 0$

$b = \mathbf{-1}$

x^0 (i.e. constant term):

$c - b = 5$

$c - (-1) = 5$

$c = \mathbf{4}$

Exercise 5.1

Find the values of the unknown constants for the identities in questions 1 to 10.

1. $a(x-2) + b(x-4) \equiv x + 2$
2. $x^3 - 2x^2 + 4x + 3 \equiv (x-1)(x^2 - x + a) + b$
3. $2x^3 - 6x + 5 \equiv (x-2)(2x^2 + ax + 2) + b$

4. $x^3 - 6x^2 - x + 30 \equiv (x - 3)(ax^2 - 3x + b)$

5. $6x^3 - 5x^2 - 3x + 2 \equiv (x - 1)(ax^2 + bx + c)$

6. $x^2 + x + A \equiv (x + A)(x - 1) + B$

7. $3x^2 + 6x - 4 \equiv p(x - q)^2 + r$

8. $x^3 + cx^2 + x + 6 \equiv (x + 1)(x - 2)(ax + b)$

9. $3x^2 - 5x + 4 \equiv A(x - 2)^2 + B(x - 2) + C$

10. $x^3 + 3x^2 - 2x + 16 \equiv Ax^2(x - 1) + B(x - 2)^2(x - 1) + C(x + 2)$

11. Given $3x^3 + 2x^2 + x + 2 = (x^2 + 2x - 1)(3x - 4) + Ax + B$ for all values of x, find the value of A and of B.

12. Given that $4x^3 - 6x^2 + 1 \equiv (x - 2)(x + 1)Q(x) + ax + b$ where $Q(x)$ is a polynomial, find a and b.

13. James solved for A, B, C and D in the identity

$$3x^3 - 2x^2 + x - 4 \equiv A(x - 1) + B(x - 1)(x + 1) + Cx(x^2 - 1) + D$$

and obtained $A = 3$, $B = -2$, $C = 5$ and $D = -2$. How can we use a graph plotter to show that he has made mistake(s)?
Please solve for A, B, C and D. Check your answers with the graph plotter.

5.2 Remainder Theorem

Recall the long division for integers.

```
                146   ← quotient
divisor → 7 ) 1025   ← dividend
               7
               32
               28
                45
                42
                 3   ← remainder
```

We note that the process stops when the remainder, 3, is less than the divisor, 7. The result may be stated as $1025 = 7 \times 146 + 3$.

dividend = divisor × quotient + remainder

The process is similar for division of polynomials.

For example, dividing $2x^4 - 3x^3 + x^2 + 1$ by $x - 2$, we have the following:

$$
\begin{array}{rrrrrrl}
 & 2x^3 & + \; x^2 & + \; 3x & + \; 6 & & \leftarrow \textbf{quotient} \\
\textbf{divisor} \rightarrow \; x - 2 \, \big) & 2x^4 - 3x^3 & + \; x^2 & + \; 0x & + \; 1 & & \leftarrow \textbf{dividend} \\
 & \underline{2x^4 - 4x^3} & & & & & \\
 & x^3 & + \; x^2 & & & & \\
 & \underline{x^3} & \underline{- \; 2x^2} & & & & \\
 & & 3x^2 & + \; 0x & & & \\
 & & \underline{3x^2} & \underline{- \; 6x} & & & \\
 & & & 6x & + \; 1 & & \\
 & & & \underline{6x} & \underline{- \; 12} & & \\
 & & & & 13 & & \leftarrow \textbf{remainder}
\end{array}
$$

We note that the process stops when **the degree of the remainder is less than the degree of the divisor**. The result may be stated as an identity.

$$2x^4 - 3x^3 + x^2 + 1 \equiv (x - 2)(2x^3 + x^2 + 3x + 6) + 13$$

i.e. $\quad$ dividend $\equiv$ divisor $\times$ quotient + remainder

Let $f(x) = 2x^4 - 3x^3 + x^2 + 1$ and $Q(x) = 2x^3 + x^2 + 3x + 6$

Then $\quad f(x) \equiv (x - 2)Q(x) + 13$

Hence $\quad f(2) = (2 - 2)Q(2) + 13$

$\quad = 0 + 13$

$\quad = 13$, the remainder

This illustrates the **Remainder Theorem** which states that:

> **If a polynomial $f(x)$ is divided by a linear divisor $x - a$, the remainder is $f(a)$.**

Proof: $\quad$ Let $Q(x)$ be the quotient and R, the remainder.

Then $\quad f(x) \equiv (x - a)Q(x) + R$

Substitute $x = a$ into $f(x)$ so that the divisor $x - a = 0$,

We get $\quad f(a) = 0 + R$

$\quad = R$, the remainder

Similarly, if the divisor is $ax - b$, we would have:

$$f(x) \equiv (ax - b)Q(x) + R$$

Substitute $x = \frac{b}{a}$ into $f(x)$ so that $ax - b = 0$, $f\left(\frac{b}{a}\right) = 0 + R = R$

Hence when the divisor is linear, the remainder can be found by simply using the Remainder Theorem instead of the tedious process of long division.

Example 2 Find the remainder when $4x^3 - 5x + 1$ is divided by
(a) $x - 2$, **(b)** $x + 3$, **(c)** $2x - 1$.

Solution:

Let $f(x) = 4x^3 - 5x + 1$

(a) When $f(x)$ is divided by $x - 2$,
remainder, $R = f(2)$
$= 4(2)^3 - 5(2) + 1$
$= \mathbf{23}$

(b) When $f(x)$ is divided by $x + 3$,
remainder, $R = f(-3)$
$= 4(-3)^3 - 5(-3) + 1$
$= \mathbf{-92}$

(c) When $f(x)$ is divided by $2x - 1$,

remainder, $R = f\left(\frac{1}{2}\right)$
$= 4\left(\frac{1}{8}\right) - 5\left(\frac{1}{2}\right) + 1$
$= \mathbf{-1}$

Example 3 The expression $4x^2 - px + 7$ leaves a remainder of -2 when divided by $x - 3$. Find the value of p.

Solution:

Let $f(x) = 4x^2 - px + 7$
By the Remainder Theorem, $f(3) = -2$
$4(3)^2 - 3p + 7 = -2$
$p = \mathbf{15}$

Example 4 Given that the expression $2x^3 + ax^2 + bx + c$ leaves the same remainder when divided by $x - 2$ or by $x + 1$, prove that $a + b = -6$.

Solution:

Let $f(x) = 2x^3 + ax^2 + bx + c$
The two remainders $f(2)$ and $f(-1)$ are equal.
Hence $f(2) = f(-1)$
$16 + 4a + 2b + c = -2 + a - b + c$
$3a + 3b = -18$
$\therefore \mathbf{a + b = -6}$

Exercise 5.2

1. By using the Remainder Theorem, find the remainder for the following operations.
   (a) $x^8 - 4x^7 + 3x^5 + x^4 + 3$ is divided by $x + 1$
   (b) $x(x - 1)^4(1 - 2x)^2 + x^2 - 3$ is divided by $2 - x$
   (c) $3(x + 4)^2 - (1 - x)^3$ is divided by x
   (d) $(2x - 1)^3 + 6(3 + 4x)^2 - 10$ is divided by $2x + 1$

2. Find k if $x^3 + 3x^2 - kx + 4$ leaves a remainder of k when divided by $x - 2$.

3. The expression $6x^2 - 2x + 3$ leaves a remainder of 3 when divided by $x - p$. Determine the values of p.

4. When $x^4 + x^3 + 2ax^2 - 14a^4$ is divided by $x + 2a$, the remainder is 32. Find the possible values of a.

5. When $ax^2 + bx - 6$ is divided by $x + 3$, the remainder is 9. Find, in terms of a only, the remainder when $2x^3 - bx^2 + 2ax - 4$ is divided by $x - 2$.

6. The expression $8x^3 + ax^2 + bx - 9$ leaves remainders -95 and 3 when divided by $x + 2$ and $2x - 3$ respectively. Calculate the value of a and of b.

7. The polynomial $x^3 + ax^2 + bx - 3$ leaves a remainder of 27 when divided by $x - 2$ and a remainder of 3 when divided by $x + 1$. Calculate the remainder when the polynomial is divided by $x - 1$.

8. The expression $x^3 + ax^2 + 7$ leaves a remainder of $2p$ when divided by $x + 1$ and a remainder of $p + 5$ when divided by $x - 2$. Calculate the value of a and of p.

9. The remainder when $ax^3 + bx^2 + 2x + 3$ is divided by $x - 1$ is twice that when it is divided by $x + 1$. Show that $b = 3a + 3$.

10. The expression $x^2 + bx + a$ leaves the same remainder when divided by $x + 2$ or by $x - a$, where $a \neq -2$. Show that $a + b = 2$.

*11. When $2x^3 - 4x^2 - 5x - 2$ is divided by $(x - 1)(x + 2)$, the remainder is $ax + b$. This result may be expressed as the identity
$$2x^3 - 4x^2 - 5x - 2 \equiv (x - 1)(x + 2)Q(x) + ax + b,$$
where $Q(x)$ is the quotient.
   (a) State the degree of $Q(x)$.
   (b) By substituting suitable values of x, find a and b.

12. $2x^2 + 6x + 3$ has the same remainder when divided by $x + p$ or by $x - 2q$ where $p \neq -2q$. Find the value of $p - 2q$.

13. Find the value of p and of q if $2x^4 + px^3 + qx - 4$ leaves a remainder of $36x + 32$ when it is divided by $x^2 - 2x - 3$.

14. Without doing the long division, find the remainder when $3x^4 - 5x^2 + 4$ is divided by $x^2 + 2$.

5.3 Factor Theorem

When $f(x)$ is divided by $x - a$, we have:

$$f(x) \equiv (x - a)Q(x) + R$$

i.e. $$f(x) \equiv (x - a)Q(x) + f(a)$$

If $f(a) = 0$, i.e. the remainder is 0,
then $$f(x) \equiv (x - a)Q(x)$$

We say $x - a$ is a **factor** of $f(x)$ or equivalently, $f(x)$ is **exactly divisible** by $x - a$.
Hence the **Factor Theorem**:

> **$x - a$ is a factor of the polynomial $f(x)$ $\Leftrightarrow$ $f(a) = 0$**

Example 5 Determine whether or not $x + 1$ is a factor of the following polynomials.

(a) $3x^4 + x^3 - x^2 + 3x + 2$ **(b)** $x^6 + 2x(x - 1) - 4$

Solution:

(a) Let $f(x) = 3x^4 + x^3 - x^2 + 3x + 2$

$$\begin{aligned} f(-1) &= 3(-1)^4 + (-1)^3 - (-1)^2 + 3(-1) + 2 \\ &= 3(1) + (-1) - 1 - 3 + 2 \\ &= 0 \end{aligned}$$

$\therefore$ **$x + 1$ is a factor of $f(x)$.**

(b) Let $g(x) = x^6 + 2x(x - 1) - 4$

$$\begin{aligned} g(-1) &= (-1)^6 + 2(-1)(-2) - 4 \\ &= 1 \end{aligned}$$

So, $g(-1) \neq 0$.

$\therefore$ **$x + 1$ is not a factor of $g(x)$.**

Example 6 The expression $ax^3 - 8x^2 + bx + 6$ is exactly divisible by $x^2 - 2x - 3$. Calculate the value of a and of b.

Solution:

Let $f(x) = ax^3 - 8x^2 + bx + 6$

Since $x^2 - 2x - 3 = (x + 1)(x - 3)$, $f(x)$ is exactly divisible by $x + 1$ and $x - 3$.

So $f(-1) = 0 \quad \Rightarrow \quad -a - 8 - b + 6 = 0$

$$a + b = -2 \quad \text{.......... (1)}$$

and $f(3) = 0 \quad \Rightarrow \quad 27a - 72 + 3b + 6 = 0$

$$27a + 3b = 66$$

$$9a + b = 22 \quad \text{.......... (2)}$$

Solving (1) and (2) gives $a = $ **3**, $b = $ **–5**.

Exercise 5.3

1. Determine whether or not each of the following is a factor of the expression $3x^3 + 2x^2 - 7x + 2$.
   (a) $x - 1$ (b) $2x - 3$ (c) $x^2 + x - 2$
2. Show that $x^4 - 3x^2 + 2x + 4$ is exactly divisible by $x + 1$ but not by $x - 2$.
3. For what value of k is $x^3 - 2kx^2 + 3x + k$ exactly divisible by $x - 2$?
4. Given that $3x^2 - 4ax - 4a^2$ has a factor $x + 2$, find the values of a.
5. Find the values of k if $3(x + 3)^4 - (k + x)^2 - 12$ has a factor $x + 1$.
6. The polynomial $2x^3 + x^2 + px - 4$ has a factor $x - 2$. Find p. Show that $2x + 1$ is also a factor and deduce the third factor.
7. The expression $ax^3 + bx^2 - 5x + 2a$ is exactly divisible by $x^2 - 3x - 4$. Calculate the value of a and of b and factorise the expression completely.
8. The expression $x^3 + ax^2 + bx + 3$ is exactly divisible by $x + 3$ but it leaves a remainder of 91 when divided by $x - 4$. What is the remainder when it is divided by $x + 2$?
9. The polynomial $x^3 + ax^2 + bx + c$ is exactly divisible by $x^2 - x - 2$. Find the value of $a + b$.
10. The polynomial $2x^2 + bx - 3$ has a factor $x - a$, where $a \neq 0$. Express b in terms of a.
11. Show that $x^2 + 2x - 3$ is a factor of $(x + 2)^3 - (x + 1)^3 - 3x - 16$.
12. $x^3 + ax^2 - x + b$ and $x^3 + bx^2 - 5x + 3a$ have a common factor $x + 2$. Find a and b.
13. For what values of p does the polynomial $(2p + 1)x^2 + px + 2p^2$ have a factor
    (a) $x - 1$,   (b) $x + 2$?
    State the value of p for which the polynomial is exactly divisible by $x + 2$ but not by $x - 1$.
14. Given that $f(a) = a^3 - ba^2 - 4b^2a + 4b^3$, show that $a - 2b$ is a factor of $f(a)$. Find, in terms of b, the remainder when $f(a)$ is divided by $a + b$.

*15. Given that $2x^2 + 3px - 2q$ and $x^2 + q$ have a common factor $x - a$, where p, q and a are non-zero constants, show that $9p^2 + 16q = 0$.

5.4 Solving Cubic Equations

Having learnt to solve linear and quadratic equations, we shall now consider **cubic equations** of the form $px^3 + qx^2 + rx + s = 0$ where p, q, r and s are constants. If $s = 0$, the equation becomes $x(px^2 + qx + r) = 0$ and the roots are 0 and those of $px^2 + qx + r = 0$.

If $s \neq 0$ and α is a root, the equation becomes $(x - \alpha)(ax^2 + bx + c) = 0$ and the other roots are obtained from $ax^2 + bx + c = 0$.
An integer value for α may be obtained by trial and error.

Consider the equation $2x^3 + 3x^2 - 11x - 6 = 0$ with an integer root α.

If we let $2x^3 + 3x^2 - 11x - 6 \equiv (x - \alpha)(ax^2 + bx + c)$,
then $\alpha \times c = 6$ (constant terms).
$\therefore$ the possible values of α are ±1, ±2, ±3 and ±6.

Let $f(x) = 2x^3 + 3x^2 - 11x - 6$ and test the possible values until we obtain $f(\alpha) = 0$.

Thus
$$f(1) = 2 + 3 - 11 - 6 \neq 0$$
$$f(-1) = -2 + 3 + 11 - 6 \neq 0$$
$$f(2) = 16 + 12 - 22 - 6 = 0 \quad \Rightarrow \quad \alpha = 2$$

and so

$$\begin{aligned} 2x^3 + 3x^2 - 11x - 6 &\equiv (x - 2)(ax^2 + bx + c) \\ &\equiv (x - 2)(2x^2 + bx + 3) \quad (\textit{coeff. of } x^3,\ x^0) \\ &\equiv (x - 2)(2x^2 + 7x + 3) \quad (\textit{coeff. of } x^2) \\ &\equiv (x - 2)(2x + 1)(x + 3) \end{aligned}$$

$$\therefore \quad f(x) = 0 \quad \Rightarrow \quad x = 2, -\frac{1}{2}, -3$$

Example 7 Solve the cubic equation $x^3 - 7x^2 + 4x + 12 = 0$.

Solution:

$$x^3 - 7x^2 + 4x + 12 = 0$$

Let
$$f(x) = x^3 - 7x^2 + 4x + 12$$
$$f(-1) = -1 - 7 - 4 + 12$$
$$= 0$$
$\Rightarrow (x + 1)$ is a factor of $f(x)$.

So, $x^3 - 7x^2 + 4x + 12 \equiv (x + 1)(x^2 + bx + 12)$

Equating the coefficients of x^2, $-7 = b + 1$

$$b = -8$$
$$x^3 - 7x^2 + 4x + 12 = 0$$
$$(x + 1)(x^2 - 8x + 12) = 0$$
$$(x + 1)(x - 2)(x - 6) = 0$$
$$\Rightarrow \quad x = -1, 2, 6$$

Note: The possible values of α for which $f(\alpha) = 0$ are ±1, ±2, ±3, ±4, ±6 and ±12 (factors of 12).

Example 8 Solve the equation $x^2(5 - 2x) = 4$, giving your answers correct to two decimal places where necessary.

Solution:

$$x^2(5 - 2x) = 4$$
$$\Rightarrow \quad 2x^3 - 5x^2 + 4 = 0$$

Let $f(x) = 2x^3 - 5x^2 + 4$

$f(2) = 16 - 20 + 4 = 0$

$\Rightarrow$ $(x - 2)$ is a factor of $f(x)$.

So, $2x^3 - 5x^2 + 4 = (x - 2)(2x^2 + kx - 2)$

Equating the coefficients of x^2, $-5 = k - 4$

$k = -1$

$\therefore \ f(x) = 0 \quad \Rightarrow \quad (x - 2)(2x^2 - x - 2) = 0$

$$\Rightarrow \quad x = 2 \quad \text{or} \quad x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-2)}}{2(2)}$$
$$= -0.78,\ 1.28$$

Exercise 5.4

1. Factorise the cubic polynomials $3x^3 - 10x^2 + 9x - 2$ and $x^3 - 2x^2 - 4x + 8$.
2. Given $f(x) = x^3 - 3x^2 - 4x$, solve the equation
   (a) $f(x) = 0$, **(b)** $f(x) = -12$.
3. Solve the following cubic equations.
   (a) $x^3 - 4x^2 + x + 6 = 0$ **(b)** $4x^3 + 3x^2 - 16x = 12$
   (c) $4x^3 + 18 = 7x^2 + 21x$ **(d)** $x^3 + 4 = x(x + 4)$
   (e) $(x - 3)^2 = \dfrac{4}{x}$ **(f)** $x(x + 3)(x - 1) = x + 8$
4. The curves $y = 2x^3$ and $y = (2 - x)(5x + 6)$ intersect in 3 points. Find the x-coordinates of these points.
5. Solve the following equations, giving the values of x correct to two decimal places where necessary.
   (a) $3x^3 + 5x^2 = 3x + 2$ **(b)** $2x^3 + 6x - 6 = (13x - 6)(x - 1)$
6. The remainder when $2x^3 + 9x^2 + 7x + 3$ is divided by $x - k$ is 9. Find k.
7. Factorise $f(x) = x^3 - 2x^2 - 7x - 4$. Hence solve the equation
   (a) $f(x) = 0$, **(b)** $f(x) = (x + 1)(x - 4)$, **(c)** $f(x) = 6(x + 1)$.

*8. Given that $P(x) = x^4 + ax^3 - x^2 + bx - 12$ has factors $x - 2$ and $x + 1$, solve the equation $P(x) = 0$.

*9. Given $f(x) = 2x^3 + ax^2 - 7a^2x - 6a^3$, determine whether or not $x - a$ and $x + a$ are factors of $f(x)$. Hence find, in terms of a, the roots of $f(x) = 0$.

*10. Complete the following table.

	f(x)	Completely factorised form of f(x)
(a)	$(4x^2 - 1)(x + 3)$	$(2x + 1)(2x - 1)(x + 3)$
(b)	$(x + 2)(6 - x - x^2)$	
(c)	$2x^3 + 11x^2 + 10x - 8$	
(d)	$-3x^3 - 5x^2 + 11x - 3$	

Use a graph plotter to plot $y = \mathrm{f}(x)$ for each of the above cases. Devise a method to sketch the graph of $y = \mathrm{f}(x)$ from the completely factorised form of f(x). Use your method to sketch the graphs of $y = (x - 4)(2x + 3)(2 - x)$ and $y = (x + 1)^2(x + 3)$. Check your answers with the graph plotter. Are you right? If not, modify your method and check it with more cubic polynomials f(x).

Important Notes

1. **Polynomials**
   $ax^2 + bx + c$ is a polynomial of degree 2. (Quadratic)
   $ax^3 + bx^2 + cx + d$ is a polynomial of degree 3. (Cubic)

2. **Identities**

   $$\mathrm{P}(x) \equiv \mathrm{Q}(x) \quad \Leftrightarrow \quad \mathrm{P}(x) = \mathrm{Q}(x) \text{ for all values of } x$$

   To find unknowns in an identity,
   (a) substitute suitable values of x, or
   (b) equate coefficients of like powers of x.

3. **Remainder theorem**
   If a polynomial f(x) is divided by $(x - a)$, the remainder is $R = \mathrm{f}(a)$.

4. **Factor theorem**
   $(x - a)$ is a factor of $\mathrm{f}(x) \quad \Leftrightarrow \quad \mathrm{f}(a) = 0$

5. **Solution of cubic equation $px^3 + qx^2 + rx + s = 0$ with an integer root α**
   ***Step 1*:** Obtain one factor $(x - \alpha)$ by trial and error.
   ***Step 2*:** Factorise $px^3 + qx^2 + rx + s = 0$ as $(x - \alpha)(ax^2 + bx + c) = 0$.
   ***Step 3*:** Solve $ax^2 + bx + c = 0$ for the other roots.

Miscellaneous Example

Example 9 Given $f(x) = 2x^3 + ax^2 + bx - 9$, find the value of a and of b if $f(x)$ has a factor $x - 3$ but leaves a remainder of 8 when divided by $x + 1$. Factorise $f(x)$ completely. Hence sketch the curve $y = f(x)$ and solve the equation $16x^3 + 4ax^2 = 9 - 2bx$.

Solution:

$f(3) = 0 \quad \Rightarrow \quad 2(3)^3 + a(3)^2 + 3b - 9 = 0$

$3a + b = -15 \text{ .... } (1)$

$f(-1) = 8 \quad \Rightarrow \quad -2 + a - b - 9 = 8$

$a - b = 19 \text{ .... } (2)$

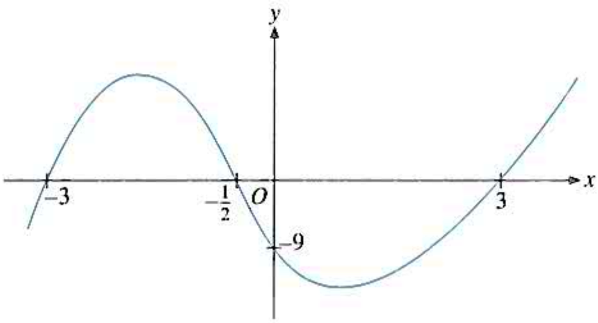
Solving (1) and (2), we get $a = \mathbf{1}$ and $b = \mathbf{-18}$.

Since $x - 3$ is a factor of $f(x)$,
$f(x) = 2x^3 + x^2 - 18x - 9 \equiv (x - 3)(2x^2 + kx + 3)$
Equating the coefficients of x^2, $1 = k - 6$
$k = 7$

$\therefore \; f(x) = (x - 3)(2x^2 + 7x + 3)$
$= \mathbf{(x - 3)(x + 3)(2x + 1)}$

For the curve $y = f(x)$, $y = 0 \quad \Rightarrow \quad x = 3, -3, -\frac{1}{2}$.

With the help of these x-intercepts, the curve is sketched as shown.



$$16x^3 + 4ax^2 = 9 - 2bx$$
$$\Rightarrow \quad 2(2x)^3 + a(2x)^2 + b(2x) - 9 = 0$$
$$2u^3 + au^2 + bu - 9 = 0, \text{ where } u = 2x$$
$$(u - 3)(u + 3)(2u + 1) = 0$$
$$u = 3, -3, -\frac{1}{2}$$
$$\Rightarrow \quad 2x = 3, -3, -\frac{1}{2}$$
$$\therefore \quad x = \frac{3}{2}, -\frac{3}{2}, -\frac{1}{4}$$

Miscellaneous Exercise 5

1. Show that the expression $x^3 + (k - 2)x^2 + (k - 7)x - 4$ has a factor $x + 1$ for all values of k. If the expression also has a factor $x + 2$, find the value of k and the third factor.

2. Given that $x^3 + x - 4 = (x^2 + x - 1)(x - 1) + Ax + B$ for all values of x, find the values of A and B. Hence or otherwise, find the remainder when $x^3 + x - 4$ is divided by $x^2 + x - 1$.

3. Given that $f(x) = x^3 - 7x + 6$, calculate the remainders when $f(x)$ is divided by $4 - x$ and $x + 3$, respectively. Factorise $f(x)$ completely. By using the substitution $y = \frac{1}{x}$, or otherwise, solve the equation $6x^3 - 7x^2 + 1 = 0$.

4. The expression $3(x + 2)^5 + (x + k)^2$ leaves a remainder of 7 when divided by $x + 1$. Determine the values of k.

5. If $x + 2$ is a factor of $x^4 + (p - 1)x^2 - p^2$ but not of $x^3 + px^2 - 3x + 10$, find the value of p.

***6.** Given that $x^5 + ax^3 + bx^2 - 3 \equiv (x^2 - 1)Q(x) - x - 2$, where $Q(x)$ is a polynomial. State the degree of $Q(x)$ and find the value of a and of b. Find also the remainder when $Q(x)$ is divided by $x + 2$.

7. The equation $6x^3 + px^2 = qx - 10$ has roots $x = 2$ and $x = \frac{1}{2}$. Find the values of the constants p and q and the third root.

8. **(a)** Solve the equation $x^3 + 3x^2 - 6x - 8 = 0$.
(b) The expressions $22x^3 + 15x^2 - 4x - 6$ and $2 + 14x - 12x^2 - 5x^3$ have the same remainder when each is divided by $x - a$. Show that $27a^3 + 27a^2 - 18a - 8 = 0$. Use the substitution $y = 3a$ to find the values of a.

9. Find the value of c for which the expression
$$x^3 + (9 - c)x^2 - c^2x - 5c^2 + 25c - 100$$
is divisible by $x + 4$ but not by $x - 5$.
What is the value of c if the expression is exactly divisible by $x^2 - x - 20$?

10. Given that $f(x) = x^3 + px^2 - 2x + 4\sqrt{3}$ has a factor $x + \sqrt{2}$, find the value of p. Show that $x - 2\sqrt{3}$ is also a factor and hence solve the equation $f(x) = 0$.

***11.** Given that $kx^3 + 2x^2 + 2x + 3$ and $kx^3 - 2x + 9$ have a common factor, what are the possible values of k?

12. Find the value of the positive integer n for which division of $x^{2n+1} + 3x^2 - 28$ by $x - 4$ gives a remainder of 52.

13. Solve the cubic equation $x^3 + x^2 = 10x - 8$. By using the substitution $y = x^2$, solve the equation $x^6 + x^4 = 10x^2 - 8$.

14. Given that $16x^4 - 4x^3 - 4b^2x^2 + 7bx + 18$ is divisible by $2x + b$,
(a) show that $b^3 - 7b^2 + 36 = 0$,
(b) find the possible values of b. *(C)*

15. The expression $2x^3 + ax^2 + bx + c$ is divisible by $x - 1$ but leaves a remainder of 3 when divided by $x + 2$. Show that $a - b = 7$. Given also that the remainder is 9 when the expression is divided by $x + 1$, calculate the value of c.

16. Find the x-coordinates of the points of intersection of the curve $y = 4x^3$ with the straight line $y = 13x + 6$. *(C)*

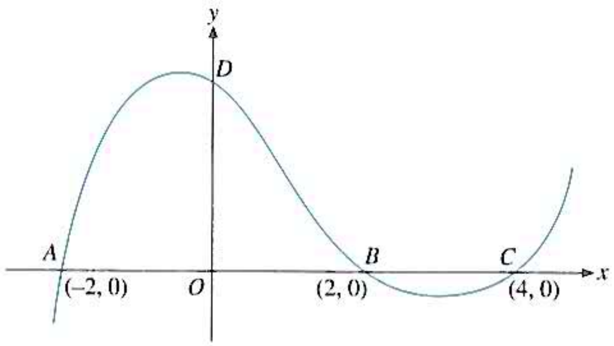
17. (a) Given $g(x) = x^3 + ax^2 + x + 5$ and that $g(x)$ leaves a remainder of 31 when divided by $x - 2$, find the value of a.
Given further that $g(x) = x(x - 1)(x - b) + cx + 5$ for all values of x, calculate the value of b and of c.
(b) Solve the equation $x^3 = 3x^2 + 6x + 2$, giving answers correct to two decimal places where necessary.

18. (a) Given $f(x) = 4x^3 - 16x^2 - 9x + 40$, find the remainder when $f(x)$ is divided by $x - 4$. Deduce a root of the equation $f(x) = x$ and find the other roots of this equation.
(b) Show that $x - 2y$ is a factor of $x^3 + 2x^2y - 5xy^2 - 6y^3$ and find the other two factors.

19. (a) Find the value of p for which $x^2 + 5px + p^2 + 5$ has a factor $x + 2$ but not $x + 3$.
(b) Solve the equation $x^2(x + 3) = 10x + 24$. Hence by using a suitable substitution, solve the equation $25x^2(5x + 3) = 50x + 24$.

20. Find the value of k for which $x^2 - 3x + k$ is a factor of $x^3 - 5x^2 + 12$. *(C)*

21.



The sketch shows part of the graph of $y = x^3 + px^2 + qx + r$, where p, q and r are constants.
The points A, B and C have coordinates $(-2, 0)$, $(2, 0)$ and $(4, 0)$ respectively. The curve crosses the y-axis at D. Evaluate p, q and r, and state the coordinates of D. *(C)*

22. (a) Given that $4x^4 - 9a^2x^2 + 2(a^2 - 7)x - 18$ is exactly divisible by $2x - 3a$, show that $a^3 - 7a - 6 = 0$ and hence find the possible values of a.
(b) The expression $2x^3 + bx^2 - cx + d$ leaves the same remainder when divided by $x + 1$ or $x - 2$ or $2x - 1$. Evaluate b and c. Given also that the expression is exactly divisible by $x + 2$, evaluate d. *(C)*

***23.** **(a)** The expression $f(x) = ax^3 - (a + 3b)x^2 + 2bx + c$ is exactly divisible by $x^2 - 2x$. When $f(x)$ is divided by $x - 1$, the remainder is 8 more than when it is divided by $x + 1$. Factorise $f(x)$ completely. Hence sketch the curve $y = f(x)$ and find the range of values of x for which $f(x) < 0$.

(b) Given $6x^2 - x - 3 = A(2x + 1)(x - 1) + B(1 - x) + C$ for all values of x, find A, B and C.

***24.** Given that $f(x) = (x - 3)^2 - 2(x - 1)^3 + ax + b$ has a factor $x^2 - x - 2$,

(a) find the value of a and of b,

(b) find the remainder when $f(x)$ is divided by x.

Hence or otherwise, solve the equation $f(x) = 0$.

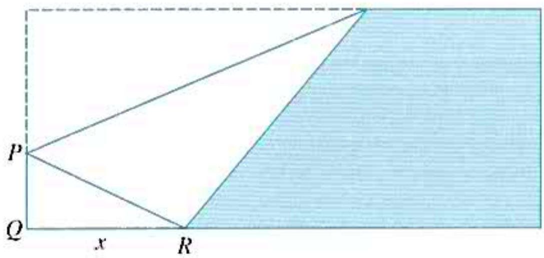
***25.** Given $f(x) = 2x^3 + (4 - 2a)x^2 - ax + 6a$, show that $x + 2$ is a factor of $f(x)$ and find the other quadratic factor. If $f(x) = 0$ has only one real root, find the range of values of a.

26. Paul worked out the following:

$$f(x) = 4x^3(x - 2)^5 - x^2 + x\sqrt{x} - 7$$
$$f(1) = 4 - 1 + 1 - 7 = -3$$

and claimed that $f(x)$ leaves a remainder of -3 when it is divided by $x - 1$. Do you agree with him? State your reason(s).

27. Take an ordinary sheet of A4 paper (size is 21.0 cm by 29.7 cm). Fold a corner over to the opposite side, thus determining a triangle PQR (see diagram) with area $A(x)$. Experiment with several folds to see how A varies with x.



(a) $A(x)$ is a cubic polynomial. Find it.

(b) Check that 0 and 21 satisfy the equation $A(x) = 0$. Why must this be so?

(c) Use a plotter to graph $y = A(x)$ and determine the value of x in the interval $0 \leqslant x \leqslant 21$ so that $A(x)$ is maximised.

***28.** A cubic polynomial $f(x)$ has a factor x and $f(x) - f(x - 1) \equiv 3x^2 - 5x$. By substituting suitable values of x, show that $f(x)$ has a factor $x + 1$ but leaves a remainder of -8 when divided by $x + 2$. What is the remainder when $f(x)$ is divided by $x - 1$? Hence or otherwise, find $f(x)$.

Revision Exercise 1

1. **(a)** Given that $A = \{x : x \text{ is an odd positive integer and } x \leqslant 7\}$ and
$B = \{x : x \text{ is a prime number and } x(x + 1) < 39\}$,
   (i) list the elements of the sets A and B.
   (ii) find an integer x such that $x \in A$ and $x \notin B$,
   (iii) find the integer x such that $x \notin A$ and $x \in B$.
   (b) Solve the simultaneous equations $2x + y = 3$, $x^2 + y^2 + 4y = 6$.

2. Solve the following equations.
   (a) $32^{5x} = 8^{13 + 4x}$
   (b) $\log_2 (y + 6) + \log_2 (y - 1) = 1 + \log_2 (3y)$
   (c) $4^{x+2} = 27$

3. **(a)** If $u = \log_4 x$, show that $\log_x 4 = \frac{1}{u}$. Hence find the values of x for which $2 \log_4 x + 3 \log_x 4 = 7$.
   (b) Given that $\log_3 x = a$ and $\log_{\sqrt{3}} y = b$, express x^2y and $\frac{x}{y^2}$ as powers of 3. Hence given that $x^2y = 1$ and $\frac{x}{y^2} = \frac{1}{243}$, determine the value of a and of b.

4. **(a)** Find the range of values of x for which $(2x - 3)^2 - (x - 1)^2 < 5$.
   (b) Express $y = 5 + 4x - 2x^2$ in the form $y = k - 2(x + h)^2$ where h and k are constants. Hence state the maximum value of y and the corresponding value of x. Sketch the curve $y = 5 + 4x - 2x^2$.

5. **(a)** Find the range of values of p for which the equation $4x^2 - 2px + p + 3 = 0$ has no real roots.
   (b) The line $y = x + c$ is a tangent to the curve $y = \frac{9}{2 - x}$. Find the values of c.

6. **(a)** Find the remainder when $(x + 2)^3(x - 1) - 6x + 3$ is divided by $x + 3$.
   (b) Given that $x - p$ is a factor of the expression $x^2 - (p + 2)x - p^2 + 4p + 8$, calculate the possible values of p.

7. **(a)** Given that the expression $2x^3 + ax^2 + bx + c$ leaves the same remainder when divided by $x + 3$ and when divided by $x - 2$, prove that $a = 14 + b$. Given also that the remainder is 9 when the expression is divided by $x + 1$, calculate the value of c.
   (b) Solve the cubic equation $x^3 - x^2 - 11x + 18 = \frac{(x - 2)(x - 3)}{2}$.

Revision Exercise 2

1. Given that $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$,
$A = \{x : x \text{ is an odd number and } x^2 > 5\}$
and $B = \{x : x \text{ is not a multiple of } 3\}$.
   (a) List the elements of the sets A, B and B'.
   (b) List the elements of the sets $A \cap B$ and $A \cup B$.
   (c) Find $n(A \cup B')$.

2. (a) Find the range of values of k for which the quadratic equation
$$2kx^2 + (8 - 4k)x + k + 1 = 0$$
has real roots. State the largest integer value of k for which this equation has no real roots.
   (b) If each of the equations $px^2 + qx + 2r = 0$, $rx^2 + px - q + 1 = 0$ have equal real roots, find a relation between p and q.

3. Solve the simultaneous equations $x + 2y = 7$, $x^2 + 4y^2 = 37$. Hence, find the possible values of a and b, correct to three significant figures where necessary, which satisfy both the equations $3^a + 2^{b+1} = 7$, $9^a + 4^{b+1} = 37$.

4. (a) Given that $y = 3(4)^{x+2}$, find, without using tables or calculators,
   (i) the value of y when $x = -\frac{1}{2}$, (ii) the value of x when $y = 96$.
   (b) Solve the following equations, giving your answers correct to three significant figures.
   (i) $\ln x^3 + 2 \ln x^2 - 5 \ln x + \ln \sqrt{x} = 5$
   (ii) $3^x = 5 \times 2^{x+1}$

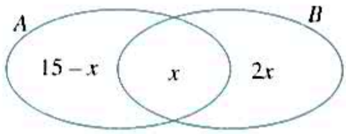
5. (a) Without using tables or calculators, evaluate $\dfrac{\log_5 9 + 2 \log_5 6 - 4 \log_5 3}{\log_5 40 - \log_5 4 - 1}$.
   (b) The curve $y = ab^x$ passes through the points $(0, 5)$ and $\left(\frac{2}{3}, \frac{5}{4}\right)$. Find the positive value of a and of b.

6. (a) Given that $4x^2 - 6x + 9 = A(x - 1)(2x + 1) + B(x - 1) + C$ for all values of x, find the values of A, B and C.
   (b) Solve the cubic equation $2x^3 + 36 = 11x^2 - 3x$.

7. (a) The expression $px^3 - 5x^2 + qx + 10$ has factor $2x - 1$ but leaves a remainder of -20 when divided by $x + 2$. Find the values of p and q and factorise the expression completely.
   (b) The quadratic equation $x^2 + ax + b = 2$ has roots -1 and 4. Find
   (i) the value of a and of b,
   (ii) the range of values of c for which the equation $x^2 + ax + b = c$ has real roots.

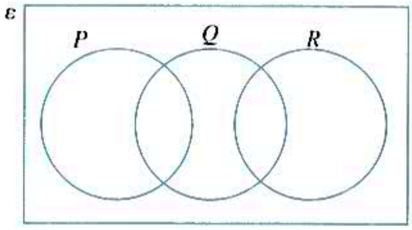
Revision Exercise 3

1. **(a)** A and B are two sets and the numbers of elements are shown in the Venn diagram. Given that $n(A \cap B') = 2n(A' \cap B)$, find



(i) the value of x,
(ii) $n(A)$ and $n(B)$.

(b) P, Q and R are three sets and $\varepsilon = P \cup Q \cup R$. It is given that $n(\varepsilon) = 56$, $n(P \cup Q) = 48$, $n(R) = 15$ and $n(P) = 32$. Find the value of



(i) $n(Q \cap R)$,
(ii) $n(P' \cap Q \cap R')$.

2. **(a)** Solve the simultaneous equations $10y - 6x = 3$ and $(x - 2)^2 + (2y - 3)^2 = 61$.

(b) Given that $\lg 2 = a$ and $\lg 3 = b$, express $\lg \sqrt[3]{972}$ in terms of a and b. Find x if $\lg x = 3a - 4b + 1$.

3. **(a)** Find the exact value of x if $8^x = \sqrt{2\sqrt{8\sqrt{2}}}$.
(b) Solve the following equations.
(i) $2 \log_2 x = 4 + \log_2 (x + 5)$
(ii) $4e^{2y} - 21 = 0$
(iii) $\lg (4^z + 2) - z \lg 2 = \lg 3$

4. **(a)** When the expression $2x^2 - (8 - p)x + (p + 1)(p - 3)$ is divided by $x + p$, the remainder is p. Calculate the possible values of p.
(b) The expressions $x^3 + ax^2 - x + b$ and $x^3 + bx^2 - 5x + 3a$ have a common factor $x + 2$. Find the value of a and of b.

5. **(a)** Given that $f(x) = 8x^3 + 4x - 3$, find the remainder, if it exists, when
(i) $f(x)$ is divided by $2x + 1$,
(ii) $f(x)$ is divided by $2 - x$,
(iii) $\dfrac{1}{f(x)}$ is divided by $x + 1$,
(iv) $f(x + 1)$ is divided by $x + 2$.
(b) Solve the equation $4x^3 + 3x^2 - 16x = 12$. Hence, find to three significant figures, the value of x such that $4e^{3x} + 3e^{2x} - 16e^x = 12$.

6. **(a)** Find the range of values of x for which $x^2 - 3x + 8$ has values between 6 and 12.
(b) Factorise $2x^2 - 3x - 5$ and hence sketch the curve $y = 2x^2 - 3x - 5$.
(c) Express $-2x^2 + 8x + 9$ in the form $a(x - h)^2 + k$. Hence state the maximum value of $-2x^2 + 8x + 9$ and sketch the curve $y = -2x^2 + 8x + 9$.

7. If the equation $x^2 - 2kx + k^2 - 2k - 6 = 0$ has real roots, show that the roots of the equation $x^2 + 6x = 3 + k$ has two distinct real roots.

Revision Exercise 4

1. Each of the 42 students in a class chooses at least one of the three subjects: Literature, History and Geography.
   Of the 25 students who choose Literature,
   3 choose History but not Geography,
   12 choose Geography but not History.
   Of the 28 students who do not choose History,
   8 choose Geography only.
   (a) Draw a Venn diagram to illustrate this information.
   (b) How many students choose Literature only.
   (c) How many students choose all the three subjects.
   (d) Find the greatest possible number of students who choose History only.

2. **(a)** Using the substitution $y = 3^x$, find the values of x such that
   $$9^x - 10(3^{x+1}) + 81 = 0.$$
   (b) Given that $(3^{x+2})(5^{x-1}) = 15^{2x}$, show that $15^x = \frac{9}{5}$. Hence or otherwise, find the value of x, correct to three significant figures.

3. **(a)** Solve the simultaneous equations $2x - 3y + 1 = 0$ and $2y^2 - xy + 2x = 3$.
   (b) Given that $2x^4 + ax^3 - 7x^2 - 6x + 2 \equiv (x^2 + 3x - 4)f(x) - bx - 6$, find
   (i) the value of a and of b,
   (ii) the remainder when $f(x)$ is divided by $2x + 1$.

4. **(a)** If the roots of the equation $3x^2 + (k - x)x + 2 - k^2 = 0$ are real, find the range of values of k.
   (b) The curves $y = 3ax^2 + ax + 3$ and $y = 2x^2 + 4x + 5$ do not intersect. Find the range of values of a.

5. **(a)** The expression $x^2 + px + r$ leaves the same remainder when divided by $x + q$ or $x - 2p$, where $q \neq -2p$. Express p in terms of q.
   (b) Solve the equation $3x^3 - 2x^2 - 7x - 2 = 0$.
   Hence sketch the curve $y = 3x^3 - 2x^2 - 7x - 2$ and find the range of values of x for which $3x^3 - 2x^2 - 7x < 2$.

6. **(a)** Find x if $\lg(2x - 1) = 1 + \lg x - \lg(x + 3)$.
   (b) Find x to three significant figures if $\frac{1}{\log_x 2} + \frac{1}{\log_x 3} + \frac{1}{\log_x 6} = 3.6$.
   (c) Solve the simultaneous equations $e\sqrt{e^x} = e^{2y}$ and $\log_4(x + 2) - \log_2 y = 1$.

7. **(a)** Find the range of values of x for which $(x - 1)(x - 3) \geqslant 15$. Hence use the substitution $y = x^2$, or otherwise, to solve the inequality $(x^2 - 1)(x^2 - 3) \geqslant 15$.
   (b) Sketch the curve $y = 2x^2 + 4x + 3$. Hence find the range of values of k for which $2x^2 + 4x + 3 > k$ for all real values of x.

6 Matrices

6.1 Represent Information as a Matrix

The following table shows the stock of pens at two stationery shops.

	Pilot	Pentel	Red Leaf
Shop 1	94	104	70
Shop 2	65	83	54

We can represent this information as a rectangular array of numbers as follows:

$$\begin{pmatrix} 94 & 104 & 70 \\ 65 & 83 & 54 \end{pmatrix}$$

This rectangular array of numbers in rows and columns is called a **matrix** (plural: **matrices**). Since this matrix

$$\begin{array}{cc} & \begin{array}{ccc} \text{column 1} & \text{column 2} & \text{column 3} \end{array} \\ \begin{array}{c} \text{row 1} \\ \text{row 2} \end{array} & \begin{pmatrix} 94 & \quad 104 \quad & 70 \\ 65 & \quad 83 \quad & 54 \end{pmatrix} \end{array}$$

has 2 rows and 3 columns, we say that it has **order**, or **dimension**, 2×3 (read '2 by 3'), where the number of rows is specified first.
Similarly, the matrices

$$\begin{pmatrix} -3 & 6 \\ 1 & 2 \\ 5 & 1 \\ 0 & -7 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 3 & -2 & 0 \\ 1 & -5 & 4 & 21 \end{pmatrix}$$

have orders 4×2 and 2×4 respectively.

The numbers in a matrix are called its **elements** or **entries**.

A matrix that has only one column is called a **column matrix**, e.g. $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \\ 6 \end{pmatrix}$.

A matrix that has only one row is called a **row matrix**, e.g. $(2 \quad 1)$ and $(4 \quad -9 \quad 7 \quad 30)$. Note that there are no commas between the elements.

Example 1 The numbers of boys and girls in the four levels at a certain secondary school are given in the table below:

	Sec 1	Sec 2	Sec 3	Sec 4
Boys	212	204	198	192
Girls	193	202	194	203

Represent the above information as a matrix. State the order of your matrix.

Solution: The information may be represented by the matrix

$$\begin{pmatrix} 212 & 204 & 198 & 192 \\ 193 & 202 & 194 & 203 \end{pmatrix}.$$

It has order 2×4.

Example 2 A survey was done to find out which colour is the most popular among a group of 150 people. 53 of them prefer blue, 48 black, 35 white and the rest red. Represent this information as a column matrix and state its order.

Solution: Number of people who prefer red = 150 − 53 − 48 − 35 = 14

$\therefore$ the column matrix is $\begin{pmatrix} 53 \\ 48 \\ 35 \\ 14 \end{pmatrix}$. It has order 4×1.

Sometimes, one may label the rows and columns of a matrix. For instance, the labelled matrices for Examples 1 and 2 are

$$\begin{array}{c} \\ \text{Boys} \\ \text{Girls} \end{array}\begin{array}{c} \begin{array}{cccc} \text{Sec 1} & \text{Sec 2} & \text{Sec 3} & \text{Sec 4} \end{array} \\ \begin{pmatrix} 212 & 204 & 198 & 192 \\ 193 & 202 & 194 & 203 \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{c} \\ \text{Blue} \\ \text{Black} \\ \text{White} \\ \text{Red} \end{array}\begin{array}{c} \text{Number of people} \\ \begin{pmatrix} 53 \\ 48 \\ 35 \\ 14 \end{pmatrix} \end{array}.$$

Equality of Matrices

If the information in Example 1 is represented by the matrix $\begin{pmatrix} a & b & 198 & 192 \\ c & 202 & d & 203 \end{pmatrix}$, then it is obvious that $\begin{pmatrix} a & b & 198 & 192 \\ c & 202 & d & 203 \end{pmatrix} = \begin{pmatrix} 212 & 204 & 198 & 192 \\ 193 & 202 & 194 & 203 \end{pmatrix}$ which implies that $a = 212$, $b = 204$, $c = 193$ and $d = 194$.

Two matrices are equal if they have the same order and if their corresponding elements are equal.

Thus, $\begin{pmatrix} a & b \\ c & 21 \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 4 & 21 \end{pmatrix} \quad \Rightarrow \quad a = 1,\ b = -5$ and $c = 4$

but $\begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix} \neq (-3 \quad 2 \quad 5)$ *(different order)*

and $\begin{pmatrix} 1 & -5 \\ 4 & 21 \end{pmatrix} \neq \begin{pmatrix} 1 & -51 \\ 4 & 21 \end{pmatrix}$ *(not all corresponding elements are equal).*

Exercise 6.1

1. State the order of each of the following matrices.

(a) $(-3 \quad 2 \quad 5)$

(b) $\begin{pmatrix} 1 & 6 & 4 & -3 \\ 0 & -9 & 2 & 5 \end{pmatrix}$

(c) $\begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix}$

(d) $\begin{pmatrix} -3 & 0 \\ 9 & 21 \\ -3 & 1 \\ 2 & 5 \end{pmatrix}$

(e) (3)

(f) $\begin{pmatrix} 7 \\ -3 \end{pmatrix}$

2. In a Mathematics test, 65% of the students in Class A scored distinctions, 55% of the students in Class B scored distinctions while 52% of the students in Class C scored distinctions. Complete the table provided and represent the given information as a matrix. State the order of your matrix.

Class	Percentage of students who scored distinctions	Percentage of students who did not score distinctions
Class A		
Class B		
Class C		

3. In a class of 30 students, 16 are boys, 7 of whom wear glasses. 8 of the girls in this class wear glasses. Complete the table provided. Represent the given information as a matrix and state the order of this matrix.

	Wear glasses	Do not wear glasses
Boys		
Girls		

4. A drinks stall sold 160 cups of coffee, 125 cups of tea and 210 glasses of soft drinks on Monday. On Tuesday, it sold 145 cups of coffee, 130 cups of tea and 275 glasses of soft drinks. On Wednesday, it sold 120 cups of coffee, 155 cups of tea and 325 glasses of soft drinks. Design a matrix to represent this information, labelling the rows and columns. State the order of your matrix.

5. A travel agent sold 143 holiday packages to Malaysia and 105 to Thailand for the month of June. In July he sold 65 holiday packages to Malaysia and 46 to Thailand. In August he sold 122 holiday packages to Malaysia and 89 to Thailand. Design a matrix to represent this information, labelling the rows and columns. State the order of your matrix.

6. The 4 by 3 matrix on the right shows the prices, in dollars, of fruit juices per glass at various stalls. The rows represent the fruit juices: watermelon, orange, papaya and honeydew in that order. The columns represent the stalls: A, B and C in that order.

$$\begin{pmatrix} 1.50 & 1.60 & 1.50 \\ 2.00 & 1.80 & 2.10 \\ 1.20 & 1.50 & 1.60 \\ 2.50 & 2.50 & 2.20 \end{pmatrix}$$

   (a) How much does a glass of orange juice cost at Stall B?
   (b) At which of these three stalls is a glass of papaya cheapest?
   (c) Lynn bought 2 glasses of honeydew juice and 3 glasses of watermelon juice from Stall C. How much did she pay for them?

7. Where possible, find the values of the letters in each of the following. In cases where it is not possible, give your reason.

(a) $\begin{pmatrix} 6 & -2 \\ a & 5 \end{pmatrix} = \begin{pmatrix} 6 & b \\ 4 & c \end{pmatrix}$

(b) $\begin{pmatrix} 1 & a & b \\ 9 & -2 & 3 \end{pmatrix} = \begin{pmatrix} c & 0 & 4 \\ 9 & d & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 6 \\ a \\ 5 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 4 & 0 \\ b & 0 \end{pmatrix}$

(d) $\begin{pmatrix} a & b \\ 9 & -2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ c & -2 \\ 3 & d \end{pmatrix}$

(e) $\begin{pmatrix} 6 & a & 5 \\ b & -4 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 7 & c \\ -4 & 2 & d \end{pmatrix}$

(f) $(a \quad 12 \quad c \quad 3) = (5 \quad 12 \quad b \quad 3 \quad 0)$

6.2 Addition, Subtraction and Scalar Multiplication of Matrices

For convenience, bold capital letters such as **A** and **B** are often used to represent matrices symbolically. For instance,

$$\mathbf{A} = \begin{pmatrix} -3 & 2 \\ 5 & 1 \\ 0 & -7 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 & -5 \\ 4 & 21 \end{pmatrix}.$$

Addition of Matrices

Example 3 The following table shows the current stock of colour pens at two shops.

	Blue	Black	Red
Shop 1	30	20	35
Shop 2	45	25	30

Represent the above information as a matrix **S**. State the order of **S**.

The following replenishments have just arrived at the shops:

	Blue	Black	Red
Shop 1	70	60	55
Shop 2	105	65	75

Find the matrices **R** and **F** which represent respectively the replenishments and the final stock of pens (after replenishment) at these shops.

Solution: The matrix for the current stock, $\mathbf{S} = \begin{pmatrix} 30 & 20 & 35 \\ 45 & 25 & 30 \end{pmatrix}$.

The matrix for the replenishments, $\mathbf{R} = \begin{pmatrix} 70 & 60 & 55 \\ 105 & 65 & 75 \end{pmatrix}$.

For Shop 1, after replenishment,
the final stock of blue pens $= 30 + 70 = 100$,
the final stock of black pens $= 20 + 60 = 80$,
the final stock of red pens $= 35 + 55 = 90$.

Similarly the final stocks for Shop 2 can be worked out. Thus, we see that the matrix for the final stock,

$$\mathbf{F} = \begin{pmatrix} 30+70 & 20+60 & 35+55 \\ 45+105 & 25+65 & 30+75 \end{pmatrix}$$
$$= \begin{pmatrix} 100 & 80 & 90 \\ 150 & 90 & 105 \end{pmatrix}.$$

From the above example, we see that

$$\begin{pmatrix} 30 & 20 & 35 \\ 45 & 25 & 30 \end{pmatrix} + \begin{pmatrix} 70 & 60 & 55 \\ 105 & 65 & 75 \end{pmatrix} = \begin{pmatrix} 30+70 & 20+60 & 35+55 \\ 45+105 & 25+65 & 30+75 \end{pmatrix}$$

and we write $\mathbf{S} + \mathbf{R} = \mathbf{F}$.

Hence we define the **addition** of matrices as:

For two matrices A and B of the same order, the matrix A + B is obtained by adding the corresponding elements of A and B.

Example 4 Given the matrices $\mathbf{A} = \begin{pmatrix} 4 & 2 & 6 \\ 1 & 3 & -7 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & 9 & 0 \\ 4 & -1 & 8 \end{pmatrix}$,

$\mathbf{C} = \begin{pmatrix} 2 & 5 \\ 3 & -4 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 1 & 6 \\ 7 & 12 \end{pmatrix}$, find the following matrices if they exist.

(a) $\mathbf{A} + \mathbf{B}$ **(b)** $\mathbf{A} + \mathbf{C}$ **(c)** $\mathbf{C} + \mathbf{D}$ **(d)** $\mathbf{D} + \mathbf{C}$

If $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, show that $\mathbf{C} + \mathbf{O} = \mathbf{C}$.

Solution:

(a) Note that **A** and **B** are of the same order. So,

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 4 & 2 & 6 \\ 1 & 3 & -7 \end{pmatrix} + \begin{pmatrix} 3 & 9 & 0 \\ 4 & -1 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 4+3 & 2+9 & 6+0 \\ 1+4 & 3+(-1) & -7+8 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{7} & \mathbf{11} & \mathbf{6} \\ \mathbf{5} & \mathbf{2} & \mathbf{1} \end{pmatrix}$$

(b) As **A** and **C** are of **different** orders, we cannot add **A** and **C** and so **A** + **C** does not exist.

(c) Note that **C** and **D** are of the same order. So,

$$\mathbf{C} + \mathbf{D} = \begin{pmatrix} 2 & 5 \\ 3 & -4 \end{pmatrix} + \begin{pmatrix} 1 & 6 \\ 7 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} 2+1 & 5+6 \\ 3+7 & -4+12 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{3} & \mathbf{11} \\ \mathbf{10} & \mathbf{8} \end{pmatrix}$$

(d) $$\mathbf{D} + \mathbf{C} = \begin{pmatrix} 1 & 6 \\ 7 & 12 \end{pmatrix} + \begin{pmatrix} 2 & 5 \\ 3 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 1+2 & 6+5 \\ 7+3 & 12+(-4) \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{3} & \mathbf{11} \\ \mathbf{10} & \mathbf{8} \end{pmatrix}$$

$$\mathbf{C} + \mathbf{O} = \begin{pmatrix} 2 & 5 \\ 3 & -4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2+0 & 5+0 \\ 3+0 & -4+0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 5 \\ 3 & -4 \end{pmatrix}$$

$\therefore$ $\mathbf{C} + \mathbf{O} = \mathbf{C}$

Notice that all the elements in the matrix $\mathbf{O}$ are zeros. Such a matrix is called a **zero** or **null matrix** and is usually denoted by $\mathbf{O}$. The following are also null matrices.

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In Example 4, observe from parts (c) and (d) that $\mathbf{C} + \mathbf{D} = \mathbf{D} + \mathbf{C}$.

In fact, the following properties hold for matrix addition:

If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{O}$ are of the same order, where $\mathbf{O}$ is a null matrix,
(1) $\mathbf{A} + \mathbf{O} = \mathbf{A}$
(2) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ (commutative property)
(3) $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$ (associative property)

These properties of matrix addition are similar to the corresponding properties of real numbers.

Subtraction of Matrices

Example 5 After replenishments, the new stock of colour pens for Shop 1 and Shop 2 are as follows:

	Blue	Black	Red
Shop 1	100	80	90
Shop 2	150	90	105

A week later, the owners found that they have sold the following numbers of pens.

	Blue	Black	Red
Shop 1	83	54	44
Shop 2	76	72	80

Find the matrices $\mathbf{A}$ and $\mathbf{B}$ which represent respectively the sales of pens and the stock of pens after these sales.

Solution: Let the matrix $\mathbf{F} = \begin{pmatrix} 100 & 80 & 90 \\ 150 & 90 & 105 \end{pmatrix}$, where $\mathbf{F}$ represents the final stock after replenishments.

The matrix for the sales, $\mathbf{A} = \begin{pmatrix} \mathbf{83} & \mathbf{54} & \mathbf{44} \\ \mathbf{76} & \mathbf{72} & \mathbf{80} \end{pmatrix}$.

For Shop 1, after the sales,
the stock of blue pens $= 100 - 83 = 17$,
the stock of black pens $= 80 - 54 = 26$,
the stock of red pens $= 90 - 44 = 46$.

Similarly the stocks for Shop 2 can be worked out.
Hence the matrix for the stock of pens after the sales is

$$\mathbf{B} = \begin{pmatrix} 100-83 & 80-54 & 90-44 \\ 150-76 & 90-72 & 105-80 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{17} & \mathbf{26} & \mathbf{46} \\ \mathbf{74} & \mathbf{18} & \mathbf{25} \end{pmatrix}$$

From Example 5, we see that

$$\begin{pmatrix} 100 & 80 & 90 \\ 150 & 90 & 105 \end{pmatrix} - \begin{pmatrix} 83 & 54 & 44 \\ 76 & 72 & 80 \end{pmatrix} = \begin{pmatrix} 100-83 & 80-54 & 90-44 \\ 150-76 & 90-72 & 105-80 \end{pmatrix}$$

and we write $\mathbf{F} - \mathbf{A} = \mathbf{B}$.

Hence we define the **subtraction** of matrices as:

For two matrices A and B of the same order, the matrix A – B is obtained by subtracting the corresponding elements of B from A.

Example 6 Given the matrices $\mathbf{A} = \begin{pmatrix} 4 & 2 & 6 \\ 1 & 3 & -7 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & 9 & 0 \\ 4 & -1 & 8 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 2 & 5 \\ 3 & -4 \end{pmatrix}$. Find the following matrices if they exist.

(a) $\mathbf{A} - \mathbf{B}$ **(b)** $\mathbf{B} - \mathbf{C}$

If $\mathbf{D} = \begin{pmatrix} p & -7 & q \\ r & 4 & -15 \end{pmatrix}$ and $\mathbf{D} + \mathbf{B} = \mathbf{A}$, find the values of p, q and r.

Solution: (a) Note that **A** and **B** are of the same order. So,

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 4 & 2 & 6 \\ 1 & 3 & -7 \end{pmatrix} - \begin{pmatrix} 3 & 9 & 0 \\ 4 & -1 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 4-3 & 2-9 & 6-0 \\ 1-4 & 3-(-1) & -7-8 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{1} & \mathbf{-7} & \mathbf{6} \\ \mathbf{-3} & \mathbf{4} & \mathbf{-15} \end{pmatrix}$$

(b) As **B** and **C** are of **different** orders, **B** – **C** does not exist.

Subtracting **B** from both sides of the equation

$$\mathbf{D} + \mathbf{B} = \mathbf{A}$$

gives $\mathbf{D} + \mathbf{B} - \mathbf{B} = \mathbf{A} - \mathbf{B}$

$$\mathbf{D} = \mathbf{A} - \mathbf{B}$$

$$\Rightarrow \begin{pmatrix} p & -7 & q \\ r & 4 & -15 \end{pmatrix} = \begin{pmatrix} 1 & -7 & 6 \\ -3 & 4 & -15 \end{pmatrix}$$

$\Rightarrow$ $p = \mathbf{1}$, $q = \mathbf{6}$ and $r = \mathbf{-3}$

Scalar Multiplication

If $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}$, then $2\mathbf{A} = \mathbf{A} + \mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}$

$$\Rightarrow 2\mathbf{A} = \begin{pmatrix} 2 \times 2 & 2 \times 3 \\ 2 \times 1 & 2 \times (-5) \end{pmatrix}$$

Similarly, $3\mathbf{A} = \mathbf{A} + \mathbf{A} + \mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}$

$$\Rightarrow 3\mathbf{A} = \begin{pmatrix} 3 \times 2 & 3 \times 3 \\ 3 \times 1 & 3 \times (-5) \end{pmatrix}$$

More generally, $k\mathbf{A} = \begin{pmatrix} k \times 2 & k \times 3 \\ k \times 1 & k \times (-5) \end{pmatrix}$

Hence for a matrix A and real number k (also called a scalar), the matrix kA is obtained by multiplying each element in A by k.

The following properties hold for scalar multiplication:

If A and B have the same order, then for any scalars h and k, we have

(1) $k(A + B) = kA + kB$

(2) $(h + k)A = hA + kA$

(3) $h(kA) = (hk)A$

Example 7

The number of ships arriving at a harbour every weekday from Monday to Friday is given in matrix **D**. The number of ships arriving on Saturday and on Sunday is given in matrix **E**.

$$\mathbf{D} = \begin{array}{c} \text{Cargo ships} \quad \text{Passenger ships} \\ \begin{pmatrix} 8 & 3 \\ 10 & 2 \end{pmatrix} \end{array} \begin{array}{l} \text{Dock 1} \\ \text{Dock 2} \end{array} \qquad \mathbf{E} = \begin{array}{c} \text{Cargo ships} \quad \text{Passenger ships} \\ \begin{pmatrix} 17 & 5 \\ 20 & 6 \end{pmatrix} \end{array} \begin{array}{l} \text{Dock 1} \\ \text{Dock 2} \end{array}$$

Find the matrix $5\mathbf{D} + 2\mathbf{E}$.
What information does this matrix give?

Solution:

$$5\mathbf{D} + 2\mathbf{E} = 5\begin{pmatrix} 8 & 3 \\ 10 & 2 \end{pmatrix} + 2\begin{pmatrix} 17 & 5 \\ 20 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \times 8 + 2 \times 17 & 5 \times 3 + 2 \times 5 \\ 5 \times 10 + 2 \times 20 & 5 \times 2 + 2 \times 6 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{74} & \mathbf{25} \\ \mathbf{90} & \mathbf{22} \end{pmatrix}$$

The matrix gives the numbers of cargo ships and passenger ships arriving at Dock 1 and Dock 2 in a week.

Exercise 6.2

1. Where possible, simplify the following. In cases where it is not possible, give your reason.

(a) $\begin{pmatrix} 6 & -2 \\ 5 & -1 \\ 4 & 6 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ -4 & 1 \\ 7 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 8 & 9 \\ 13 & 21 & 4 \end{pmatrix} + \begin{pmatrix} 5 & -8 \\ 0 & -2 \\ 3 & 9 \end{pmatrix}$

(c) $\begin{pmatrix} 6 & -2 \\ 3 & 5 \end{pmatrix} + \begin{pmatrix} -4 & 2 \\ 4 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 10 \\ 8 \end{pmatrix} + \begin{pmatrix} -4 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

(e) $\begin{pmatrix} 6 & -2 & 5 \\ 3 & -4 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 7 & 3 \\ -1 & 2 & c \end{pmatrix}$ (f) $(0 \quad 12 \quad -2 \quad 3) + (5 \quad 2 \quad 8 \quad 3 \quad 0)$

2. Where possible, simplify the following. In cases where it is not possible, give your reason.

(a) $\begin{pmatrix} 6 & 3 & 5 \\ 9 & 14 & -5 \end{pmatrix} - \begin{pmatrix} 2 & 7 & 5 \\ -4 & 20 & 1 \end{pmatrix}$ (b) $(16 \quad 12 \quad -8 \quad 3) - (5 \quad 8 \quad 2 \quad -3)$

(c) $\begin{pmatrix} 3 & 20 & 0 \\ -2 & 3 & 5 \end{pmatrix} - \begin{pmatrix} -9 & 13 \\ 5 & -2 \end{pmatrix}$ (d) $\begin{pmatrix} 18 & 23 \\ 35 & -9 \end{pmatrix} - \begin{pmatrix} 7 & 6 \\ 5 & 12 \end{pmatrix} - \begin{pmatrix} 9 & 13 \\ 21 & -20 \end{pmatrix}$

3. Simplify the following matrices.

(a) $2\begin{pmatrix} -2 \\ 3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ (b) $\frac{(8 \quad 10)}{2} - 4\left(\frac{1}{2} \quad 2\right)$

(c) $8 \times \frac{1}{4}\begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}$ (d) $\dfrac{4\begin{pmatrix} 9 & 6 \\ -3 & 12 \end{pmatrix}}{3}$

4. Given the matrices $\mathbf{A} = \begin{pmatrix} 3 & 21 \\ 16 & -4 \\ 3 & 8 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 9 & 12 \\ -9 & 14 \\ -7 & 9 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 4 & -9 \\ 3 & 10 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 5 & 1 \\ -13 & 5 \end{pmatrix}$,

find the following matrices if they exist:

(a) $3\mathbf{A} + \mathbf{B}$ (b) $\mathbf{B} - \mathbf{A} + \mathbf{C}$ (c) $2\mathbf{C} + 5\mathbf{C} - 4\mathbf{D}$
(d) $-\mathbf{A} + 3(\mathbf{B} - \mathbf{A})$ (e) $2(3\mathbf{C}) - 7\mathbf{D}$ (f) $-\mathbf{A} + 2(\mathbf{D} + \mathbf{C})$

5. Given the matrices $\mathbf{P} = \begin{pmatrix} 2 & 15 & -6 \\ 7 & -3 & 3 \end{pmatrix}$, $\mathbf{R} = \begin{pmatrix} 4 & 0 & 8 \\ 1 & -4 & -1 \end{pmatrix}$ and the 2×3 null matrix

O, find the matrix **Q** if

(a) $\mathbf{Q} = 2(5\mathbf{P}) - 7\mathbf{P}$, (b) $\mathbf{R} - 3\mathbf{P} = 2\mathbf{Q}$,
(c) $6\mathbf{Q} = 4(\mathbf{P} + \mathbf{Q}) + \mathbf{P}$, (d) $\mathbf{R} + \mathbf{Q} = 2\mathbf{P}$,
(e) $2(\mathbf{P} - \mathbf{R}) + 4\mathbf{P} = \mathbf{Q} + 2\mathbf{O}$, (f) $3\mathbf{R} - 2\mathbf{Q} = \mathbf{O}$.

6. A traffic census was carried out at a particular street for a 3-hour period on *one* morning. The results are shown in the matrix below.

$$\begin{array}{l} \\ \text{7 am – 8 am} \\ \text{8 am – 9 am} \\ \text{9 am – 10 am} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Cars} & \text{Buses} & \text{Motorcycles} & \text{Others} \end{array} \\ \begin{pmatrix} 67 & 72 & 52 & 24 \\ 83 & 79 & 62 & 34 \\ 34 & 50 & 26 & 33 \end{pmatrix} \end{array}$$

Write a matrix which shows the total figures that might be expected for the *five* mornings of a working week.

7. A factory has several lorries, vans and cars. At times these vehicles need spare parts. At the beginning of the month, the storekeeper notes the number of spare parts, and he counts them again at the end of the month to see how many are left.

$$\begin{array}{ccc} & \text{Beginning of month} & & \text{End of month} \\ & \begin{array}{ccc}\text{Tyres} & \text{Bulbs} & \text{Fan belts}\end{array} & & \begin{array}{ccc}\text{Tyres} & \text{Bulbs} & \text{Fan belts}\end{array} \\ \begin{array}{l}\text{Lorries}\\ \text{Vans}\\ \text{Cars}\end{array} & \begin{pmatrix} 40 & 35 & 15 \\ 25 & 40 & 6 \\ 19 & 23 & 7 \end{pmatrix} & \begin{array}{l}\text{Lorries}\\ \text{Vans}\\ \text{Cars}\end{array} & \begin{pmatrix} 22 & 27 & 9 \\ 18 & 29 & 4 \\ 11 & 15 & 3 \end{pmatrix} \end{array}$$

(a) How many tyres did the lorries use?
(b) How many tyres did the cars use?
(c) How many fan belts did the vans use?
(d) Write a matrix which shows the number of spare parts used on these vehicles during the month.

8. The list prices, in dollars, of 'ten-year series' for these subjects from two publishers are shown below:

	E Math	**A Math**	**English**	**Physics**
Excel	7.80	7.00	6.50	7.60
Distinction	7.50	7.20	7.00	7.70

Represent the information contained in the table as a matrix **P**.
These publishers sell them to a bookshop at 15% discount. Find the matrix **D** which represents the discounted prices, to the nearest cent, given to the bookshop.

9. A university track and field team participated in a three-day competition with events for men and women. The team recorded the following successes.

$$\begin{array}{lcc} & & \begin{array}{ccc}\text{1st place} & \text{2nd place} & \text{3rd place}\end{array} \\ \text{Wednesday} & \begin{array}{r}\text{Men}\\ \text{Women}\end{array} & \begin{pmatrix} 3 & 5 & 2 \\ 4 & 3 & 1 \end{pmatrix} \\ \text{Thursday} & \begin{array}{r}\text{Men}\\ \text{Women}\end{array} & \begin{pmatrix} 4 & 1 & 3 \\ 5 & 3 & 0 \end{pmatrix} \\ \text{Friday} & \begin{array}{r}\text{Men}\\ \text{Women}\end{array} & \begin{pmatrix} 6 & 0 & 3 \\ 2 & 6 & 2 \end{pmatrix} \end{array}$$

Write down the matrices **W**, **T** and **F** which represent the team's successes on Wednesday, Thursday and Friday respectively.
Hence find a matrix to show the men's and women's placings for the whole competition.

10. A newspaper man delivered the following papers to two vendors *every* working day (that is, from Monday to Friday).

$$\begin{array}{l} \\ \\ \text{Vendor 1} \\ \text{Vendor 2} \end{array} \begin{array}{cccc} \text{Straits} & \text{Zao Bao} & \text{Berita} & \text{Business} \\ \text{Times} & & \text{Harian} & \text{Times} \\ \left(\begin{array}{cccc} 60 & 50 & 20 & 40 \\ 50 & 70 & 30 & 30 \end{array}\right) \end{array}$$

Write down a matrix **W** to represent the above information.
On Saturday and on Sunday, he delivered one-and-a-half times the number of papers he delivers on a weekday. Write down a matrix **S** to represent the number of papers delivered on Saturday.
Find the matrix 5**W** + 2**S** and explain what the elements in this matrix represent.

11. Kiam Lui is in Malaysia visiting her relatives. At a supermarket there, she noticed a packet of rice costs 8.90 *ringgit* and a packet of soap powder costs 6.50 *ringgit*. Represent this information as a column matrix **M**.
She recalled that the prices of these same items in Singapore are S\$5.20 and S\$3.20 respectively. Represent this information as a column matrix **S**.
The exchange rate is currently 1 Malaysian *ringgit* to 0.55 Singapore dollar. Find the matrix 0.55**M** – **S** and explain what the numbers in your answer represent.

12. Mrs Tan has three children. Four times during their June vacation, she gives her children the pocket money shown in matrix **M**. Matrix **L** shows how much of the pocket-money was left at the end of the vacation.

$$\mathbf{M} = \begin{array}{c} \$ \\ \begin{pmatrix} 25.00 \\ 20.00 \\ 30.00 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{Donny} \\ \text{Marie} \\ \text{Sherry} \end{array} \qquad \mathbf{L} = \begin{array}{c} \$ \\ \begin{pmatrix} 5.20 \\ 7.40 \\ 14.50 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{Donny} \\ \text{Marie} \\ \text{Sherry} \end{array}$$

(a) Write down the matrix 4**M** – **L**.
(b) Explain what the numbers given in your answer to part (a) represent.

• **13.** A bookshop has two branches.
The following table shows the stocks on 1st June.

	Hardback	Paperback
Orchard Road	260	547
Raffles Place	155	483

During the month, 180 hardbacks and 250 paperbacks were added to the stock at its Orchard Road branch while 200 hardbacks and 320 paperbacks were added to the stock at its Raffles Place branch.

On 30th June, the stocks were found to be:

	Hardback	Paperback
Orchard Road	283	388
Raffles Place	142	295

It is given that the matrix $\mathbf{B} = \begin{pmatrix} 260 & 547 \\ 155 & 483 \end{pmatrix}$ represents the stocks on 1st June.

(a) Represent the stocks on 30th June as a matrix **E**.
(b) Write down the matrix **A** to represent the stocks added during the month.
(c) Find $\mathbf{B} + \mathbf{A} - \mathbf{E}$ and explain what the numbers obtained represent.

6.3 Multiplication of Matrices

Tammy and her friends wish to throw a party for the children from a children's home. They decided to buy chocolate bars, biscuits and sweets for the children.
Two proposals are shown in Table 6.1.

	Chocolate (bars)	Biscuits (packets)	Sweets (packets)
Proposal 1	40	10	5
Proposal 2	35	15	8

Table 6.1

The prices of these items at two shops are given in Table 6.2 below.

	Price ($) at	
	Shop 1	Shop 2
Chocolate (per bar)	0.30	0.35
Biscuits (per packet)	1.45	1.50
Sweets (per packet)	1.80	1.70

Table 6.2

For each proposal, the cost of buying these items at the two shops is shown in Table 6.3 below.

	Cost ($) at	
	Shop 1	**Shop 2**
Proposal 1	$40 \times 0.30 + 10 \times 1.45 + 5 \times 1.80 = 35.50$	$40 \times 0.35 + 10 \times 1.50 + 5 \times 1.70 = 37.50$
Proposal 2	$35 \times 0.30 + 15 \times 1.45 + 8 \times 1.80 = 46.65$	$35 \times 0.35 + 15 \times 1.50 + 8 \times 1.70 = 48.35$

Table 6.3

The above result may be represented as a product of two matrices:

$$\begin{array}{l} \\ \text{Proposal 1} \\ \text{Proposal 2} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Chocolate} & \text{Biscuits} & \text{Sweets} \end{array} \\ \begin{pmatrix} 40 & 10 & 5 \\ 35 & 15 & 8 \end{pmatrix} \end{array} \begin{array}{c} \begin{array}{cc} \textbf{Shop 1} & \textbf{Shop 2} \end{array} \\ \begin{pmatrix} 0.30 & 0.35 \\ 1.45 & 1.50 \\ 1.80 & 1.70 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{Chocolate} \\ \text{Biscuits} \\ \text{Sweets} \end{array} = \begin{array}{c} \textbf{Cost} \\ \begin{array}{cc} \textbf{Shop 1} & \textbf{Shop 2} \end{array} \\ \begin{pmatrix} 35.50 & 37.50 \\ 46.65 & 48.35 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \text{Proposal 1} \\ \text{Proposal 2} \end{array}$$

Let $\mathbf{A} = \begin{pmatrix} 40 & 10 & 5 \\ 35 & 15 & 8 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0.30 & 0.35 \\ 1.45 & 1.50 \\ 1.80 & 1.70 \end{pmatrix}$ and $\mathbf{P} = \begin{pmatrix} 35.50 & 37.50 \\ 46.65 & 48.35 \end{pmatrix}$, then the above result can be written as

$$\mathbf{AB} = \mathbf{P}.$$

Let us examine how the elements in matrix $\mathbf{P}$ are obtained.
Consider the element 35.50 from matrix $\mathbf{P}$.
It is obtained as follows:

$$\begin{pmatrix} 40 & 10 & 5 \\ 35 & 15 & 8 \end{pmatrix} \begin{pmatrix} \mathbf{0.30} & 0.35 \\ \mathbf{1.45} & 1.50 \\ \mathbf{1.80} & 1.70 \end{pmatrix} = \begin{pmatrix} \mathbf{35.50} & 37.50 \\ 46.65 & 48.35 \end{pmatrix}$$

elements in the 1st row of **A**

$$35.50 = 40 \times \mathbf{0.30} + 10 \times \mathbf{1.45} + 5 \times \mathbf{1.80}$$

elements in the **1st column of B**

That is, we sum the product of each element in the 1st row of A by the 'corresponding' element in the **1st column of B**.

Similarly, to obtain the element in the 1st row, **2nd column of P** (i.e. 37.50), we sum the product of each element in the 1st row of A by the 'corresponding' element in the **2nd column of B** as follows:

$$\begin{pmatrix} 40 & 10 & 5 \\ 35 & 15 & 8 \end{pmatrix} \begin{pmatrix} 0.30 & \mathbf{0.35} \\ 1.45 & \mathbf{1.50} \\ 1.80 & \mathbf{1.70} \end{pmatrix} = \begin{pmatrix} 35.50 & \mathbf{37.50} \\ 46.65 & 48.35 \end{pmatrix}$$

i.e. $40 \times \mathbf{0.35} + 10 \times \mathbf{1.50} + 5 \times \mathbf{1.70} = 37.50$

For the element in the 2nd row, **1st column of P** (i.e. 46.65), we sum the product of each element in the 2nd row of A by the 'corresponding' element in the **1st column of B** as follows:

$$\begin{pmatrix} 40 & 10 & 5 \\ 35 & 15 & 8 \end{pmatrix} \begin{pmatrix} \mathbf{0.30} & 0.35 \\ \mathbf{1.45} & 1.50 \\ \mathbf{1.80} & 1.70 \end{pmatrix} = \begin{pmatrix} 35.50 & 37.50 \\ \mathbf{46.65} & 48.35 \end{pmatrix}$$

i.e. $35 \times \mathbf{0.30} + 15 \times \mathbf{1.45} + 8 \times \mathbf{1.80} = 46.65$

For the element of the 2nd row, **2nd column of P** (i.e. 48.35) we sum the product of each element in the 2nd row of A by the 'corresponding' element in the **2nd column of B** as follows:

$$\begin{pmatrix} 40 & 10 & 5 \\ 35 & 15 & 8 \end{pmatrix} \begin{pmatrix} 0.30 & \mathbf{0.35} \\ 1.45 & \mathbf{1.50} \\ 1.80 & \mathbf{1.70} \end{pmatrix} = \begin{pmatrix} 35.50 & 37.50 \\ 46.65 & \mathbf{48.35} \end{pmatrix}$$

i.e. $35 \times \mathbf{0.35} + 15 \times \mathbf{1.50} + 8 \times \mathbf{1.70} = 48.35$

What do you observe when you take a closer look at the labels of the rows and columns of the 3 matrices, **A**, **B** and **P**?

$$\begin{array}{l} \\ \text{Proposal 1} \\ \text{Proposal 2} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Chocolate} & \text{Biscuits} & \text{Sweets} \end{array} \\ \begin{pmatrix} 40 & 10 & 5 \\ 35 & 15 & 8 \end{pmatrix} \end{array} \begin{array}{c} \begin{array}{cc} \textbf{Shop 1} & \textbf{Shop 2} \end{array} \\ \begin{pmatrix} 0.30 & 0.35 \\ 1.45 & 1.50 \\ 1.80 & 1.70 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{Chocolate} \\ \text{Biscuits} \\ \text{Sweets} \end{array} = \begin{array}{c} \textbf{Cost} \\ \begin{array}{cc} \textbf{Shop 1} & \textbf{Shop 2} \end{array} \\ \begin{pmatrix} 35.50 & 37.50 \\ 46.65 & 48.35 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \text{Proposal 1} \\ \text{Proposal 2} \end{array}$$

$$\begin{array}{llcccc}
\text{Notice that} & & \mathbf{A} & \cdot & \mathbf{B} & = & \mathbf{P} \\
\text{Order:} & & 2 \times 3 & & 3 \times 2 & & 2 \times 2
\end{array}$$

equal

> **In general,** $\quad \mathbf{A} \cdot \mathbf{B} = \mathbf{P}$
>
> **Order:** $\quad m \times \underbrace{e \quad e}_{\textbf{equal}} \times t \qquad m \times t$

If the number of columns in **A** $\neq$ number of rows in **B**, then the product **AB** does not exist!

Let us quickly go over the procedure with another product:

$$\mathbf{MN} = \begin{pmatrix} 2 & 4 \\ 1 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 & 4 \\ 1 & 5 & 2 & 7 \end{pmatrix} = \begin{pmatrix} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

$$\begin{array}{ccccc}
\text{order} & \text{order} & & \text{product has order} \\
3 \times 2 & 2 \times 4 & \Rightarrow & 3 \times 4
\end{array}$$

equal

To get the elements in the 1st row of **MN**, we combine the elements in the 1st row of **M** with the elements in successive columns of **N** as follows:

$$\begin{pmatrix} \boxed{2 \quad 4} \\ 1 \quad 3 \\ 2 \quad -1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 & 4 \\ 1 & 5 & 2 & 7 \end{pmatrix} = \begin{pmatrix} 10 & 24 & 10 & 36 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

Step 1: $(2 \times 3) + (4 \times 1) = 10$ (1st row, 1st column of **MN**)
Step 2: $(2 \times 2) + (4 \times 5) = 24$ (1st row, 2nd column of **MN**)
Step 3: $(2 \times 1) + (4 \times 2) = 10$ (1st row, 3rd column of **MN**)
Step 4: $(2 \times 4) + (4 \times 7) = 36$ (1st row, 4th column of **MN**)

To get the elements in the 2nd row of **MN**, we compute the elements in the 2nd row of **M** with the elements in successive columns of **N** as follows:

$$\begin{pmatrix} 2 & 4 \\ 1 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 & 4 \\ 1 & 5 & 2 & 7 \end{pmatrix} = \begin{pmatrix} 10 & 24 & 10 & 36 \\ 6 & 17 & 7 & 25 \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Step 1: $(1 \times 3) + (3 \times 1) = 6$ (2nd row, 1st column of **MN**)
Step 2: $(1 \times 2) + (3 \times 5) = 17$ (2nd row, 2nd column of **MN**)
Step 3: $(1 \times 1) + (3 \times 2) = 7$ (2nd row, 3rd column of **MN**)
Step 4: $(1 \times 4) + (3 \times 7) = 25$ (2nd row, 4th column of **MN**)

Finally, to get the elements in the 3rd row of **MN**, we compute the elements in the 3rd row of **M** with the elements in successive columns of **N** as follows:

$$\begin{pmatrix} 2 & 4 \\ 1 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 & 4 \\ 1 & 5 & 2 & 7 \end{pmatrix} = \begin{pmatrix} 10 & 24 & 10 & 36 \\ 6 & 17 & 7 & 25 \\ 5 & -1 & 0 & 1 \end{pmatrix}$$

Step 1: $(2 \times 3) + (-1 \times 1) = 5$
Step 2: $(2 \times 2) + (-1 \times 5) = -1$
Step 3: $(2 \times 1) + (-1 \times 2) = 0$
Step 4: $(2 \times 4) + (-1 \times 7) = 1$

Example 8

Given that $\mathbf{A} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ -3 & 7 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 & -2 & 1 \\ 2 & 4 & -6 \end{pmatrix}$, find the following if they exist.

(a) **AB**
(b) **BA**
(c) **CA**
(d) **BC**
(e) **CB**

Solution:

(a) Note the order:

$$\begin{array}{ccc} \mathbf{A} & & \mathbf{B} \\ 2 \times \underbrace{1 & & 2}_{\text{not equal}} \times 2 \end{array}$$

AB does not exist.

(b) Note the order:

$$\begin{array}{cc} \mathbf{B} & \mathbf{A} \\ 2\times 2 & 2\times 1 \end{array}$$

equal

BA exists and has order 2×1.

$$\mathbf{BA} = \begin{pmatrix} 2 & 1 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2\times 4 + 1\times 3 \\ -3\times 4 + 7\times 3 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{11} \\ \mathbf{9} \end{pmatrix}$$

(c) Note the order:

$$\begin{array}{cc} \mathbf{C} & \mathbf{A} \\ 2\times 3 & 2\times 1 \end{array}$$

not equal

CA does not exist.

(d) Note the order:

$$\begin{array}{cc} \mathbf{B} & \mathbf{C} \\ 2\times 2 & 2\times 3 \end{array}$$

equal

BC exists and has order 2×3.

$$\mathbf{BC} = \begin{pmatrix} 2 & 1 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} 3 & -2 & 1 \\ 2 & 4 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times 3 + 1\times 2 & 2\times(-2) + 1\times 4 & 2\times 1 + 1\times(-6) \\ -3\times 3 + 7\times 2 & -3\times(-2) + 7\times 4 & -3\times 1 + 7\times(-6) \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{8} & \mathbf{0} & \mathbf{-4} \\ \mathbf{5} & \mathbf{34} & \mathbf{-45} \end{pmatrix}$$

(e) Note the order:

$$\begin{array}{cc} \mathbf{C} & \mathbf{B} \\ 2\times 3 & 2\times 2 \end{array}$$

not equal

CB does not exist.

Example 9 Given that $\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 7 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 2 & -1 \\ -2 & 4 \end{pmatrix}$, find

(a) **AB**, **(b)** **BA**, **(c)** $\mathbf{A}^2$ (i.e. **AA**), **(d)** **(AB)C**.
Is **AB** = **BA**?

Solution:

(a) $\mathbf{AB} = \begin{pmatrix} 4 & 3 \\ 7 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 4 \times 1 + 3 \times 5 & 4 \times 2 + 3 \times 3 \\ 7 \times 1 + 1 \times 5 & 7 \times 2 + 1 \times 3 \end{pmatrix}$

$$= \begin{pmatrix} 19 & 17 \\ 12 & 17 \end{pmatrix}$$

(b) $\mathbf{BA} = \begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 7 & 1 \end{pmatrix} = \begin{pmatrix} 18 & 5 \\ 41 & 18 \end{pmatrix}$

(c) $\mathbf{A}^2 = \begin{pmatrix} 4 & 3 \\ 7 & 1 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 7 & 1 \end{pmatrix} = \begin{pmatrix} 37 & 15 \\ 35 & 22 \end{pmatrix}$

(d) $(\mathbf{AB})\mathbf{C} = \begin{pmatrix} 19 & 17 \\ 12 & 17 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 49 \\ -10 & 56 \end{pmatrix}$

No, **AB** ≠ **BA**.

In general, for matrix multiplication:
(1) AB ≠ BA (not commutative)
(2) A(BC) = (AB)C (associative)

Example 10 The results of four soccer teams after each has played 8 games are summarised in the table below.

	Wins	Draws	Losses
Tigers	4	1	3
Lions	5	2	1
Leopards	2	4	2
Cheetahs	4	0	4

A team is awarded 2 points, 1 point and 0 point for each win, draw and loss respectively.

Given that $\mathbf{G} = \begin{pmatrix} 4 & 1 & 3 \\ 5 & 2 & 1 \\ 2 & 4 & 2 \\ 4 & 0 & 4 \end{pmatrix}$ and $\mathbf{P}$ is a column matrix representing the point system, find

(a) $\mathbf{P}$, (b) $\mathbf{GP}$.

Explain what the numbers in the product $\mathbf{GP}$ represent.

Solution:

(a) $\mathbf{P} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

(b) $\mathbf{GP} = \begin{pmatrix} 4 & 1 & 3 \\ 5 & 2 & 1 \\ 2 & 4 & 2 \\ 4 & 0 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 12 \\ 8 \\ 8 \end{pmatrix}$

The numbers in the product $\mathbf{GP}$ represent the scores of the teams.

Exercise 6.3

1. Where possible, simplify the following:

(a) $\begin{pmatrix} 1 & -2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix}$

(b) $\begin{pmatrix} 2 \\ -3 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 5 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 6 & -3 \end{pmatrix}\begin{pmatrix} 3 & -1 & -4 \\ 0 & 4 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 2 & -3 & 4 \\ -1 & 6 & 3 \end{pmatrix}$

(e) $\begin{pmatrix} 2 & 5 & -1 \end{pmatrix}\begin{pmatrix} 6 & 4 \\ -2 & 1 \\ 2 & 3 \end{pmatrix}$

(f) $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}\begin{pmatrix} 5 & 12 \\ 3 & 0 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & -1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 2 & -6 \\ 1 & -2 \\ 3 & -1 \end{pmatrix}$

(h) $\begin{pmatrix} 2 & -6 \\ 1 & -2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ 5 & -2 & 4 \end{pmatrix}$

(i) $\begin{pmatrix} 2 & -3 & 4 \\ -1 & 6 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

(j) $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & -3 & 4 \\ -1 & 6 & 3 \end{pmatrix}$

(k) $\begin{pmatrix} 1 & -1 & 2 \\ 3 & -2 & 4 \end{pmatrix}\begin{pmatrix} 5 & -8 \\ 0 & -2 \\ 3 & 9 \end{pmatrix}$

(l) $\begin{pmatrix} 5 & -8 \\ 0 & -2 \\ 3 & 9 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ 3 & -2 & 4 \end{pmatrix}$

2. Simplify

(a) $\begin{pmatrix} -2 & 7 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -7 \\ 1 & -2 \end{pmatrix}$,   (b) $\begin{pmatrix} 8 & 5 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ -3 & 8 \end{pmatrix}$.

(These products illustrate an interesting result which shall be dealt with in more detail in the next section.)

3. Given that $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \\ 4 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ -3 & 7 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 & -2 & 1 \\ 2 & 4 & -6 \end{pmatrix}$, find

(a) **AB**,   (b) **BC**,   (c) $\mathbf{B}^2$,
(d) **AC**,   (e) **CA**,   (f) **(CA)B**.

4. Find the value of each unknown in the following matrix equations.

(a) $(2 \quad -2 \quad 5)\begin{pmatrix} 1 \\ a \\ 3 \end{pmatrix} = (21)$ (b) $\begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} a \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ b \end{pmatrix}$

(c) $(1 \quad 3)\begin{pmatrix} p & 2 \\ -1 & q \end{pmatrix} = (7 \quad 11)$ (d) $(x \quad 3 - x)\begin{pmatrix} x \\ 2 \end{pmatrix} = (14)$

5. Simplify

(a) $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$,   (b) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

If $\mathbf{AB} = \mathbf{O}$, can we conclude that either $\mathbf{A} = \mathbf{O}$ or $\mathbf{B} = \mathbf{O}$?

6. $\mathbf{M} = \begin{pmatrix} 1 & s \\ r & 6 \end{pmatrix}$, $\mathbf{N} = \begin{pmatrix} 2 & -3 \\ 0 & 8 \end{pmatrix}$.

(a) Express $4\mathbf{M} - 3\mathbf{N}$ in terms of r and s.
(b) Find $\mathbf{N}^2$.
(c) Given that $\mathbf{NM} = 8\mathbf{M}$, find the value of r and the value of s. *(C)*

7. Given that $\mathbf{A} = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$, find

(a) **AB**,   (b) **BA**,   (c) **BAB**,   (d) **ABBA**.

8. Given that $\mathbf{A} = \begin{pmatrix} 1 & -2 & 5 \\ -1 & 4 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ -3 & 7 \end{pmatrix}$, find

(a) **BA**,   (b) $\mathbf{BA} - 2\mathbf{A}$.

If $\mathbf{B}^2\mathbf{A} = \begin{pmatrix} a & 34 & 32 \\ -73 & b & 3 \end{pmatrix}$, find the value of a and of b.

9. Given that $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, find $\mathbf{A}^2$, $\mathbf{A}^3$ and $\mathbf{A}^4$. Hence deduce $\mathbf{A}^{23}$.

10. Given that $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, find $\mathbf{A}^2$, $\mathbf{A}^3$ and $\mathbf{A}^4$. Hence deduce $\mathbf{A}^{12}$.

11. A postman delivers 20 letters and 5 parcels whose masses, in grams, are given in the second matrix. Find the total mass which he carries.

$$\begin{array}{l} \\ \text{Number delivered} \end{array} \begin{array}{c} \begin{array}{cc} \text{Letters} & \text{Parcels} \end{array} \\ \begin{pmatrix} 20 & 5 \end{pmatrix} \end{array} \begin{array}{c} \text{Mass in g} \\ \begin{pmatrix} 15 \\ 3000 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{Each letter} \\ \text{Each parcel} \end{array}$$

12. The quantity of each type of vegetables bought by two women, Mrs Lim and Mrs Ng, is shown in the first matrix. The cost, in dollars, per kg is given in the second matrix. Find how much each woman spends.

$$\begin{array}{l} \\ \text{Mrs Lim} \\ \text{Mrs Ng} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Cabbage} & \text{Beans} & \text{Peas} \end{array} \\ \begin{pmatrix} 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} \end{array} \begin{array}{c} \text{Cost per kg} \\ \begin{pmatrix} 2.40 \\ 1.10 \\ 1.35 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{Cabbage} \\ \text{Beans} \\ \text{Peas} \end{array} = \begin{array}{c} \text{Money spent} \\ \begin{pmatrix} \cdots \\ \cdots \end{pmatrix} \end{array} \begin{array}{l} \\ \text{Mrs Lim} \\ \text{Mrs Ng} \end{array}$$

13. A café sells lemonade and orangeade, each in small and large glasses. The cost of a small glass of either drink is 30 cents and the cost of a large glass is 50 cents. During a period of five minutes the following numbers of glasses of drink were sold.

	Small	**Large**
Lemonade	6	2
Orangeade	5	1

Given that $\mathbf{P} = \begin{pmatrix} 6 & 2 \\ 5 & 1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 30 \\ 50 \end{pmatrix}$,

(a) find $\mathbf{PQ}$,

(b) explain what the numbers given in your answer to part (a) represent. *(C)*

14. The following table shows an hour's output in a toy factory.

	Production line 1	**Production line 2**
Car	20	16
Lorry	25	30
Truck	15	18

Given that $\mathbf{P} = \begin{pmatrix} 20 & 16 \\ 25 & 30 \\ 15 & 18 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$ denotes the numbers of hours in a day which the two production lines operate. Compute **PQ** and give its physical interpretation.

15. Two public libraries purchased the quantities of different newspapers as follows:

$$\begin{array}{r} \\ \\ \text{Stamford} \\ \text{Tampines} \end{array} \begin{array}{ccc} \text{Straits Times} & \text{Zao Bao} & \text{Berita Harian} \\ \\ \multicolumn{3}{c}{\begin{pmatrix} 6 & \quad 4 \quad & 2 \\ 4 & \quad 4 \quad & 3 \end{pmatrix}} \end{array}$$

The prices of the Strait Times, Zao Bao and Berita Harian are 60¢, 65¢ and 50¢ respectively.

Write a 2×3 matrix **Q** to represent the quantities of different newspapers bought.

Given further that $\mathbf{C} = \begin{pmatrix} 60 \\ 65 \\ 50 \end{pmatrix}$, find **QC** and interpret your answer.

16. The prices of three types of light bulbs are given below.

50 watts – \$1.20 100 watts – \$2.00 150 watts – \$2.35

(a) Represent the above information as a column matrix **C**.

A factory has a workshop and offices. The number of each type of bulbs used are given in the table below.

	50 W	100 W	150 W
Workshop	12	40	25
Offices	3	30	10

(b) Represent the information contained in the table as a matrix **N**.

(c) Find the product **NC** and explain what the number(s) obtained represents.

6.4 Determinant and Inverse of a 2 x 2 Matrix

When the number of rows equals the number of columns, the matrix is called a **square matrix**. For instance, $\begin{pmatrix} 3 & 1 \\ -4 & 1 \end{pmatrix}$ and $\begin{pmatrix} 10 & 5 & -9 \\ 3 & 6 & -1 \\ 2 & 4 & -6 \end{pmatrix}$ are square matrices.

In this section, we shall consider 2×2 square matrices. The 2×2 null matrix is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Example 11 Given that $\mathbf{A} = \begin{pmatrix} 4 & 3 \\ -1 & 2 \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, find

(a) $\mathbf{AI}$, **(b)** $\mathbf{IA}$.

Solution:

(a)
$$\mathbf{AI} = \begin{pmatrix} 4 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 4 \times 1 + 3 \times 0 & 4 \times 0 + 3 \times 1 \\ -1 \times 1 + 2 \times 0 & -1 \times 0 + 2 \times 1 \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{4} & \mathbf{3} \\ \mathbf{-1} & \mathbf{2} \end{pmatrix}$$

(b)
$$\mathbf{IA} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ -1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \times 4 + 0 \times (-1) & 1 \times 3 + 0 \times 2 \\ 0 \times 4 + 1 \times (-1) & 0 \times 3 + 1 \times 2 \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{4} & \mathbf{3} \\ \mathbf{-1} & \mathbf{2} \end{pmatrix}$$

The matrix $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the **identity matrix**. In this matrix, the elements along the **principal diagonal** (shaded) are all 1's while the other elements are all 0's. The above example illustrates an interesting and important property:

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}$$

Example 12 Given that $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, where $ad - bc \neq 0$, find

(a) **MN**, **(b)** **NM**.

Hence deduce the matrix **P** such that **PM** = **I**, where **I** is the 2×2 identity matrix.

Solution:

(a) $$\mathbf{MN} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & -ab + ba \\ cd - dc & -cb + da \end{pmatrix}$$

$$= \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}$$

(b) $$\mathbf{NM} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} da - bc & db - bd \\ -ca + ac & -cb + ad \end{pmatrix}$$

$$= \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}$$

From our result in (b),

$$\mathbf{NM} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}$$

$$= (ad - bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \frac{1}{ad - bc}\mathbf{NM} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Comparing this with **PM** = **I**, we have $\mathbf{P} = \dfrac{1}{ad - bc}\mathbf{N}$

$$= \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

In Example 12, $\mathbf{PM} = \mathbf{I}$. It can also be shown that $\mathbf{MP} = \mathbf{I}$. We call $\mathbf{P}$ the **(multiplicative) inverse** of $\mathbf{M}$ which is usually denoted by $\mathbf{M}^{-1}$. Hence we have:

(1) $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ **has inverse** $\mathbf{M}^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ **provided** $ad - bc \neq 0$.

(2) $\mathbf{MM}^{-1} = \mathbf{I}$ **and** $\mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$

The expression $ad - bc$ is called the **determinant** of the matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and is denoted by det $\mathbf{M}$ or $|\mathbf{M}|$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

The determinant of $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ **is given by** $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

In summary, to get the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we do as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{\text{Interchange the 2 elements in the principal diagonal}} \begin{pmatrix} d & b \\ c & a \end{pmatrix} \xrightarrow{\text{Next, change the signs of the remaining 2 elements}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \xrightarrow{\text{Then, multiply by } \frac{1}{\text{determinant}}} \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example 13 For each of the following matrices, find its determinant and inverse, if it exists.

(a) $\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 2 & 2 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 3 & -2 \\ 1 & -4 \end{pmatrix}$ (c) $\mathbf{C} = \begin{pmatrix} -2 & 3 \\ -4 & 6 \end{pmatrix}$

Solution:

(a) $\det \mathbf{A} = \begin{vmatrix} 4 & 3 \\ 2 & 2 \end{vmatrix} = 4(2) - 3(2) = \mathbf{2}$

Since the determinant, $ad - bc \neq 0$, $\mathbf{A}^{-1}$ exists.

$$\mathbf{A}^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -3 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & -\frac{\mathbf{3}}{\mathbf{2}} \\ \mathbf{-1} & \mathbf{2} \end{pmatrix}$$

(b) $\det \mathbf{B} = \begin{vmatrix} 3 & -2 \\ 1 & -4 \end{vmatrix} = 3(-4) - (-2)1 = \mathbf{-10}$

Since $\det \mathbf{B} \neq 0$, $\mathbf{B}^{-1}$ exists.

$$\mathbf{B}^{-1} = \frac{1}{-10}\begin{pmatrix} -4 & 2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} \mathbf{0.4} & \mathbf{-0.2} \\ \mathbf{0.1} & \mathbf{-0.3} \end{pmatrix}$$

(c) $\det \mathbf{C} = \begin{vmatrix} -2 & 3 \\ -4 & 6 \end{vmatrix} = -2(6) - 3(-4) = \mathbf{0}$

Since $\det \mathbf{C} = 0$, $\mathbf{C}^{-1}$ does not exist.

Note: A matrix whose determinant is zero is said to be **singular** and has **no inverse**. For **non-singular** matrices, their determinants are not zero and they have inverses. Hence the matrix **C** above is singular and has no inverse while matrices **A** and **B** are non-singular and have inverses.

Recall the following properties of the inverse matrix and identity matrix:

$$\mathbf{MM}^{-1} = \mathbf{I} \quad \text{and} \quad \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$$
$$\mathbf{IA} = \mathbf{A} \quad \text{and} \quad \mathbf{AI} = \mathbf{A}$$

Consider the matrix equation $\mathbf{MX} = \mathbf{N}$, where the non-singular matrix $\mathbf{M} = \begin{pmatrix} 2 & -3 \\ 4 & 3 \end{pmatrix}$, $\mathbf{N} = \begin{pmatrix} -4 & 14 \\ 10 & 10 \end{pmatrix}$ and $\mathbf{X}$ is an unknown matrix. Take note where the above properties are used in the solution of the equation.

$$\mathbf{MX} = \mathbf{N}$$

We *pre*-multiply both sides by the inverse $\mathbf{M}^{-1}$:

$$\begin{aligned}
\mathbf{M}^{-1}(\mathbf{MX}) &= \mathbf{M}^{-1}\mathbf{N} \\
(\mathbf{M}^{-1}\mathbf{M})\mathbf{X} &= \mathbf{M}^{-1}\mathbf{N} \\
\mathbf{IX} &= \mathbf{M}^{-1}\mathbf{N} \\
\mathbf{X} &= \mathbf{M}^{-1}\mathbf{N} \\
&= \frac{1}{18}\begin{pmatrix} 3 & 3 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} -4 & 14 \\ 10 & 10 \end{pmatrix} \\
&= \frac{1}{18}\begin{pmatrix} 18 & 72 \\ 36 & -36 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 4 \\ 2 & -2 \end{pmatrix}
\end{aligned}$$

Again, if the equation is

$$\mathbf{XM} = \mathbf{N},$$

we *post*-multiply both sides by the inverse $\mathbf{M}^{-1}$,

$$\begin{aligned}
(\mathbf{XM})\mathbf{M}^{-1} &= \mathbf{NM}^{-1} \\
\mathbf{X}(\mathbf{MM}^{-1}) &= \mathbf{NM}^{-1} \\
\mathbf{XI} &= \mathbf{NM}^{-1} \\
\mathbf{X} &= \mathbf{NM}^{-1} \\
&= \begin{pmatrix} -4 & 14 \\ 10 & 10 \end{pmatrix}\frac{1}{18}\begin{pmatrix} 3 & 3 \\ -4 & 2 \end{pmatrix} \\
&= \frac{1}{18}\begin{pmatrix} -68 & 16 \\ -10 & 50 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{34}{9} & \frac{8}{9} \\ -\frac{5}{9} & \frac{25}{9} \end{pmatrix}
\end{aligned}$$

Example 14 Find the values of p and q in the matrix equation

$$\begin{pmatrix} 3 & 4 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 17 \\ -7 \end{pmatrix}.$$

Solution: Let $A = \begin{pmatrix} 3 & 4 \\ 2 & -1 \end{pmatrix}$. Then $A^{-1} = -\frac{1}{11}\begin{pmatrix} -1 & -4 \\ -2 & 3 \end{pmatrix}$

$$\begin{pmatrix} 3 & 4 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 17 \\ -7 \end{pmatrix}$$

We *pre*-multiply both sides by the inverse $\mathbf{A}^{-1}$:

$$-\frac{1}{11}\begin{pmatrix}-1 & -4\\ -2 & 3\end{pmatrix}\begin{pmatrix}3 & 4\\ 2 & -1\end{pmatrix}\begin{pmatrix}p\\ q\end{pmatrix} = -\frac{1}{11}\begin{pmatrix}-1 & -4\\ -2 & 3\end{pmatrix}\begin{pmatrix}17\\ -7\end{pmatrix}$$

$$\mathbf{I}\begin{pmatrix}p\\ q\end{pmatrix} = -\frac{1}{11}\begin{pmatrix}11\\ -55\end{pmatrix}$$

$$\begin{pmatrix}p\\ q\end{pmatrix} = \begin{pmatrix}-1\\ 5\end{pmatrix}$$

$$\Rightarrow \quad p = -1, \quad q = 5$$

Exercise 6.4

1. Evaluate the following determinants.

(a) $\begin{vmatrix}5 & 2\\ -3 & 1\end{vmatrix}$ **(b)** $\begin{vmatrix}-1 & 4\\ 2 & 8\end{vmatrix}$ **(c)** $\begin{vmatrix}3 & 4\\ -1 & 2\end{vmatrix}$

(d) $\begin{vmatrix}10 & -7\\ -5 & -2\end{vmatrix}$ **(e)** $\begin{vmatrix}-12 & -4\\ 7 & -2\end{vmatrix}$ **(f)** $\begin{vmatrix}0.9 & 0.8\\ -5 & -10\end{vmatrix}$

(g) $\begin{vmatrix}2 & \pi\\ -2 & -1\end{vmatrix}$ **(h)** $\begin{vmatrix}x+2 & x^2\\ 1 & x-2\end{vmatrix}$

2. Determine which of the following matrices are non-singular and find their inverses.

(a) $\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$ **(b)** $\begin{pmatrix}6 & -1\\ -5 & 1\end{pmatrix}$ **(c)** $\begin{pmatrix}-12 & 9\\ 4 & -3\end{pmatrix}$

(d) $\begin{pmatrix}4 & -3\\ 10 & -5\end{pmatrix}$ **(e)** $\begin{pmatrix}6 & 3\\ 10 & -5\end{pmatrix}$ **(f)** $\begin{pmatrix}-5 & 10\\ -3 & 6\end{pmatrix}$

(g) $\begin{pmatrix}0 & 2\\ -5 & 10\end{pmatrix}$ **(h)** $\begin{pmatrix}-2 & 1\\ 6 & -4\end{pmatrix}$ **(i)** $\begin{pmatrix}3 & 2\\ -2 & 3\end{pmatrix}$

(j) $\begin{pmatrix}-2 & 4\\ -1 & 2\end{pmatrix}$

3. The determinant of $\begin{pmatrix}2a & 3\\ -2 & -2\end{pmatrix}$ is –6. Find the value of a and write down the inverse of the matrix.

4. If the matrix $\begin{pmatrix} 3 & a \\ 2 & -4 \end{pmatrix}$ is singular, find the value of a.

5. Given that $\mathbf{S} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, find $\mathbf{S}^2$ and **deduce** the inverse of $\mathbf{S}$.

6. The matrix $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix}$. Calculate

   (a) the value of the determinant of $\mathbf{A}$,
   (b) the inverse of $\mathbf{A}$,
   (c) $\mathbf{A}\mathbf{A}^{-1}\begin{pmatrix} 4 \\ 2 \end{pmatrix}$. *(C)*

7. $\mathbf{A} = \begin{pmatrix} -2 & 2 \\ -3 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 & -5 \\ 2 & h \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -2 & 1 \\ k & 1 \end{pmatrix}$.

   (a) If $\mathbf{AC} = \mathbf{I}$, find the value of k.
   (b) If $|\mathbf{B}| = 2|\mathbf{A}|$, find the value of h.
   (c) Find the inverse of $\mathbf{A}^2$.

8. $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 0 \\ -2 & 4 \end{pmatrix}$.

   (a) Evaluate $3\mathbf{A} - 2\mathbf{B}$.
   (b) Find $\mathbf{A}^{-1}$, the inverse of $\mathbf{A}$.
   (c) Given that $\mathbf{AX} = \mathbf{B}$, find the matrix $\mathbf{X}$. *(C)*

9. Find the matrix $\mathbf{X}$ in the following equations.

   (a) $\begin{pmatrix} 2 & 3 \\ -1 & 6 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 0 \\ -15 \end{pmatrix}$ **(b)** $\begin{pmatrix} 5 & -2 \\ -3 & 2 \end{pmatrix}\mathbf{X} = \begin{pmatrix} -11 \\ 9 \end{pmatrix}$

   (c) $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\mathbf{X} = 2\begin{pmatrix} 3 & 1 \\ -1 & -1 \end{pmatrix}$

10. Find the values of the unknowns in the following equations.

   (a) $\begin{pmatrix} 10 & 3 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} a \\ -1 \end{pmatrix} = \begin{pmatrix} 15 \\ b \end{pmatrix}$

   (c) $\begin{pmatrix} -1 & a \\ b & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 3 & 0 \end{pmatrix}$

11. In the equation $\mathbf{M}\begin{pmatrix}1 & 0\\2 & 3\end{pmatrix} = 3\begin{pmatrix}-1 & -2\\0 & -1\\2 & 3\end{pmatrix}$, state the order of the matrix $\mathbf{M}$.

Find the inverse of $\begin{pmatrix}1 & 0\\2 & 3\end{pmatrix}$ and use it to solve the equation for the matrix $\mathbf{M}$.

12. $\mathbf{A} = \begin{pmatrix}-2 & -1\\6 & 2\end{pmatrix}$. Find the inverse of $\mathbf{A}$ and use it to find the matrices $\mathbf{P}$ and $\mathbf{Q}$ in the equation

(a) $\mathbf{AP} = \begin{pmatrix}-1 & -7\\4 & 18\end{pmatrix}$.

(b) $\mathbf{QA} = \begin{pmatrix}0 & -2\\12 & 5\end{pmatrix} - 2\mathbf{A}$.

6.5 Solving Simultaneous Equations by a Matrix Method

The matrix equation

$$\begin{pmatrix}2 & -2\\4 & -3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}6\\-5\end{pmatrix} \Rightarrow \begin{pmatrix}2x-2y\\4x-3y\end{pmatrix} = \begin{pmatrix}6\\-5\end{pmatrix}$$

Equating the elements, we have the pair of simultaneous equations:

$$\begin{aligned}2x - 2y &= 6\\4x - 3y &= -5\end{aligned}$$

Conversely, we could express the simultaneous equations

$$\begin{aligned}3x - 4y &= -5\\-2x + 5y &= 8\end{aligned}$$

as a matrix equation

$$\begin{pmatrix}3 & -4\\-2 & 5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-5\\8\end{pmatrix}.$$

The matrix $\begin{pmatrix}3 & -4\\-2 & 5\end{pmatrix}$ is called the **coefficient matrix** as its elements are the coefficients.

The inverse of $\begin{pmatrix}3 & -4\\-2 & 5\end{pmatrix}$ is $\frac{1}{7}\begin{pmatrix}5 & 4\\2 & 3\end{pmatrix}$.

Pre-multiplying both sides by this inverse, we get:

$$\frac{1}{7}\begin{pmatrix}5 & 4\\2 & 3\end{pmatrix}\begin{pmatrix}3 & -4\\-2 & 5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{7}\begin{pmatrix}5 & 4\\2 & 3\end{pmatrix}\begin{pmatrix}-5\\8\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{7}\begin{pmatrix}7\\14\end{pmatrix}$$

$$= \begin{pmatrix}1\\2\end{pmatrix}$$

$$\Rightarrow \quad x = 1, y = 2$$

Example 15 Use a matrix method to solve the simultaneous equations:

$$3x + 2y = 8$$
$$x - y = 6$$

Solution: The simultaneous equations

$$3x + 2y = 8,$$
$$1x - 1y = 6,$$

can be written as the matrix equation

$$\begin{pmatrix}3 & 2\\1 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}8\\6\end{pmatrix}.$$

Pre-multiplying both sides by the inverse of $\begin{pmatrix}3 & 2\\1 & -1\end{pmatrix}$, we get

$$-\frac{1}{5}\begin{pmatrix}-1 & -2\\-1 & 3\end{pmatrix}\begin{pmatrix}3 & 2\\1 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = -\frac{1}{5}\begin{pmatrix}-1 & -2\\-1 & 3\end{pmatrix}\begin{pmatrix}8\\6\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix} = -\frac{1}{5}\begin{pmatrix}-20\\10\end{pmatrix}$$

$$= \begin{pmatrix}4\\-2\end{pmatrix}$$

$$\Rightarrow \quad x = 4 \text{ and } y = -2$$

The simultaneous equations

$$3x - 2y = 5,$$
$$6x - 4y = 8,$$

yield two parallel lines. The lines do not intersect and hence the pair of equations has no solution.

On the other hand, the simultaneous equations

$$3x - 2y = 5,$$
$$6x - 4y = 10,$$

yield the same line and hence the pair of equations has infinitely many solutions.

In matrix form, these pairs of equations are respectively

$$\begin{pmatrix} 3 & -2 \\ 6 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix} \text{ and } \begin{pmatrix} 3 & -2 \\ 6 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}.$$

Observe that $\begin{vmatrix} 3 & -2 \\ 6 & -4 \end{vmatrix} = 3(-4) - (-2)6 = 0$ and so the coefficient matrix $\begin{pmatrix} 3 & -2 \\ 6 & -4 \end{pmatrix}$ is singular and has no inverse.

Thus, we see that when the coefficient matrix is singular, the pair of simultaneous equations has no unique solution.

Exercise 6.5

1. For each of the following pairs of simultaneous equations, find the coefficient matrix and state whether the equations have a unique solution. Find the unique solution if it exists.

(a) $4x + 2y = 3$
$12x + 6y = 9$

(b) $2x + 3y = 13$
$3x - y = 14$

(c) $4x + 2y = 2$
$7x + 3y = 4$

(d) $4x - 5y = 13$
$-8x + 10y = 18$

2. Use a matrix method to solve the following pairs of simultaneous equations.

(a) $x - 2y = 7$
$4x + 3y = 6$

(b) $5x + y = -8$
$-x + 2y = 17$

(c) $3x + 7y = -6$
$4x - 2y = 26$

(d) $2x - y = 7$
$3x + y = 8$

(e) $3x + 2y = 29$
$2x - y = 10$

(f) $6x - 11y = -28$
$-3x + 5y = 13$

Important Notes

1. Order (Dimension) of a matrix

A matrix that has m rows and n columns is said to be of order $m \times n$.

For example, the row matrix $(3 \quad -2 \quad 1)$ has order 1×3,

while the column matrix $\begin{pmatrix} -4 \\ 7 \end{pmatrix}$ has order 2×1.

2. Equality

Two matrices are equal if they are of the same order and if their corresponding elements are equal.

3. **Addition**

To add two matrices, we add their corresponding elements.

For example, $\begin{pmatrix} 2 & 4 & 3 \\ 1 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 5 & 6 \\ 3 & 10 & 2 \end{pmatrix} = \begin{pmatrix} 2+1 & 4+5 & 3+6 \\ 1+3 & -1+10 & 0+2 \end{pmatrix}$.

4. **Subtraction**

To subtract two matrices, we subtract their corresponding elements.

For example, $(4 \quad 8 \quad -3) - (3 \quad 1 \quad 4) = (4-3 \quad 8-1 \quad -3-4)$.

5. **Scalar multiplication**

To multiply a matrix by k, we multiply each element by k.

For example, $k\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 2k & 4k \\ 3k & -k \end{pmatrix}$.

6. **Matrix multiplication**

(a)

	A	.	**B**	=	**P**
Order :	$m \times e$		$e \times t$		$m \times t$

For example, $\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 2 & 4 & 3 \\ 1 & -1 & 0 \end{pmatrix}$ exists and is of order 2×3

whereas the product $\begin{pmatrix} 2 & 4 & 3 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$ does not exist.

(b) To find the element in the ith row, jth column of **AB**, we sum the product of each element in the ith row of **A** by the corresponding element in the jth column of **B**. For example, to find the element in the 2nd row, 3rd column of **AB**, we do as follows:

$$\begin{pmatrix} \cdots & \cdots & \cdots \\ 2 & 4 & 3 \\ \cdots & \cdots & \cdots \end{pmatrix}\begin{pmatrix} \cdots & \cdots & 5 & \cdots \\ \cdots & \cdots & 8 & \cdots \\ \cdots & \cdots & 7 & \cdots \end{pmatrix} = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 2\times5+4\times8+3\times7 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

7. **2×2 Matrices**

(a) For the identity matrix $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{IA} = \mathbf{A}$ and $\mathbf{AI} = \mathbf{A}$.

(b) For $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its determinant, $\det \mathbf{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

(c) If **A** is non-singular, i.e. $\det \mathbf{A} \neq 0$, then the inverse of **A** exists and

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

8. Solving simultaneous linear equations by a matrix method

$$\begin{aligned} ax + by &= h \\ cx + dy &= k \end{aligned} \quad \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} h \\ k \end{pmatrix}$$

If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, then $\begin{pmatrix} x \\ y \end{pmatrix} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix}$.

If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$, the equations have no unique solution.

Miscellaneous Examples

Example 16 The labour and material costs for manufacturing two guitar models are given in the table below.

	Model A	Model B
Labour cost	\$35	\$40
Material cost	\$20	\$30

A total of \$2500 a week is allowed for labour and material costs. In three forthcoming weeks, \$1450, \$1500 and \$1550 of the weekly allocations will go to labour. Find a 2×3 matrix $\mathbf{C}$ to represent the weekly allocations for the costs of labour and material.

It is given that $\mathbf{P} = \begin{pmatrix} 35 & 40 \\ 20 & 30 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$.

(a) Write down an equation connecting **P**, **Q** and **C**.
(b) Find $\mathbf{P}^{-1}$.
(c) Calculate $\mathbf{P}^{-1}\mathbf{C}$.
(d) Explain the significance of your answer to part (c).

Solution: The allocation for material = \$2500 – the allocation for labour and so we have:

	Allocation (\$)		
	Week 1	Week 2	Week 3
Labour	1450	1500	1550
Material	1050	1000	950

Hence $\mathbf{C} = \begin{pmatrix} 1450 & 1500 & 1550 \\ 1050 & 1000 & 950 \end{pmatrix}$.

(a) $\mathbf{PQ} = \mathbf{C}$.

(b) $|\mathbf{P}| = \begin{vmatrix} 35 & 40 \\ 20 & 30 \end{vmatrix} = 35(30) - 40(20) = 250$

$$\text{So, } \mathbf{P}^{-1} = \frac{1}{250}\begin{pmatrix} 30 & -40 \\ -20 & 35 \end{pmatrix}$$

$$= \begin{pmatrix} 0.12 & -0.16 \\ -0.08 & 0.14 \end{pmatrix}$$

(c) $$\mathbf{P}^{-1}\mathbf{C} = \begin{pmatrix} 0.12 & -0.16 \\ -0.08 & 0.14 \end{pmatrix}\begin{pmatrix} 1450 & 1500 & 1550 \\ 1050 & 1000 & 950 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 20 & 34 \\ 31 & 20 & 9 \end{pmatrix}$$

(d) From (a), $\mathbf{PQ} = \mathbf{C} \quad \Rightarrow \quad \mathbf{P}^{-1}\mathbf{P}\,\mathbf{Q} = \mathbf{P}^{-1}\mathbf{C}$

$\Rightarrow \quad \mathbf{Q} = \mathbf{P}^{-1}\mathbf{C}$

Hence the matrix in part (c) gives the quantities of Model *A* and Model *B* to be produced in each of the three weeks.

Miscellaneous Exercise 6

1. Express each of the following as a single matrix.

(a) $3\begin{pmatrix} -1 \\ 4 \end{pmatrix} + \frac{5}{2}\begin{pmatrix} 4 \\ -2 \end{pmatrix} - 2\begin{pmatrix} -2 \\ 5 \end{pmatrix}$

(b) $\begin{pmatrix} 4 & -8 \end{pmatrix}\begin{pmatrix} 2 & -4 & 3 \\ 1 & -3 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 12 & 4 & -2 \\ 2 & -3 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 7 & 4 \\ 6 & 5 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & -2 & 4 \end{pmatrix}\begin{pmatrix} 20 \\ 5 \\ -6 \end{pmatrix} + (15)$

2. Given that $\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} q & -7 \\ p & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ 6 & 3r \end{pmatrix}$, find the value of p, of q and of r. *(C)*

3. Given that $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ -3 & -1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 7 & -2 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 2 & 1 \\ -4 & 8 \end{pmatrix}$, find

   (a) $2\mathbf{A} + 3\mathbf{B}$, **(b)** $\mathbf{C}^2 - 4\mathbf{A}$,
   (c) $\mathbf{A}^{-1}$, the inverse of $\mathbf{A}$, **(d)** $\mathbf{A}\mathbf{A}^{-1}\mathbf{B}$.

4. $\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 0 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} \frac{1}{4} & k \\ 0 & \frac{1}{3} \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 12 & 4 \\ -9 & m \end{pmatrix}$.

   (a) Evaluate $\mathbf{A}^2$.
   (b) Find the value of k which makes $\mathbf{AB}$ the identity matrix.
   (c) Find the value of m which makes the determinant of $\mathbf{A}$ equal to the determinant of $\mathbf{C}$. *(C)*

5. **(a)** Find $\mathbf{B}$ if $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ and $\mathbf{BA} = \begin{pmatrix} 5 & 5 \\ 4 & 18 \end{pmatrix}$.

   (b) Given that $\mathbf{A} = \begin{pmatrix} 7 & 7 \\ -3 & -3 \end{pmatrix}$ and $\mathbf{A}^2 = k\mathbf{A}$, find the value of k.

6. $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ -4 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 6 & h \\ -3 & k \end{pmatrix}$.

   (a) If det $\mathbf{C}$ = det $\mathbf{A}$ + 2(det $\mathbf{B}$), find h in terms of k.
   (b) Is it possible for $\mathbf{C} - 2(-3\mathbf{A})$ to be equal to $4(\mathbf{B} - \mathbf{C})$? Give your reason.
   (c) Find $\mathbf{X}$ if $\mathbf{XA} = \mathbf{B}$.

7. **(a)** Express as a single matrix

   (i) $2\begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix} - \begin{pmatrix} -5 & 2 \\ 0 & 3 \end{pmatrix}$, **(ii)** $\begin{pmatrix} 1 & 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 4 \\ 3 & 1 \\ 5 & 0 \end{pmatrix}$.

   (b) The determinant of the matrix $\begin{pmatrix} w & 2w + 5 \\ -1 & w + 1 \end{pmatrix}$ is 15.

   (i) Form an equation in w and show that it reduces to
   $$w^2 + 3w - 10 = 0.$$
   (ii) Solve this equation. *(C)*

8. Given that $\mathbf{S} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, show that $\mathbf{S}$ is self-inverse (i.e. $\mathbf{S}^{-1} = \mathbf{S}$). Hence deduce $\mathbf{S}^{20}$.

9. Given that $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -5 & -4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$, find

(a) $\mathbf{AB}$ and $(\mathbf{AB})^{-1}$,
Is $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$?

(b) $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$ and $\mathbf{B}^{-1}\mathbf{A}^{-1}$.

10. Use a matrix method to solve the following pairs of simultaneous equations.

(a) $3x + 7y = 23$
$2x + 4y = 14$

(b) $2x + 3y = 11$
$x - y = 3$

(c) $2x - y = -8$
$x + 2y = 11$

(d) $10x + 33y = 14$
$5x + 16y = 8$

11. Find the inverse of the matrix $\begin{pmatrix} 7 & -3 \\ 3 & 2 \end{pmatrix}$. Hence solve the simultaneous equations

$$7x - 3y = -29,$$
$$3x + 2y = 4.$$

12. Write down $\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}^{-1}$. Hence solve the simultaneous equations

$$2x + 3y = 13,$$
$$x - y = 4.$$

13. One of your friends spent \$7.70 on some 22-cent and 40-cent stamps. If he bought 26 stamps in all, use a matrix method to find how many of each type he bought.

14. The amount of vegetables, in kg, which one lorry can hold is given in matrix **L** and the amount, in kg, that a van can hold is given in matrix **V**.
A shop has a delivery from 2 lorries and 3 vans.

$$\mathbf{L} = \begin{array}{c} \begin{matrix} \text{sacks} & \text{boxes} \end{matrix} \\ \begin{pmatrix} 23 & 7 \\ 30 & 10 \\ 25 & 8 \end{pmatrix} \end{array} \begin{matrix} \text{onions} \\ \text{potatoes} \\ \text{carrots} \end{matrix} \qquad \mathbf{V} = \begin{array}{c} \begin{matrix} \text{sacks} & \text{boxes} \end{matrix} \\ \begin{pmatrix} 5 & 6 \\ 4 & 3 \\ 5 & 2 \end{pmatrix} \end{array} \begin{matrix} \text{onions} \\ \text{potatoes} \\ \text{carrots} \end{matrix}$$

Write the matrix $2\mathbf{L} + 3\mathbf{V}$ and give its physical significance.

15. The price of tickets in each category at a cinema is given below.
Child – \$3; Adult – \$8; Senior citizen – \$5.

(a) Represent the above information as a column matrix **C**.
The number of tickets sold on one weekend is summarised in the table below.

	Child	Adult	Senior citizen
Saturday	205	160	70
Sunday	310	200	65

(b) Represent the information contained in the table as a matrix **N**.

(c) Find the product **NC**, and explain what the number(s) obtained represents.

16. A store sells large and small tins of both yellow and blue paint. The selling price of a large tin of either colour is $\$l$ and of a small tin of either colour is $\$s$. The number of tins of each type sold one day is given in the following table.

	Large	Small
Yellow	5	3
Blue	6	4

The total income from the sale of the yellow paint was \$84 and that from the blue paint was \$104.

It is given that $\mathbf{A} = \begin{pmatrix} 5 & 3 \\ 6 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} l \\ s \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 84 \\ 104 \end{pmatrix}$.

(a) Write down an equation connecting **A**, **B** and **C**.
(b) Find $\mathbf{A}^{-1}$.
(c) Calculate $\mathbf{A}^{-1}\mathbf{C}$.
(d) Explain the significance of your answer to part (c). *(C)*

17. The two outlets of a furniture company sold the following items in October.

	Table	Chair
Outlet 1	23	64
Outlet 2	17	52

It cost the company \$65 to make a table and \$5 to make a chair. Each table was sold for \$160 and each chair for \$12.

Given that $\mathbf{Q} = \begin{pmatrix} 23 & 64 \\ 17 & 52 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 65 \\ 5 \end{pmatrix}$ and $\mathbf{S} = \begin{pmatrix} 160 \\ 12 \end{pmatrix}$,

(a) find **QC** and interpret the results,
(b) find $\mathbf{S} - \mathbf{C}$ and interpret the results,
(c) find $\mathbf{Q}(\mathbf{S} - \mathbf{C})$ and interpret the results.

18. A business makes toy buses and toy lorries.
The following table is used in calculating the cost of manufacturing each toy.

	Labour (Hours)	Wood (Blocks)	Paint (tins)
Bus	6	4	3
Lorry	3	4	2

Labour costs \$8 per hour, wood costs \$1 per block and paint costs \$2 per tin.

It is given that $\mathbf{A} = \begin{pmatrix} 6 & 4 & 3 \\ 3 & 4 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 8 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{C} = \mathbf{AB}$.

(a) **(i)** Evaluate **C**.
(ii) Explain what the numbers in your answer represent.

(b) In addition, $\mathbf{D} = (100 \quad 200)$.
(i) Evaluate **DC**.
(ii) Explain what your answer represents. *(C)*

19. In order to control a certain type of crop disease, it is necessary to use 29 gallons of chemical A and 20 gallons of chemical B. The dealer can order commercial spray I, each container of which holds 5 gallons of chemical A and 2 gallons of chemical B, and commercial spray II, each container of which holds 3 gallons of chemical A and 4 gallons of chemical B.

Given that $\mathbf{S} = \begin{pmatrix} 5 & 3 \\ 2 & 4 \end{pmatrix}$, $\mathbf{U} = \begin{pmatrix} 29 \\ 20 \end{pmatrix}$ and $\mathbf{SX} = \mathbf{U}$, evaluate **X** and state what information the matrix **X** gives.

20. A biologist in a nutrition experiment wants to prepare a special diet for her experimental animals. She requires a food mixture that contains, among other things, 29 ounces of protein and 6 ounces of fat. Food mixes are available with the compositions shown in the table. How many ounces of each mix should be used to prepare the diet mix?

Mix	Protein (%)	Fat (%)
A	20	2
B	10	6

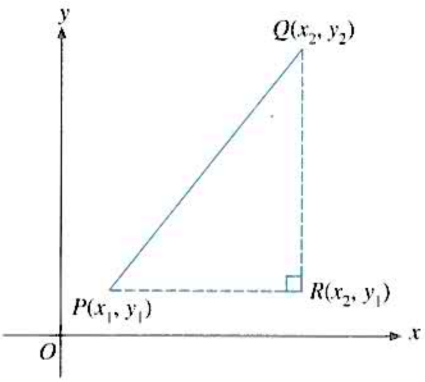
21. A company manufactures three styles of kitchen cabinets and each style comes in two grades. Style 1 requires 4 square units of plywood and 14 hours to build. Style 2 requires 5 square units of plywood and 10 hours to build. Style 3 requires 3 square units of plywood and 8 hours to build. For grade A cabinets, the wood costs \$26 per square unit and labour costs \$35 per hour. For grade B, with cheaper material and less experienced cabinetmakers, the cost drops to \$20 per square unit and \$31 per hour. Display these figures in two matrices in such a way that their product shows the cost of materials and time required for each grade and style of cabinet. Calculate their product.

7 Coordinate Geometry

Coordinate geometry is the study of points, straight lines and curves defined by algebraic expressions. In this chapter, we shall study the basic knowledge of distance between two points, midpoint of two points, straight lines with equations of the form $y = mx + c$, parallel lines and perpendicular lines.

7.1 Distance between Two Points

Three points $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_2, y_1)$ are on the cartesian plane with x-axis (the horizontal axis) and y-axis (the vertical axis) as shown in the diagram. Since $x_1 < x_2$, $PR = x_2 - x_1$. Similarly, $y_1 < y_2$ and $RQ = y_2 - y_1$.



By Pythagoras' Theorem,

$$PQ^2 = PR^2 + RQ^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 1 The vertices of a triangle are $A(-2, 0)$, $B(5, 0)$ and $C(1, 4)$. Find
(a) the length AB,
(b) the area of the triangle ABC.

Solution:

(a) $AB = 5 - (-2) =$ **7 units**

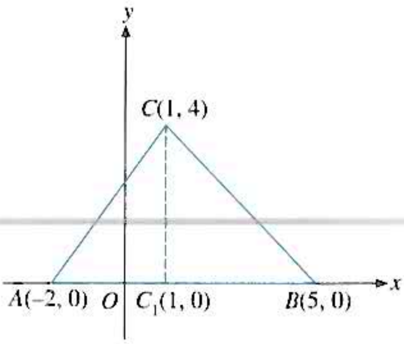
(b) $C_1C = 4 - 0 = 4$

Area of $\triangle ABC$

$= \frac{1}{2} \times AB \times C_1C$

$= \frac{1}{2} \times 7 \times 4$

$=$ **14 sq. units**

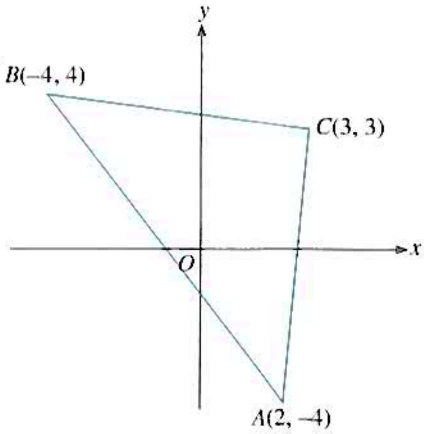


Example 2

A triangle has vertices at $A(2, -4)$, $B(-4, 4)$ and $C(3, 3)$. Calculate the length of AC and of BC and show that the triangle is an isosceles triangle.

Solution:

$$AC = \sqrt{(3-2)^2 + (3-(-4))^2}$$
$$= \sqrt{50}$$
$$= \mathbf{5\sqrt{2}\ units}$$



$$BC = \sqrt{(3-(-4))^2 + (3-4)^2}$$
$$= \sqrt{50}$$
$$= \mathbf{5\sqrt{2}\ units}$$

Since $AC = BC$, the triangle ABC is isosceles.

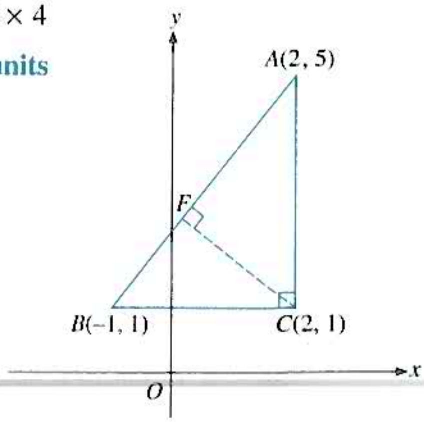
Example 3

Three points have coordinates $A(2, 5)$, $B(-1, 1)$ and $C(2, 1)$. Calculate the length of BC and of CA, and the area of the triangle ABC. Also, calculate the length of AB and of CF where F is on AB and CF is perpendicular to AB.

Solution:

$BC = 2 - (-1) = \mathbf{3\ units}$
$CA = 5 - 1 = \mathbf{4\ units}$

$$\text{Area of the triangle} = \frac{1}{2} \times 3 \times 4$$
$$= \mathbf{6\ sq.\ units}$$



$$AB = \sqrt{BC^2 + CA^2}$$
$$= \sqrt{3^2 + 4^2}$$
$$= \mathbf{5\ units}$$

$$\therefore\ \frac{1}{2} \times CF \times AB = 6$$

$$CF = \frac{6 \times 2}{5}$$
$$= \mathbf{2.4\ units}$$

Exercise 7.1

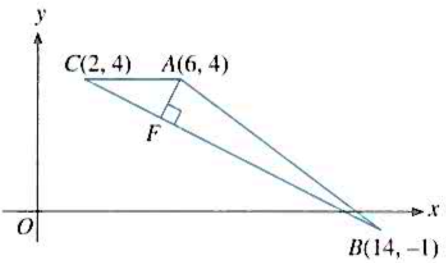
1. Find the perimeter and the area of the rectangle $ABCD$ where $A(4, 5)$, $B(4, -2)$, $C(-3, -2)$ and $D(-3, 5)$.

2. Find the area of the triangle ABC given the following coordinates.
   (a) $A(2, 1)$, $B(5, 1)$ and $C(6, 5)$
   (b) $A(a, 2)$, $B(a, 6)$ and $C(-2, a)$

3. Find the area of the trapezium $ABCD$ where $A(4, -1)$, $B(2, 3)$, $C(-1, 3)$ and $D(-2, -1)$. Show that the perimeter is 17.6 units (correct to 1 decimal place).

4. Find the lengths of the sides of the triangle with vertices $A(-4, 0)$, $B(3, 4)$ and $C(4, 1)$. Show that the triangle ABC is isosceles.

5. Calculate the area of the triangle ABC shown in the diagram. Hence, find the length AF.



6. If the point $(4, t)$ is equidistant from the points $A(4, 1)$ and $B(-1, 2)$, find the value of t.

7. Given $A(-1, 4)$ and $B(5, 2)$, find
   (a) the point P on the x-axis such that $AP = BP$,
   (b) the point Q on the y-axis such that $AQ = BQ$.

8. A trapezium with vertices $A(t, 0)$, $B(t + 2, 0)$, $C(t + 2, 3t + 2)$ and $D(t, 2t + 1)$, where $t > 0$, has an area of 18 square units. Find the value of t.

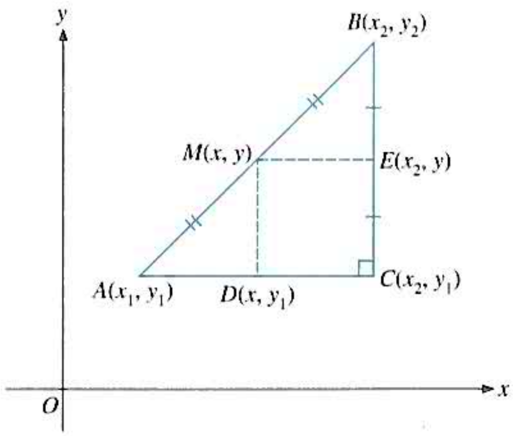
9. Which of the points $A(10, 4)$, $B(7, 6)$, $C(-3, 5)$ is nearest to $P(3, -2)$?

10. A point $P(x, y)$ is such that its distance from the y-axis is equal to its distance from the point $A(3, 2)$. Show that $y^2 - 4y - 6x + 13 = 0$.

7.2 Midpoint of the Line Joining Two Points

In the diagram, $M(x, y)$ is the midpoint of the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$. Consider the points A, D and C.

$$\begin{aligned} AM &= MB \\ \Rightarrow \quad AD &= DC \\ x - x_1 &= x_2 - x \\ x &= \frac{x_1 + x_2}{2} \end{aligned}$$



Next, we consider the points B, E and C.

$$\begin{aligned} AM &= MB \\ \Rightarrow \quad CE &= EB \\ y - y_1 &= y_2 - y \\ y &= \frac{y_1 + y_2}{2} \end{aligned}$$

The coordinates of the midpoint of $A(x_1, y_1)$ and $B(x_2, y_2)$ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Example 4

M is the midpoint of the line joining the points A and B. Find

(a) the coordinates of M if A and B have coordinates $(-1, 2)$ and $(3, -4)$ respectively,

(b) the coordinates of A if M and B have coordinates $(2, -1)$ and $(3, 2)$ respectively.

Solution:

(a) M has the coordinates $\left(\frac{(-1) + 3}{2}, \frac{2 + (-4)}{2}\right) = $ **(1, –1)**.

(b) Suppose the coordinates of A are (a, b). Since $M(2, -1)$ is the midpoint of AB

$$(2, -1) = \left(\frac{a + 3}{2}, \frac{b + 2}{2}\right)$$

i.e. $\quad 2 = \frac{a + 3}{2}$ and $-1 = \frac{b + 2}{2}$

$\quad a = 1$ and $b = -4$

The coordinates of A are **(1, –4)**.

Midpoints of the Diagonals of a Parallelogram

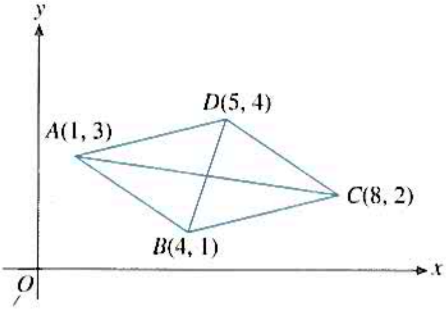
Consider the parallelogram $ABCD$ as shown below.

Midpoint of BD is $\left(\frac{4+5}{2}, \frac{4+1}{2}\right)$

$= \left(\frac{9}{2}, \frac{5}{2}\right)$

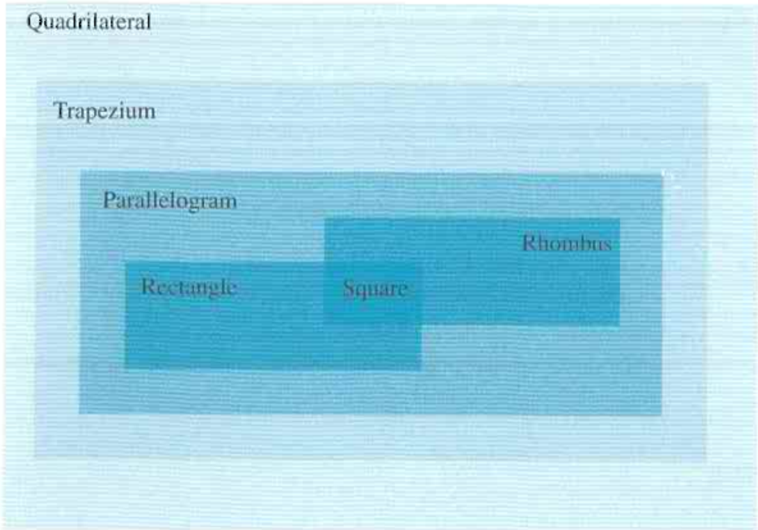
Midpoint of AC is $\left(\frac{1+8}{2}, \frac{3+2}{2}\right)$

$= \left(\frac{9}{2}, \frac{5}{2}\right)$



The above show that the diagonals BD and AC bisect each other at $M\left(\frac{9}{2}, \frac{5}{2}\right)$. That is, the midpoint of BD is the midpoint of AC. This result holds for any parallelogram.

Before proceeding to study further, we recall the relationship between some special quadrilaterals as illustrated in the following diagram.



Note that rectangles, rhombuses and squares are special parallelograms and their diagonals bisect each other, i.e. the diagonals have a common midpoint.

Example 5 $P(4, -4)$, $Q(9, 6)$, $R(-2, 4)$ and S are the vertices of a parallelogram. Find

(a) the midpoint of the diagonal PR,

(b) the coordinates of S.

Show that $PQRS$ is a rhombus.

Solution: **(a)** Midpoint of the diagonal PR is $M = \left(\frac{4 + (-2)}{2}, \frac{(-4) + 4}{2}\right)$

$= (1, 0)$

(b) Let (a, b) be the coordinates of S. Since $M(1, 0)$ is the midpoint of QS, we have

$$1 = \frac{9 + a}{2} \text{ and } 0 = \frac{6 + b}{2}$$

which give $a = -7$ and $b = -6$ and so the coordinates of S are $(-7, -6)$.

$PQ = \sqrt{(9 - 4)^2 + (6 - (-4)^2)} = \sqrt{125}$

$QR = \sqrt{(9 - (-2))^2 + (6 - 4)^2} = \sqrt{125}$

$PQ = QR \Rightarrow$ **the parallelogram $PQRS$ is a rhombus.**

Exercise 7.2

1. Find the coordinates of the midpoints of the line segments joining the following pairs of points.
(a) $(4, 5)$ and $(6, 9)$
(b) $(2a, -a)$ and $(4a, 5a)$
(c) $(2t, 5)$ and $(4, 1 - 2t)$
(d) $(a, -b)$ and $(-b, a)$

2. If $M(3, 5)$ is the midpoint of the line joining $A(-3, 7)$ and $B(p, q)$, find the value of p and of q.

3. The coordinates of the midpoint of the line joining $A(p^2, p)$ and $B(q^2, q)$ are $(5, 1)$. Find the possible values of p and q.

4. Three points have coordinates $A(-1, 6)$, $B(3, 2)$ and $C(-5, -4)$. Given that D and E are the midpoints of AB and AC respectively, calculate
(a) the midpoint of DE,
(b) the length DE.

5. Three of the vertices of a parallelogram $ABCD$ are $A(9, 3)$, $B(-2, 1)$ and $C(8, 6)$.
(a) Write down the coordinates of the midpoint of AC and hence, or otherwise, find the coordinates of the fourth vertex D.
(b) Prove that $ABCD$ is a rhombus. *(C)*

6. If $A(2, 0)$, $B(p, -2)$ and $C(-1, 1)$ and $D(3, r)$ are the vertices of a parallelogram $ABCD$, calculate the value of p and of r.

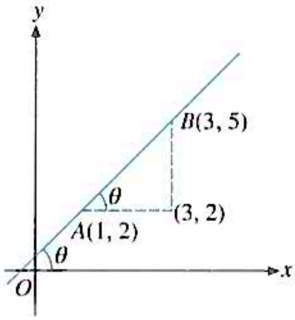
7. Four points have coordinates $A(2, -3)$, $B(3, 0)$, $C(0, 1)$ and $D(-1, -2)$.
(a) Show that $ABCD$ is a parallelogram.
(b) Calculate the length of AC and of BD.
Is $ABCD$ a rectangle?

7.3 Gradient of a Line Passing through Two Points

Consider the line passing through the points $A(1, 2)$ and $B(3, 5)$ as shown. The straight line makes an angle θ with the positive direction of the x-axis and this angle is a measure of the slope of the line with respect to the horizontal x-axis. In coordinate geometry, we use the gradient m given by:

$$m = \frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}}$$

$$= \frac{5 - 2}{3 - 1} = \frac{3}{2}$$

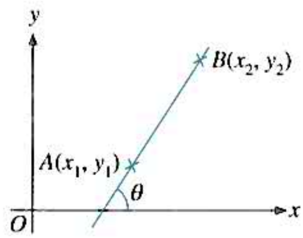


In general, the gradient of a line passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

In fact, the angle of inclination θ with the positive direction of the x-axis, and the gradient m, are related by:

$$m = \tan\theta$$



For the line AB in the above diagram, $\tan\theta = \frac{3}{2}$ and $\theta = 56.3°$, an acute angle.

Example 6 Find the gradient of the line passing through each of the following pairs of points.

(a) $A(2, 3)$ and $B(3, 6)$

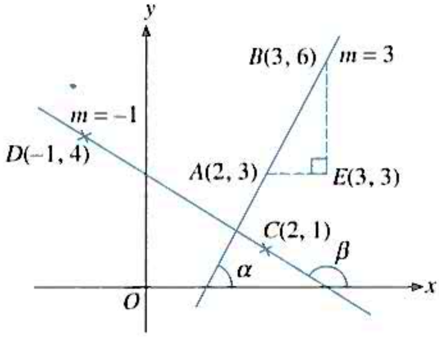
(b) $C(2, 1)$ and $D(-1, 4)$

Solution:

(a) Gradient of the line $AB = \frac{6 - 3}{3 - 2} = 3$

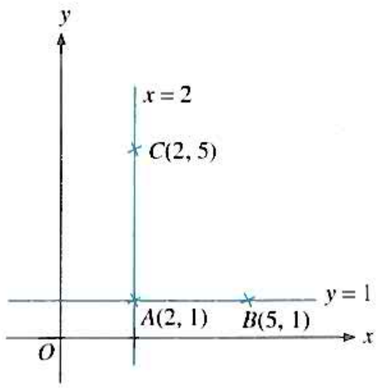
(b) Gradient of the line $CD = \frac{4 - 1}{-1 - 2} = -1$

Note: In (a), the gradient of the line AB is positive and its angle of inclination, α, is acute.
In (b), the gradient of the line CD is negative and β is an obtuse angle as shown in the diagram.



Example 7 A, B and C are the points (2, 1), (5, 1) and (2, 5) respectively. Draw the lines AB and AC on a cartesian plane. Calculate, if possible, the gradients of these lines.

Solution: The diagram shows the line AB, parallel to the x-axis and the line AC parallel to the y-axis.



Gradient of the line AB is

$$m = \frac{1-1}{5-2} = \mathbf{0}.$$

Gradient of the line AC cannot be calculated by the formula

$m = \frac{y_2 - y_1}{x_2 - x_1}$ since $x_1 = 2$, $x_2 = 2$ and $x_2 - x_1 = 0$.

Note: The equation of the line AB is $y = 1$ and the equation of the line AC is $x = 2$.

The gradient of a line passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

or $$m = \frac{y_2 - y_1}{x_2 - x_1}, \text{ provided that } x_1 \neq x_2.$$

If $x_1 = x_2$, then the gradient cannot be calculated and the line is parallel to the y-axis.

Example 8

Given that the distinct points $A(t, 1)$, $B(2, 4)$ and $C(1, t)$ are collinear, find the value of t.

Solution:

Since A, B and C lie on the same straight line, gradient of AB = gradient of BC.

$$\frac{4-1}{2-t} = \frac{t-4}{1-2}$$

$$\frac{3}{2-t} = \frac{t-4}{-1}$$

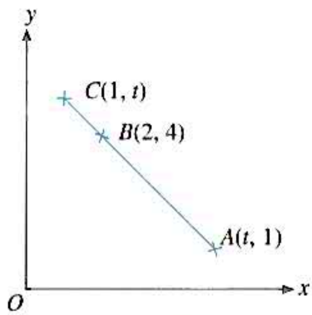
$$-3 = (2-t)(t-4)$$

$$-3 = -t^2 + 6t - 8$$

$$t^2 - 6t + 5 = 0$$

$$(t-1)(t-5) = 0$$

$$t = 1 \text{ or } 5$$



Since A and C are distinct points, $t \neq 1$ and so $t = 5$.

Exercise 7.3

1. Find the gradient of the line passing through each of the following pairs of points.
   (a) $A(5, -2)$ and $B(2, 1)$
   (b) $A(3, 5)$ and $B(-1, -2)$
   (c) $A(t, t)$ and $B(t^2, t^3)$
   (d) $A(t, t+2)$ and $B(3t, 5t+2)$

2. Let P, Q, R be the points $(2, 3)$, $(4, -1)$ and $(7, 6)$ and let M, N be the midpoints of PQ and PR respectively. Write down the coordinates of M and N and show that the line MN is parallel to the line QR.

3. Given that the points $(1, -1)$, $(2, 2)$ and $(4, t)$ are collinear, find the value of t.

4. Show that the points $A(0, -3)$, $B(4, -2)$ and $C(16, 1)$ are collinear.

5. Given that the points $A(1, -1)$, $B(t, 2)$ and $C(t^2, t+3)$ are collinear, find the possible values of t.

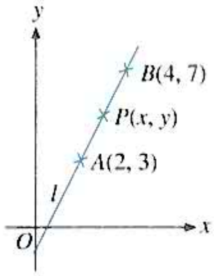
6. $A(t, 3t)$, $B(t^2, 2t)$, $C(t-2, t)$ and $D(1, 1)$ are four distinct points. If AB is parallel to CD, find the possible values of t.

7. The line joining the points $A(3, 3p)$ and $B(4, p^2+1)$ has gradient -1. Find the values of p.

8. Given that the points $A(2, 4)$, $B(6, 1)$ and $C(p, q)$ are collinear, show that $3p + 4q = 22$.

7.4 Equations of Straight Lines

Consider the straight line l passing through the points $A(2, 3)$ and $B(4, 7)$ as shown below.



Gradient of the line, $m = \dfrac{7-3}{4-2} = 2$

Let $P(x, y)$ be any point on the line l.

$$\text{Gradient of } AP = \frac{y-3}{x-2} = 2$$

$$\Rightarrow y - 3 = 2(x - 2)$$

or

$$y = 2x - 1$$

which is called the **cartesian equation** of l.

> **The cartesian equation of a straight line passing through (x_1, y_1) and (x_2, y_2) is**
> $$\frac{y-y_1}{x-x_1} = m, \quad \text{where} \quad m = \frac{y_2-y_1}{x_2-x_1}.$$

Example 9 Find the equation of the line joining the points $A(2, 5)$ and $B(5, -1)$.

Solution: Gradient of the line AB, $m = \dfrac{(-1)-5}{5-2}$

$$= -2$$

Equation of the line is $\dfrac{y-5}{x-2} = -2$

$$y - 5 = -2(x - 2)$$

$$\mathbf{y = -2x + 9}$$

From the above, we have an immediate result as follows:

> **With gradient m and a point (x_1, y_1) on a straight line, the cartesian equation of the line is**
> $$y - y_1 = m(x - x_1).$$

Example 10 Find the equation of the straight line that passes through $(2, -1)$ and has gradient 2.

Solution:

$$y - y_1 = m(x - x_1)$$

$$y - (-1) = 2(x - 2)$$

$$\mathbf{y = 2x - 5}$$

Example 11 The line $y = 3x + 6$ passes through the point $(t, 3)$. Find the value of t. If the line meets the y-axis at point P and the x-axis at Q, find the coordinates of P and Q.

Solution: Since $(t, 3)$ lies on the line $y = 3x + 6$, $3 = 3t + 6$

$\Rightarrow t = -1$

At P, $x = 0 \Rightarrow y = 6$

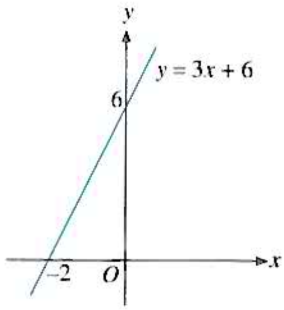
$\therefore$ the coordinates of P are $(0, 6)$.

Similarly, at Q, $y = 0$

$\Rightarrow 0 = 3x + 6$

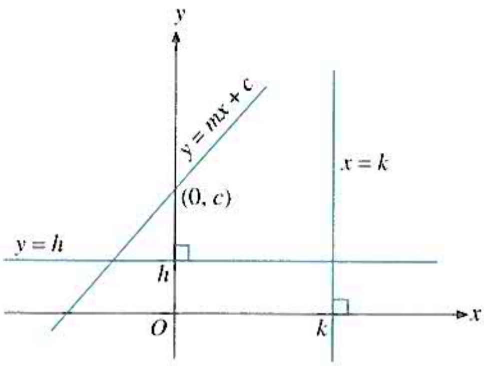
$x = -2$

$\therefore$ the coordinates of Q are $(-2, 0)$.



In Example 11, the value of $y(= 6)$ at which the line $y = 3x + 6$ intersects the y-axis is called the **y-intercept**. Similarly, $x = -2$ is the **x-intercept**.

Equations of vertical lines are of the form $x = k$ and those of horizontal lines are of the form $y = h$ as shown in the diagram below.



Equations of non-vertical straight lines are in the form

$$y = mx + c,$$

where m is the gradient of the line and c is the y-intercept.

Example 12 For each of the following lines, find the gradient and y-intercept and hence sketch the lines on the cartesian plane.

(a) $y + 2x = 3$ **(b)** $3x - 2y = 1$ **(c)** $y = 5$

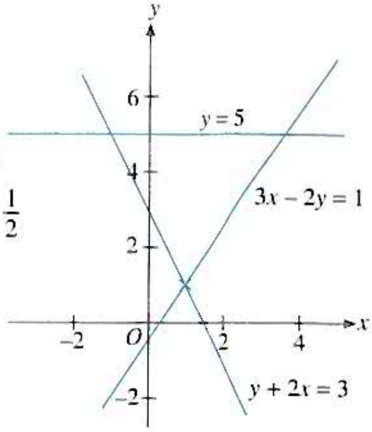
Solution:

(a) $y + 2x = 3 \Rightarrow y = -2x + 3$
Gradient, $m = -2$
y-intercept, $c = 3$

(b) $3x - 2y = 1 \Rightarrow 2y = 3x - 1$
$$y = \frac{3}{2}x - \frac{1}{2}$$
Gradient, $m = \frac{3}{2}$

y-intercept, $c = -\frac{1}{2}$

(c) $y = 5 \Rightarrow y = 0 \times x + 5$
Gradient, $m = 0$
y-intercept, $c = 5$

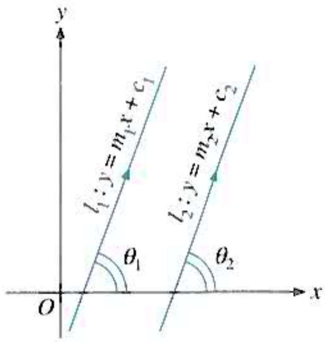


Exercise 7.4

1. Find the equation of the line passing through $(-3, 2)$ with gradient 2.
2. The line l has gradient 3 and cuts the x-axis at 4. Find its equation.
3. Find the equation of the line passing through each of the following pairs of points.
   (a) $A(1, 5)$, $B(2, 4)$ (b) $A(-1, 3)$, $B(1, 2)$
   (c) $A(0, -1)$, $B(5, 0)$ (d) $A(a, 0)$, $B(2a, 3a)$
4. Write down the equation of the straight line given
   (a) gradient -3 and y-intercept 2,
   (b) gradient -2 and y-intercept 5.
5. Find where the following lines cut the x-axis and the y-axis. Hence sketch these lines.
   (a) $y = 2x - 4$ (b) $2y = 3x + 6$ (c) $4x - 3y - 12 = 0$
6. The line joining the points $A(-1, 3)$ and $B(5, 15)$ meets the axes at P and Q. Find the equation of AB and calculate the length of PQ. *(C)*
7. Find in terms of t, the equation of the line through the point $(t, 0)$ with gradient t. Find the values of t for which the line passes through the point $(5, 6)$.
8. Find the equation of the line with gradient $\frac{1}{t}$, $(t \neq 0)$ which passes through the point $(t^2, 2t)$. If this line passes through the point $(-2, 1)$, find the possible values of t.
9. If the line through the point $A(-2, 3)$ with gradient $\frac{1}{2}$ also passes through the point $(3, k)$, find the value of k.
10. A line of gradient 3 passing through $A(-1, 6)$ meets the x-axis at B. Another line through A meets the x-axis at $C(2, 0)$.
    (a) Find the equation of AB and of AC.
    (b) Calculate the area of $\triangle ABC$.

7.5 Equations of Parallel and Non-Parallel Lines

For the lines $l_1 : y = m_1x + c_1$
and $l_2 : y = m_2x + c_2$
shown in the diagram,
l_1 and l_2 are parallel $\Leftrightarrow \theta_1 = \theta_2$
$\Leftrightarrow m_1 = m_2$.



$y = m_1x + c_1$ and $y = m_2x + c_2$ are parallel $\Leftrightarrow$ $m_1 = m_2$.

Example 13 Find the equation of the line passing through the point $A(2, 1)$ and parallel to the line $y = 2x + 5$.

Solution: With common gradient $m = 2$, the equation of the straight line is

$$y - 1 = 2(x - 2)$$
$$\mathbf{y = 2x - 3}$$

Example 14 Find the equation of the line which passes through the point $A(2, 1)$ and is parallel to the line $2x + 3y = 5$.

Solution:

$$2x + 3y = 5$$

$$\Rightarrow \quad y = -\frac{2}{3}x + \frac{5}{3}$$

$$\Rightarrow \quad \text{gradient, } m = -\frac{2}{3}$$

Then, the required equation is $y - 1 = -\frac{2}{3}(x - 2)$

i.e. $3y - 3 = -2x + 4$

or $\mathbf{2x + 3y - 7 = 0}$.

Equations of Non-Parallel Lines

If the gradients of the two lines $l_1 : y = m_1x + c$ and $l_2 : y = m_2x + d$ are not equal (i.e. they are not parallel), then the two lines intersect at a point. The point of intersection may be found by solving the equations of the two lines simultaneously.

Example 15 Find the coordinates of the point of intersection of the lines $y = 3x + 4$ and $3x + y = 10$.

Solution:

$y = 3x + 4$ .... (1)

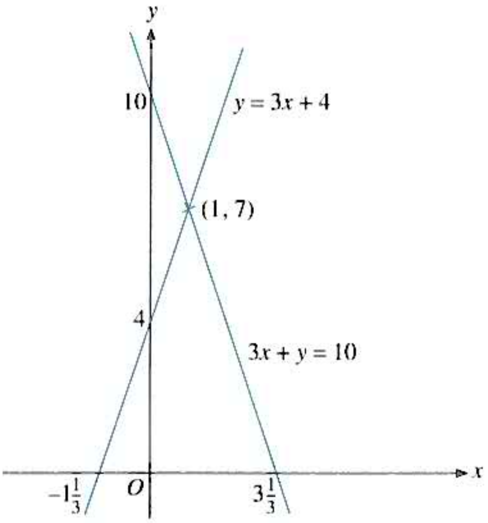
$3x + y = 10$ .... (2)

Substitute $y = 3x + 4$ into (2), we get:

$$3x + (3x + 4) = 10$$
$$6x = 6$$
$$\therefore\ x = 1 \text{ and } y = 7$$

The coordinates of the point of intersection are **(1, 7)**.

Note: Graphically, we have:



Exercise 7.5

1. Find the equation of the line which passes through the point (–1, 3) and is parallel to the line $y = 4x - 1$.
2. Find the equation of the line passing through the point (3, –2) and parallel to the line $2x - 3y - 2 = 0$.
3. Find the equation of the line with y-intercept –2 and parallel to the line $y - 3x = 4$.
4. Find the equation of the line passing through the point (–2, 3) and parallel to the line $4x + 3y - 5 = 0$. If $(p, -5)$ lies on this line, find the value of p.
5. Find the equation of the line passing through the point (3, 1) and parallel to the line which joins (3, –2) and (6, 4).
6. Find the coordinates of the point of intersection of each of the following pairs of non-parallel lines.

   (a) $4x + y - 2 = 0$
   $3x - y - 5 = 0$

   (b) $2x + 4y = 10$
   $6y - 3x + 9 = 0$

7. The straight line passing through the points $(-1, 3)$ and $(5, -3)$ intersects the line $2x - 3y - 9 = 0$ at the point P. Find the coordinates of P.

8. Find the equation of the line parallel to $3x + 2y - 6 = 0$ and passing through the point where $x + y + 2 = 0$ intersects $3x - 2y + 1 = 0$.

9. The line parallel to $y = 2x + 3$ and passing through the point $(3, -1)$ intersects another line $y = 3x - 11$ at the point P. Find the coordinates of P.

10. Find the equation of the line parallel to $x + 3y + 1 = 0$ and passing through the point where $3x - 2y + 6 = 0$ cuts the x-axis.

11. The line $y = ax + b$ is parallel to the line $y = 2x - 6$ and passes through the point $(-1, 7)$. Find the value of a and of b.

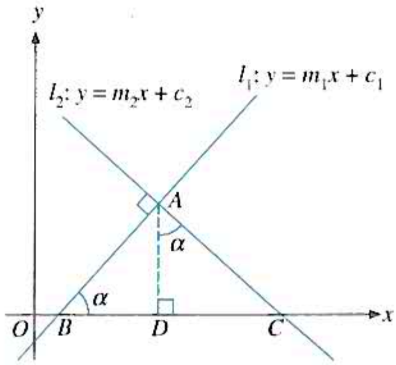
12. Prove that the lines $2y - x = 2$, $y + x = 7$ and $y = 2x - 5$ are concurrent (i.e. they intersect at only one point).

7.6 Equations of Perpendicular Lines

In the diagram, two straight lines l_1 and l_2 with gradients m_1 and m_2 intersect at right angles at A and meet the x-axis at B and C respectively. If AD is perpendicular to the x-axis and $\angle ABD = \alpha$, then it follows that $\angle CAD = \alpha$.

Since l_1 has positive gradient, we consider the right-angled $\triangle ADB$ and we have

$$m_1 = \frac{AD}{DB} = \tan\alpha$$



Since l_2 has negative gradient, we consider the right-angled $\triangle ADC$ and we have

$$m_2 = -\frac{AD}{DC} = -\frac{1}{\frac{DC}{AD}} = -\frac{1}{\tan\alpha}$$

Therefore, $m_1 \times m_2 = \tan\alpha \times \left(-\frac{1}{\tan\alpha}\right) = -1.$

This gives the following result:

Two non-vertical lines with gradients m_1 and m_2 are perpendicular $\Leftrightarrow$ $m_1m_2 = -1$.

In other words, the line perpendicular to $y = mx + c$ has gradient $-\frac{1}{m}$, where $m \neq 0$.

Example 16 The points A, B and C have coordinates $(t, 1)$, $(3, 4)$ and $(6, t + 1)$ respectively. Show that the line AB is perpendicular to the line BC for all values of t.

Solution:

$$\text{Gradient of } AB,\ m = \frac{4-1}{3-t} = \frac{3}{3-t},\ t \neq 3$$

$$\text{Gradient of } BC,\ m' = \frac{(t+1)-4}{6-3} = \frac{t-3}{3}$$

$$mm' = \frac{3}{3-t} \times \frac{t-3}{3} = -1$$

For $t = 3$, we have $A(3, 1)$, $B(3, 4)$ and $C(6, 4)$ and so the line AB is a vertical line and the line BC is a horizontal line. Therefore the line AB is perpendicular to the line BC for all values of t.

Example 17 Find the equation of the straight line passing through $A(2, -3)$ and perpendicular to the line $y = 3x + 1$. The two lines intersect at F. Find

(a) the coordinates of F, **(b)** the distance AF.

Solution:

$y = 3x + 1$ ....(1)

The gradient of this line is $m = 3$ and so the gradient of the perpendicular line is $m' = -\frac{1}{m} = -\frac{1}{3}$.

Equation of the perpendicular line is

$$y - (-3) = -\frac{1}{3}(x - 2)$$

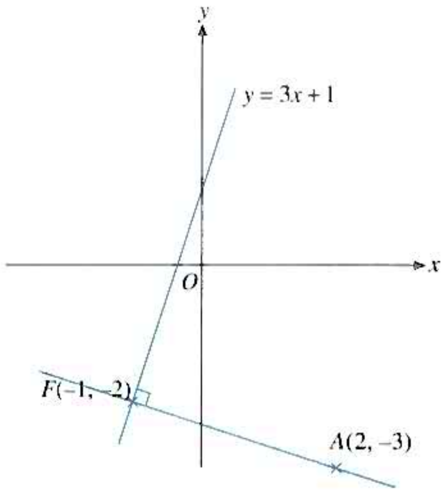
i.e. $3y + x + 7 = 0$..............................(2)

(a) Solving (1) and (2), we get $x = -1$ and $y = -2$.

The coordinates of the point F are **(–1, –2)**.

(b) $AF = \sqrt{(2-(-1))^2 + (-3-(-2))^2} = \sqrt{10}$ **units**

For the above example, the perpendicular line and the points A and F are shown in the diagram. The point F is called the **foot of the perpendicular** from the point A to the straight line $y = 3x + 1$. The distance AF is called the **perpendicular distance** from A to the line $y = 3x + 1$.



Example 18

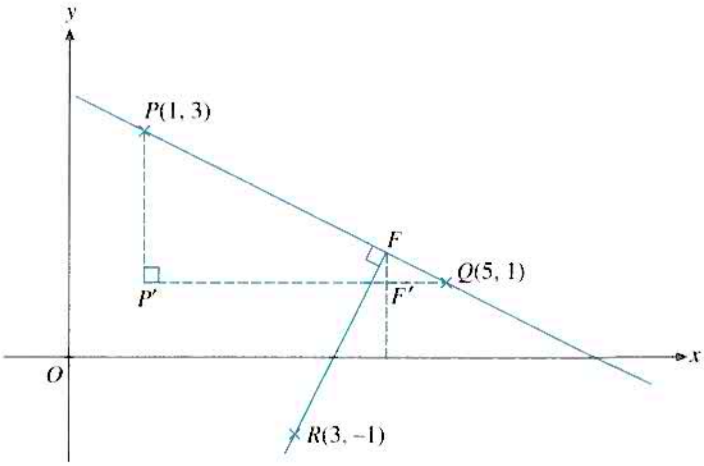
Three points have coordinates $P(1, 3)$, $Q(5, 1)$ and $R(3, -1)$. The foot of the perpendicular from R to the line joining P and Q is F. Find

(a) the coordinates of F,

(b) the ratio $PF : FQ$,

(c) the perpendicular distance from R to PQ.

Solution:



(a) Gradient of PQ, $m = \dfrac{3-1}{1-5} = -\dfrac{1}{2}$

Equation of PQ is $y - 3 = -\dfrac{1}{2}(x - 1)$

$2y = 7 - x$ .... (1)

Gradient of RF, $m' = \frac{-1}{m} = 2$

Equation of RF is $y - (-1) = 2(x - 3)$

$$y = 2x - 7 \quad \text{.............................} (2)$$

Substitute (2) into (1), we get:

$$2(2x - 7) = 7 - x$$

$$x = \frac{21}{5}$$

$$y = \frac{7}{5}$$

$\therefore$ the coordinates of F are $\left(\frac{21}{5}, \frac{7}{5}\right)$.

(b) By similar triangles,

$$\frac{PF}{FQ} = \frac{P'F'}{F'Q} = \frac{\frac{21}{5} - 1}{5 - \frac{21}{5}} = \frac{\frac{16}{5}}{\frac{4}{5}} = \frac{4}{1}$$

The ratio $PF : FQ$ is $4 : 1$.

(c) The perpendicular distance,

$$FR = \sqrt{\left(\frac{21}{5} - 3\right)^2 + \left(\frac{7}{5} - (-1)\right)^2} = \frac{6\sqrt{5}}{5} \text{ units}$$

Example 19

Three points have coordinates $A(-1, 4)$, $B(2, 2)$ and $C(7, 3)$. Find

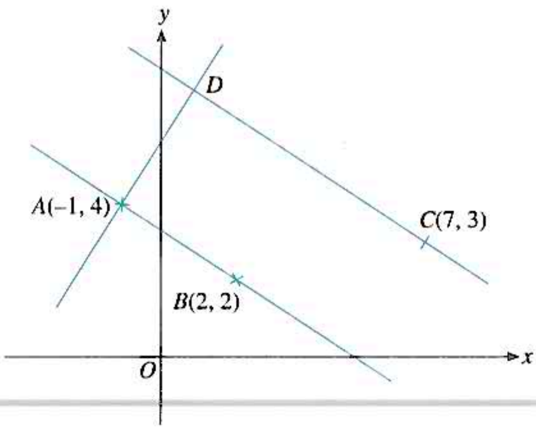
(a) the equation of the line through A perpendicular to AB,

(b) the equation of the line through C parallel to AB,

(c) the coordinates of the point D at which these two lines intersect,

(d) the distance AD.

Solution:



Gradient of AB, $m = \frac{2 - 4}{2 - (-1)} = -\frac{2}{3}$

(a) Gradient of the perpendicular line, $m' = -\frac{1}{m} = \frac{3}{2}$

Equation of the perpendicular line through A is

$$y - 4 = \frac{3}{2}(x - (-1))$$

or **$2y = 3x + 11$** .... (1)

(b) Equation of the parallel line through C is $y - 3 = -\frac{2}{3}(x - 7)$

$3y = 23 - 2x$ .... (2)

(c) Solving (1) and (2), we have:

$$3\left(\frac{3x + 11}{2}\right) = 23 - 2x$$

$$x = 1 \text{ and } y = 7$$

Hence D has coordinates **(1, 7)**.

(d) $AD = \sqrt{(1 -(-1))^2 + (7 - 4)^2} = \sqrt{13}$ **units**

In the above example, AD is also known as the perpendicular distance between the pair of parallel lines AB and CD.

Exercise 7.6

1. Given the points $A(3, 7)$, $B(6, 1)$ and $C(20, 8)$, prove that AB is perpendicular to BC.

2. Show that $A(2, -1)$, $B(5, 4)$ and $C(15, -2)$ form a right-angled triangle and state which angle is the right angle.

3. The line joining $A(a, 3)$ to $B(2, -3)$ is perpendicular to the line joining $C(10, 1)$ to B. Find the value of a.

4. Three points have coordinates $A(1, 2)$, $B(9, 0)$ and $C(6, t)$. Calculate the value(s) of t if
   (a) $\angle ABC = 90°$,
   (b) AC is perpendicular to BC.

5. Find the equation of the straight line passing through $A(4, 5)$ and perpendicular to the line $x + 2y - 4 = 0$. These two lines intersect at F. Find the coordinates of F.

6. The line $y = ax + b$ is perpendicular to the line $y - 3x = 4$ and passes through the point $(1, -2)$. Find the value of a and of b.

7. Three points have coordinates $A(3, 6)$, $B(2, -1)$ and $C(6, 7)$. The point F is the foot of the perpendicular from A to BC. Find the coordinates of F.
   Hence or otherwise, find the perpendicular distance from A to the line BC.

8. Three points have coordinates $A(6, 6)$, $B(-3, 3)$ and $C(9, k)$. The foot of the perpendicular from A to BC is the midpoint of BC. Calculate the possible values of k.

9. Calculate the perpendicular distance from the given point to the given line.
(a) $(4, 1)$, $3x - 4y + 2 = 0$ (b) $(2, 3)$, $x + y + 1 = 0$
(c) $(4, -3)$, $2x - y + 5 = 0$ (d) $(-1, 1)$, $3y + 4x = 2$

10. Calculate the perpendicular distance from the point $(-1, 4)$ to the line joining $A(4, 1)$ and $B(8, 3)$.

11. The equations of two parallel lines are $ax + y - 1 = 0$ and $2x + y + 11 = 0$.
(a) Find the value of a.
(b) Calculate the perpendicular distance between the lines.

12. The line $y = ax + 7$ is parallel to the line $y = 2x - 3$. The line $y = bx + 7$ is perpendicular to the line $y = 2x - 3$.
(a) State the value of a and of b.
(b) Calculate the perpendicular distance between the pair of parallel lines. *(C)*

13. The line through $A(3, 1)$ perpendicular to $x - 4y = 8$ meets the x-axis at P and the y-axis at Q. Calculate the ratio $PA : AQ$.

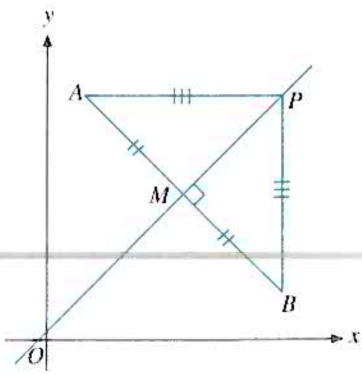
14. Three points have coordinates $A(4, 13)$, $B(9, 3)$ and $C(10, 8)$. Find the equation of
(a) the line AB,
(b) the line through C perpendicular to the line $y - 4x = 5$.
The line through C meets the line AB at point P. Calculate the coordinates of P and the ratio $AP : PB$.

15. The equations of the sides AB, BC, CA of the triangle ABC are $y = 2x$, $y = 3x$, $x + y = 8$ respectively. Find the equation of the line through C perpendicular to AB.
This line meets the x-axis in P and the y-axis in Q. Calculate the ratio $\dfrac{PC}{CQ}$. *(C)*

16. The line $y = 2x + 3$ intersects the y-axis at A. The points B and C on this line are such that $AB = BC$. The line through B perpendicular to AC passes through the point $D(-1, 6)$. Calculate
(a) the equation of BD, (b) the coordinates of B,
(c) the coordinates of C. *(C)*

7.7 Perpendicular Bisector

Let M be the midpoint of the line AB as shown in the diagram. The straight line through M and perpendicular to AB is called the **perpendicular bisector** of AB. Any point on the perpendicular bisector is equidistant from A and B.



Example 20 Two points have coordinates $A(1, 2)$ and $B(5, -1)$. Find
(a) the midpoint of AB,
(b) the equation of the perpendicular bisector of AB.

Solution:

(a) Midpoint of AB is $\left(\frac{1+5}{2}, \frac{2+(-1)}{2}\right) = \left(3, \frac{1}{2}\right)$

(b) Gradient of AB, $m = \frac{2-(-1)}{1-5} = -\frac{3}{4}$

Gradient of perpendicular bisector, $m' = -\frac{1}{m} = \frac{4}{3}$

Equation of the perpendicular bisector is $y - \frac{1}{2} = \frac{4}{3}(x - 3)$

or $\quad$ $\mathbf{6y = 8x - 21}$

Example 21 Two points have coordinates $A(-2, 3)$ and $B(4, 15)$. Find the equation of the perpendicular bisector of AB. Hence or otherwise, calculate the coordinates of the point P on the line $3y = x + 1$ if P is equidistant from A and B.

Solution:

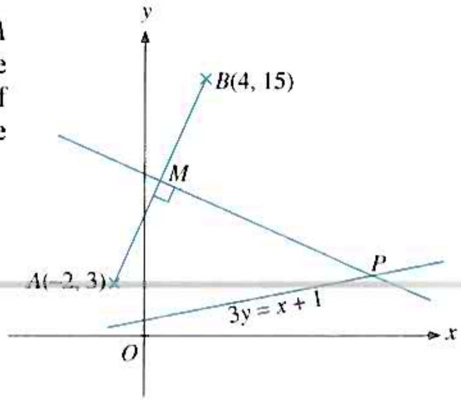
Midpoint of AB is $M = \left(\frac{-2+4}{2}, \frac{3+15}{2}\right)$
$= (1, 9)$

Gradient of AB, $m = \frac{15-3}{4-(-2)}$
$= 2$

So gradient of the bisector, $m' = -\frac{1}{m} = -\frac{1}{2}$

Equation of the bisector is $y - 9 = -\frac{1}{2}(x - 1)$
$\mathbf{2y = -x + 19}$

As P is equidistant from A and B, P must lie on the perpendicular bisector of AB, as shown in the diagram.



Hence P can be found by solving simultaneously

$$2y = -x + 19 \quad \text{.......(1)}$$
$$3y = x + 1 \quad \text{.......(2)}$$

(1) + (2) gives $5y = 20$

$$y = 4$$

Substitute $y = 4$ into either (1) or (2), we get:

$$x = 11$$

Therefore, coordinates of P are (11, 4).

Exercise 7.7

1. Given the coordinates of A and B in each of the following, find the equation of the perpendicular bisector of AB.
   (a) $A(1, 2)$, $B(3, 4)$ (b) $A(1, 8)$, $B(7, -4)$ (c) $A(2, 3)$, $B(-1, 9)$
   (d) $A(4, 5)$, $B(-2, 3)$ (e) $A(3, -6)$, $B(-2, 4)$ (f) $A(a, 7a)$, $B(3a, a)$

2. Find the equation of the perpendicular bisector of AB given the points $A(5, 4)$ and $B(3, -2)$. Hence or otherwise, find the coordinates of the point P on the x-axis, if P is equidistant from A and B.

3. Two points have coordinates $A(5, -2)$ and $B(3, 6)$. By finding the equation of the perpendicular bisector of AB, or otherwise, find the coordinates of the points on the axes which are equidistant from A and B.

4. Two points have coordinates $A(1, -3)$ and $B(-3, 5)$. Find the equation of the perpendicular bisector of AB. Find the coordinates of the point P on the line $y = x - 2$ if P is equidistant from A and B. Calculate also the length of AP.

5. Three points have coordinates $A(2, 9)$, $B(9, 7)$ and $C(2, 0)$. Find
   (a) the equation of the perpendicular bisector of AB,
   (b) the equation of the perpendicular bisector of BC,
   (c) the point of intersection of the two perpendicular bisectors.

6. Two points have coordinates $A(2, 3)$, and $B(6, 7)$. If $C(7, t)$ lies on the perpendicular bisector of AB, find the value of t. Find the coordinates of D such that the line AB is the perpendicular bisector of CD.

7. Show that the equation of the perpendicular bisector of the points $(t, t + 1)$ and $(3t, t + 3)$ is $y + tx = 2t^2 + t + 2$. If this perpendicular bisector passes through the point $(5, 2)$, calculate the values of t.

7.8 Intersection of a Straight Line and a Curve

Recall that two non-parallel lines intersect at one point. But a straight line and a curve may intersect at more than one point. The coordinates of the point of intersection can be obtained by solving the equations of the line and the curve simultaneously.

Example 21 The line $y = 2x - 2$ intersects the curve $y = x^2 - 2x + 1$ at two points. Find the coordinates of these points.

Solution: We have $y = 2x - 2$ .... (1)

$y = x^2 - 2x + 1$ .... (2)

Substitute $y = x^2 - 2x + 1$ into (1), we get:

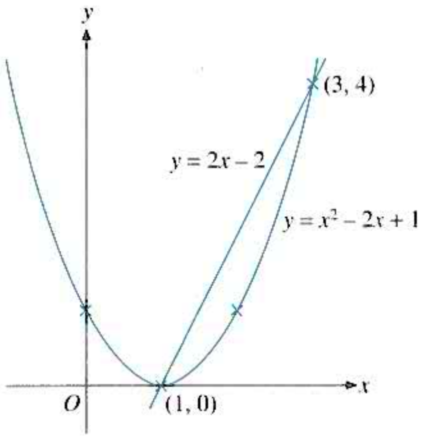
$$x^2 - 2x + 1 = 2x - 2$$
$$x^2 - 4x + 3 = 0$$
$$(x - 1)(x - 3) = 0$$

Thus $x = 1$ and $y = 0$

or $x = 3$ and $y = 4$

The points of intersection are **(1, 0)** and **(3, 4)**.

***Note*:** A sketch of the line $y = 2x - 2$ and the curve of $y = x^2 - 2x + 1$ together with the two points of intersection is shown below.



Exercise 7.8

1. Find the coordinates of the points of intersection of the line $y = 2x + 4$ and the curve $y = x^2 - 4$.

2. The line $y = x - 6$ meets the curve $y^2 = 8x$ at two points. Calculate the coordinates of these points.

3. The line $y - x = 1$ intersects the curve $xy = 6 - 2y$ at P and Q. Calculate the midpoint of PQ.

4. The line $y - x - 5 = 0$ meets the curve $y = x^2 - 3x$ at A and B. Calculate
   (a) the midpoint of AB, **(b)** the length of AB.

5. The line $3x + y = 8$ intersects the curve $3x^2 + y^2 = 28$ at A and B. Calculate
   (a) the length of AB,
   (b) the equation of the perpendicular bisector of AB.

6. Find the coordinates of the points of intersection of the curve $x^2 + 2y^2 + xy = 4$ and the line $2y = x + 1$.

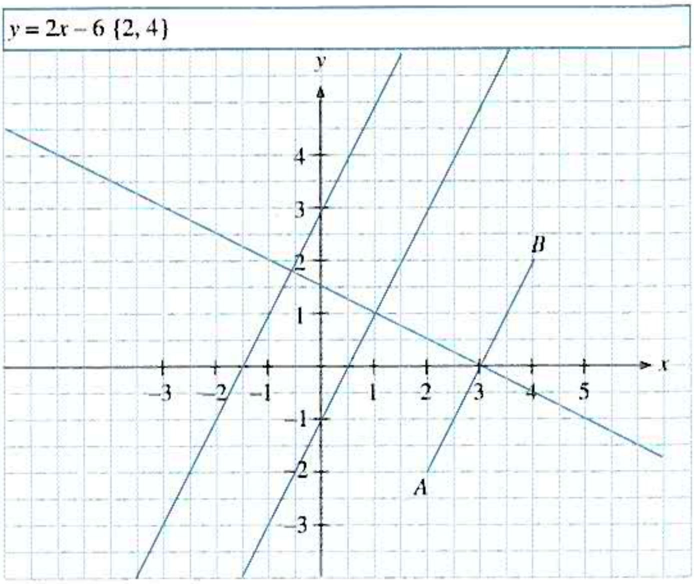
7. A straight line of gradient 3 is drawn through the point $(2, -2)$ on the curve $y = 3x^2 - 7x$. Find the coordinates of the point at which the line meets the curve again.

8. A straight line through the point $(0, -3)$ intersects the curve $x^2 + y^2 - 27x + 41 = 0$ at $(2, 3)$. Calculate the coordinates of the point at which the line again meets the curve. *(C)*

The diagram shows an application of a graph plotter, Graphmatica, to draw the lines

$$y = 2x + 3,\ y = 2x - 1,\ y = -\frac{1}{2}x + \frac{3}{2}$$

to illustrate parallel lines, perpendicular lines and perpendicular bisector.
Use a graph plotter to draw the line segment AB and lines in the above examples.



Important Notes

1. **Formulae concerning two points**
   For the points $A(x_1, y_1)$ and $B(x_2, y_2)$,
   (a) the distance $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$,
   (b) the midpoint of AB is $\left(\frac{x_1 + x_2}{2}, \frac{y_2 + y_1}{2}\right)$.

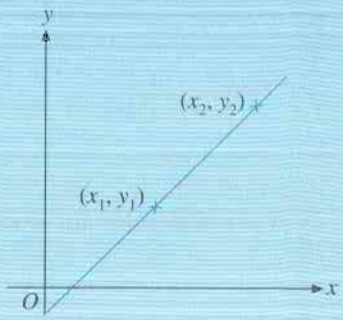
2. **Parallelogram**
   $ABCD$ is a parallelogram $\Leftrightarrow$ diagonals AC and BD have a common midpoint.
   Note that rhombuses, rectangles and squares are special parallelograms.

3. Equations of straight lines

(a) For a line of gradient m, passing through $A(x_1, y_1)$ and $B(x_2, y_2)$,

(i) gradient of the line, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$,

(ii) equation of the line is
$y - y_1 = m(x - x_1)$.

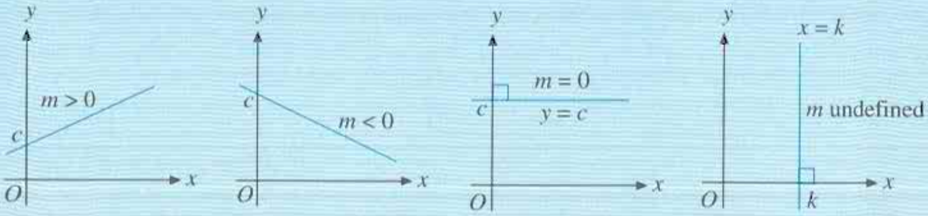


(b) The equation of a line may be written in the form

$$y = mx + c,$$

where m = gradient and c = y-intercept, i.e. the line cuts the y-axis at $C(0, c)$.

Straight lines of different gradients are shown as follows:



4. Equations of parallel lines

The lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are parallel $\Leftrightarrow$ $m_1 = m_2$.

5. Equations of perpendicular lines

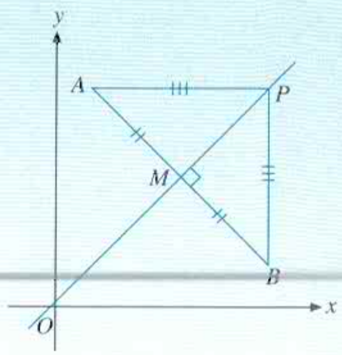
(a) The lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are perpendicular $\Leftrightarrow$ $m_1m_2 = -1$.

(b) The line passes through the point (x_1, y_1), perpendicular to the line $y = mx + c$, has gradient $m' = -\dfrac{1}{m}$ and the equation of the line is $y - y_1 = -\dfrac{1}{m}(x - x_1)$.

6. Perpendicular bisector

The line that passes through the midpoint of A and B, and perpendicular to AB is known as the perpendicular bisector of AB.

For any point P on the line, $PA = PB$.

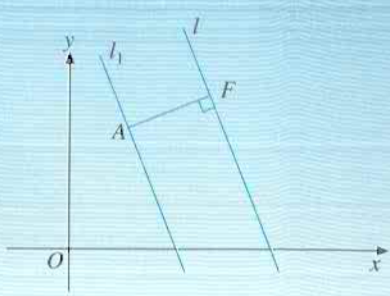


7. Foot of perpendicular and perpendicular distance

(a) The point F is the foot of the perpendicular from the point A to the line l.

(b) The distance AF is the perpendicular distance from the point A to the line l.

(c) The distance AF is the perpendicular distance between the pair of parallel lines l and l_1.



8. Points of intersection

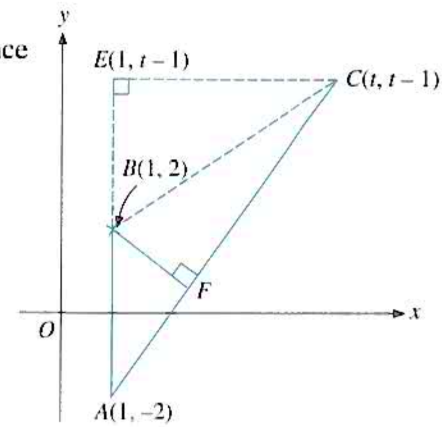
The coordinates of the point(s) of intersection of a line and a non-parallel line or a curve can be obtained by solving their equations simultaneously.

Example 23

Three points have coordinates $A(1, -2)$, $B(1, 2)$ and $C(t, t-1)$ where $t > 1$. Given that the area of triangle ABC is 12 square units, calculate

(a) the value of t,

(b) the perpendicular distance from B to AC.



Solution:

(a) $AB = 2 - (-2) = 4$

$CE = t - 1$

Area of $\triangle ABC = 12$

$\frac{1}{2} \times \text{base} \times \text{height} = 12$

i.e. $\frac{1}{2} \times AB \times CE = 12$

$\frac{1}{2} \times 4 \times (t-1) = 12$

$\therefore t = \mathbf{7}$

(b) Now $t = 7 \Rightarrow C$ has coordinates $(7, 6)$

$$AC = \sqrt{(7-1)^2 + (6-(-2))^2} = 10$$

Area of $\triangle ABC = 12$

i.e. $\frac{1}{2} \times AC \times BF = 12$

So, $\frac{1}{2} \times 10 \times BF = 12$

$BF = 2.4$

$\therefore$ the perpendicular distance is **2.4 units**.

Example 24 $ABCD$ is a trapezium in which AB is parallel to DC and AD is perpendicular to both AB and DC. The coordinates of A, B and C are $(3, 0)$, $(7, b)$ and $(7, 7)$ respectively. Given that the gradient of the line AB is $\frac{1}{2}$, find

(a) the value of b,
(b) the equation of DC,
(c) the equation of AD,
(d) the coordinates of D,
(e) the area of the trapezium.

Solution:

(a) Gradient of $AB = \frac{1}{2} \quad \Rightarrow \quad \frac{b-0}{7-3} = \frac{1}{2}$

$$b = \mathbf{2}$$

(b) Equation of DC is $y - 7 = \frac{1}{2}(x - 7)$

$$\mathbf{2y = x + 7} \text{ .... (1)}$$

(c) Equation of AD is $y - 0 = -2(x - 3)$

$$\mathbf{y = 6 - 2x} \text{ .... (2)}$$

(d) Solving (1) and (2), we have:

$$2(6 - 2x) = x + 7$$
$$x = 1 \text{ and } y = 4$$

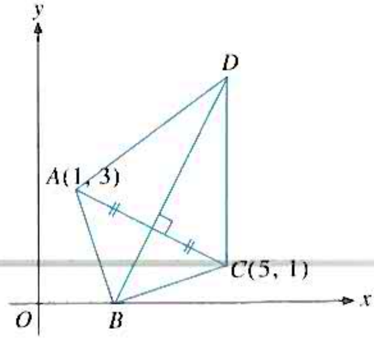
The coordinates of D are **(1, 4)**.

(e) Since $AD = 2\sqrt{5}$, $AB = 2\sqrt{5}$ and $DC = 3\sqrt{5}$:

$$\text{Area of trapezium } ABCD = \frac{1}{2}(AB + DC) \times AD$$
$$= \frac{1}{2}(2\sqrt{5} + 3\sqrt{5}) \times 2\sqrt{5}$$
$$= \mathbf{25 \text{ sq. units}}$$

Example 25 $ABCD$ is a quadrilateral where A is $(1, 3)$, C is $(5, 1)$ and B lies on the x-axis. BD is the perpendicular bisector of AC and CD is a vertical line. Find

(a) the equation of BD,
(b) the coordinates of B and of D,
(c) the area of $ABCD$.



Solution: **(a)** Gradient of $AC = \frac{1-3}{5-1} = -\frac{1}{2}$ and so gradient of $BD = 2$.

Midpoint of AC is $\left(\frac{1+5}{2}, \frac{3+1}{2}\right) = (3, 2)$.

Equation of BD is $y - 2 = 2(x - 3)$

$$y = 2x - 4 \quad \text{.................................... (1)}$$

(b) At B, $y = 0$ and so $2x - 4 = 0$, i.e. $x = 2$.
The coordinates of B are **(2, 0)**.

Since CD is a vertical line, $x = 5$ and $y = 2(5) - 4 = 6$ at D.
The coordinates of D are **(5, 6)**.

(c) Since $AC = 2\sqrt{5}$ and $BD = 3\sqrt{5}$:

$$\begin{aligned}\text{Area of } ABCD &= \frac{1}{2} \times AC \times BD \\ &= \frac{1}{2} \times 2\sqrt{5} \times 3\sqrt{5} \\ &= \mathbf{15 \text{ sq. units}}\end{aligned}$$

Miscellaneous Exercise 7

1. Given $A(1, 2)$, $B(2, -3)$ and $C(6, 1)$, show that $AB = AC$. Find the coordinates of the midpoint of AC. If the figure $ABCD$ is a rhombus, find the coordinates of D.

2. Given that $P(h, k)$ is a point equidistant from the points $A(3, 5)$ and $B(7, -1)$, prove that $3k - 2h + 4 = 0$.

3. Three points have coordinates $A(-2, 1)$, $B(10, 6)$ and $C(a, -6)$. Given $AB = BC$, find the possible values of a. If the figure $ABCD$ is a rhombus, find the coordinates of D.

4. Given the points $A(a, a + 1)$, $B(-6, -3)$ and $C(5, -1)$, find the possible values of a if the length of AB is twice the length of AC.

5. Three points have coordinates $O(0, 0)$, $A(5, 0)$ and $B(7, 6)$. If P is the point (x, y), where $y > 0$, calculate the value of x and of y given that $AP = BP$, and that the area of triangle AOP is 10 sq. units.

6. **(a)** Prove that the distance from the point $(1, -1)$ to the point $(t + 1, \sqrt{1 - t^2} - 1)$ is the same for all values of t where $-1 \leqslant t \leqslant 1$.

   (b) Find the area of the triangle whose vertices are $(a, 0)$, $(0, b)$ and $\left(\frac{a+b}{2}, \frac{a+b}{2}\right)$.

7. The points $A(6, 7)$, $B(0, 1)$ and $C(9, 4)$ are vertices of a triangle. Find the length AB and the area of the triangle ABC and hence obtain the perpendicular distance from C to the line AB.

8. Two points have coordinates $A(3, -1)$ and $B(-1, 5)$. Calculate the coordinates of P if P is on the x-axis and is equidistant from A and from B. If the figure $APBQ$ is a rhombus, find the coordinates of Q.

9. Given that the four points $A(1, 2)$, $B(2, -5)$, $C(7, 0)$ and $D(\alpha, \beta)$ are the vertices of the rhombus $ABCD$, calculate
   (a) the value of α and of β, **(b)** the length of AC,
   (c) the length of BD, **(d)** the area of the rhombus.

10. The points A, B, C and D have coordinates $(3, -2)$, $(p, 3)$, $(6, 2)$ and (q, r) respectively. The figure $ABCD$ is a rhombus. Find
   (a) the values of p, q, r, **(b)** the area of the rhombus. *(C)*

11. Three points have coordinates $A(-1, 2)$, $B(3, 10)$ and $C(p, 8)$. Find the value(s) of p if
   (a) A, B and C are collinear,
   (b) AC is perpendicular to BC.

12. **(a)** Find the equation of the line passing through the points $(-1, 3)$ and $(3, 9)$. If the line also passes through the point $(t, t + 2)$, find the value of t.
   (b) Find the equation of the line through the point $(5, 2)$ and parallel to the line $2y - 3x = 4$. Find the value of a if the line $ax - (a + 1)y + 2 = 0$ does not intersect the pair of parallel lines.

13. The lines $x - y - 2 = 0$ and $2x - 5y - 7 = 0$ intersect at the point P. Find the coordinates of P. The line through P with gradient 2 meets the axes at A and B. Calculate the area of triangle AOB.

14. Find the equation of the line with gradient m which passes through the point $(2, 3)$. If $m < 0$ such that the line meets the positive x-axis at A and the y-axis at B, find
   (a) the coordinates of A and of B in terms of m,
   (b) the values of m for which the area enclosed by this line and the axes is 16 square units.

15. Given that points $A(1, 8)$, $B(4, 7)$ and $O(0, 0)$, find
   (a) the equation of the line through A parallel to OB,
   (b) the equation of the line through B perpendicular to AB,
   (c) the coordinates of the point C at which these two lines intersect,
   (d) the area of the triangle ABC. *(C)*

16. **(a)** The line $y = ax + b$ is perpendicular to the line $y + 2x = 8$ and meets the line $y + x + 3 = 0$ on the y-axis. Find the value of a and of b.
   (b) In the triangle ABC the equations of AB and BC are $y = 3x + 1$ and $y = x - 1$ respectively. The foot of the perpendicular from A to BC is $(3, 2)$. Find the coordinates of A.

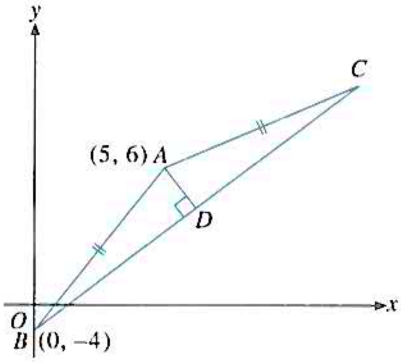
17. Three points have coordinates $A(2, 6)$, $B(6, -2)$ and $C(3, -1)$. Find the equation of
   (a) the line AB,
   (b) the line through C perpendicular to AB.
   Hence find the coordinates of the foot of the perpendicular from C to AB.

18. Given three points $A(-1, -2)$, $B(3, 4)$ and $C(3, 6)$, find the equation of the perpendicular bisector of

(a) AB, (b) BC.

Hence find the coordinates of the point equidistant from A, B and C.

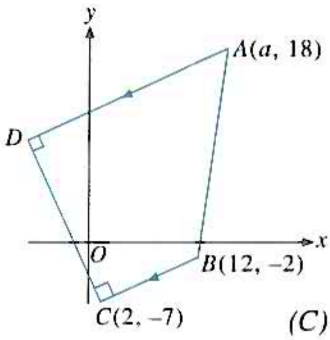
19. The diagram, which is not drawn to scale, shows an isosceles triangle ABC in which $AB = AC$. The coordinates of A and B are $(5, 6)$ and $(0, -4)$ respectively.



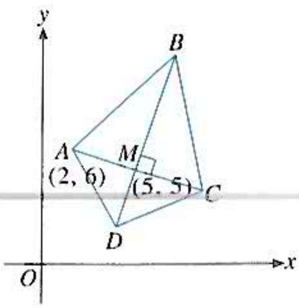
Given that the gradient of BC is $\frac{3}{4}$ and that the perpendicular from A to BC meets BC at D, find

(a) the equation of BC and of AD, (b) the coordinates of D,
(c) the coordinates of C, (d) the length of the perpendicular AD,
(e) the area of $\triangle ABC$. *(C)*

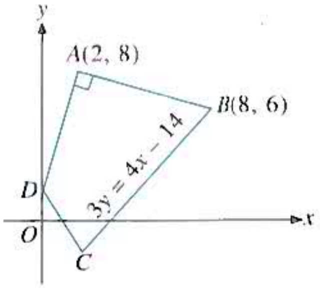
20. The diagram shows a trapezium in which AD is parallel to BC and angle ADC = angle $BCD = 90°$. The points A, B and C are $(a, 18)$, $(12, -2)$ and $(2, -7)$ respectively.
Given that $AB = 2BC$, find
(a) the value of a,
(b) the equation of AD,
(c) the equation of CD,
(d) the coordinates of D,
(e) the area of the trapezium. *(C)*



21. In the quadrilateral $ABCD$ shown in the diagram, BD is the perpendicular bisector of AC. The midpoint of AC is M. The coordinates of A and M are $(2, 6)$ and $(5, 5)$ respectively. AB is parallel to OM and ODC is a straight line. Find
(a) the coordinates of C,
(b) the equation of AB, of BD and of CD,
(c) the coordinates of B and of D,
(d) the area of $ABCD$. *(C)*



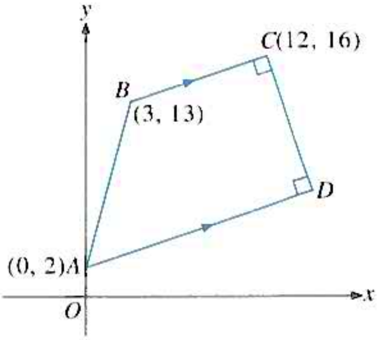
22. The diagram shows a quadrilateral $ABCD$ in which A is $(2, 8)$ and B is $(8, 6)$. The point C lies on the perpendicular bisector of AB and the point D lies on the y-axis. The equation of BC is $3y = 4x - 14$ and the angle $DAB = 90°$. Find



(a) the equation of AD,
(b) the coordinates of D,
(c) the equation of the perpendicular bisector of AB,
(d) the coordinates of C.

Show that the area of triangle ADC is 10 unit^2 and find the area of the quadrilateral $ABCD$. *(C)*

23. The diagram, which is not drawn to scale, shows a trapezium $ABCD$ in which BC is parallel to AD and CD is perpendicular to both BC and AD. The coordinates of A, B and C are $(0, 2)$, $(3, 13)$ and $(12, 16)$ respectively. Find



(a) the equation of AD and of CD,
(b) the coordinates of D.

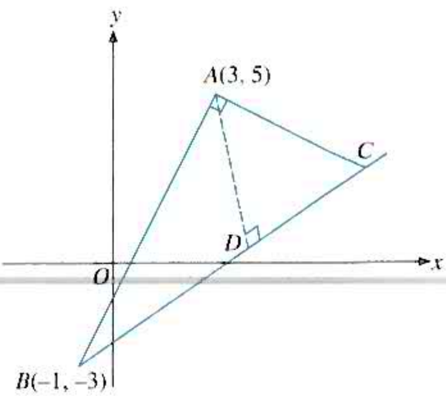
The line AB produced meets the line DC produced at E. Find

(c) the coordinates of E,
(d) the ratio $AE : BE$,
(e) the ratio of the area of the triangle BEC to the area of the trapezium $ABCD$. *(C)*

24. Two points have coordinates $A(1, 3)$ and $C(7, 7)$. Find the equation of the perpendicular bisector of AC.
B is the point on the y-axis equidistant from A and C and $ABCD$ is a rhombus. Find the coordinates of B and of D.
Show that the area of the rhombus is 52 square units and hence, or otherwise, calculate the perpendicular distance of A from BC. *(C)*

25. Find the equation of the line joining the points $A(5, -1)$ and $C(-1, 7)$ in the form $lx + my = n$ where l, m, n are integers. Given a point $B(p, 0)$ such that $BA = BC$, find the value of p. Find the coordinates of D, the fourth vertex of the square $ABCD$. Calculate the area of this square.

26. The diagram, which is not drawn to scale, shows a right-angled triangle ABC in which $B\hat{A}C = 90°$. The coordinates of A and B are $(3, 5)$ and $(-1, -3)$ respectively.
Given that the gradient of BC is $\frac{1}{2}$ and D is the foot of the perpendicular from A to BC, find



(a) the equation of BC and of AC,
(b) the coordinates of C,
(c) the coordinates of D,
(d) the length of the perpendicular AD.

27. Three points of a quadrilateral $ABCD$ are $A(1, 2)$, $B(6, 7)$ and $C(9, 6)$. Given that $AB = AD$ and $BC = DC$ and F is the foot of the perpendicular from B to the line AC. Find

(a) the coordinates of F,

(b) the coordinates of D,

(c) the area of the quadrilateral $ABCD$.

28. A trapezium $PQRS$, with parallel sides PQ and SR, has vertices $P(1, -1)$, $Q(7, 1)$ and $S(3, 3)$. Given that $P\hat{Q}R = 90°$, find

(a) the coordinates of R,

(b) the area of the trapezium $PQRS$.

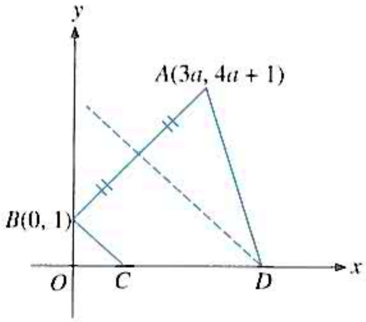
If T is a point such that $PQTS$ is a parallelogram, find the coordinates of T and show that $RT = SR$.

29. The diagram shows the quadrilateral $ABCD$. The coordinates of A and B are $(3a, 4a + 1)$, where $a > 0$, and $(0, 1)$ respectively.

(a) The length of AB is 5 units. Calculate the value of a.

(b) AB is perpendicular to BC and C lies on the x-axis. Find the equation of BC and the coordinates of C.

(c) The point D lies on the x-axis and also on the perpendicular bisector of AB. Find the coordinates of D and the area of the quadrilateral $ABCD$.



***30.** Given the points $O(0, 0)$, $A(-1, 1)$ and $B\left(\frac{1}{\lambda}, \frac{1}{\lambda^2}\right)$, where $\lambda > 0$, show that the area of $\triangle OAB$ is $\frac{1 + \lambda}{2\lambda^2}$ sq. units. Find AB in terms of λ. Show that the height from O to the line joining A and B is $\frac{1}{\sqrt{2\lambda^2 - 2\lambda + 1}}$ units and hence find the value of λ for which this height is the largest.

31. Given the points $P(at^2, 2at)$, $Q(a, 0)$ and $R\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$ where a is a positive constant and $t > 0$, show that P, Q and R are collinear. Find, in terms of a and t,

(a) the area of the triangle OPR where O is the origin,

(b) the length PR.

Hence deduce that the perpendicular distance from O to the line PR is $\frac{2at}{1 + t^2}$.

8 Linear Law

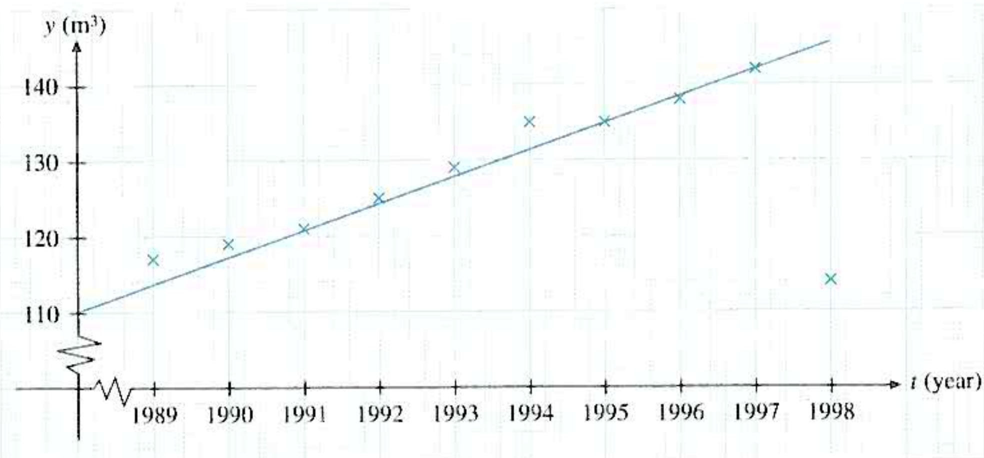
In science, we often study how two variables x and y are related. If the variables are related by an equation of the form $y = mx + c$, then the graph of y against x is a straight line whose gradient is m and y-intercept is c. The equation $y = mx + c$ is known as the **linear law** relating the variables x and y. The variables x and y are said to be **linearly related**.

For example, the water consumption per capita in Singapore over ten years is shown in the following table.

t (Year)	1989	1990	1991	1992	1993	1994	1995	1996	1997	1998
y (m^3)	117	119	121	125	129	135	135	138	142	114

(Source: Public Utilities Board Annual Report 1998)

By plotting y against t with the data, we obtain a straight line graph as shown. This suggests a trend in water consumption with y being linearly related to t.

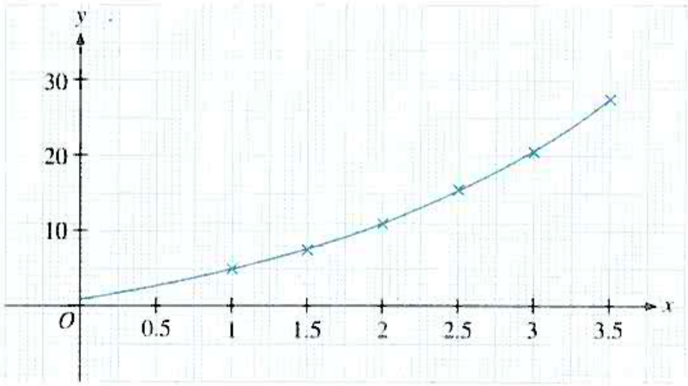


The abnormal data in 1998 is explained as follows: 1998 consumption per capita is based on total population of Singapore as at 30.6.1998 of 3 865 600. Consumption per capita prior to 1998 was based on resident population of Singapore.

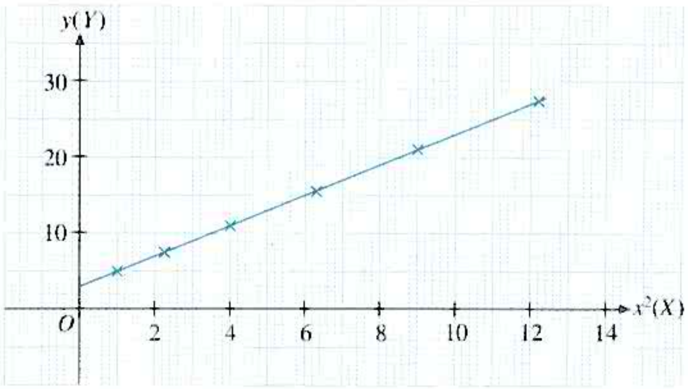
Suppose two variables x and y are related by the equation $y = 2x^2 + 3$. The table below shows some values of x and the corresponding values of x^2 and y.

x	1	1.50	2	2.50	3	3.50
x^2	1	2.25	4	6.25	9	12.25
y	5	7.50	11	15.5	21	27.50

By plotting y against x, we have a curve as shown below. The curve is not a straight line and the variables x and y are not linearly related.



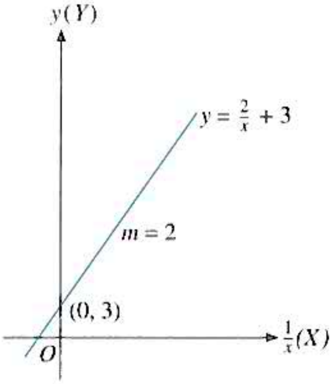
However, if we plot y against x^2, we have a straight line of gradient 2 with y-intercept 3 and so the variables y and x^2 are linearly related.



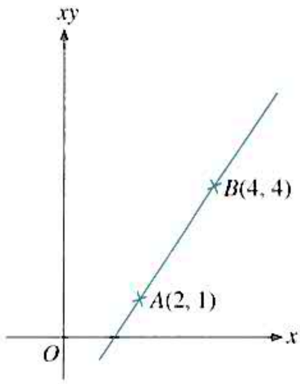
Observe that the equation $y = 2x^2 + 3$ is of the form $Y = mX + c$ where $Y = y$, $X = x^2$, $m = 2$ and $c = 3$.

Example 1 Two variables x and y are related by the equation $y = \frac{2}{x} + 3$. Sketch the graph of y against $\frac{1}{x}$.

Solution: The given equation is of the form $Y = mX + c$ where $Y = y$ and $X = \frac{1}{x}$.
Hence the graph of $y(Y)$ against $\frac{1}{x}(X)$ is a straight line of gradient $m = 2$ and with y-intercept 3 as shown.



Example 2 The diagram shows two particular points on a straight line on a cartesian plane with xy-axis and x-axis. Express y as a function of x.



Solution: The variables xy and x are related by the equation

$$xy = mx + c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

where $m = \frac{4-1}{4-2}$ *(gradient of the line AB)*

$$= \frac{3}{2}$$

So (1) becomes

$$xy = \frac{3}{2}x + c$$

At A, $x = 2$ and $xy = 1$,

i.e. $1 = \frac{3}{2} \times 2 + c$

$$c = -2$$

$$xy = \frac{3}{2}x - 2$$

$$\mathbf{y = \frac{3}{2} - \frac{2}{x}}$$

Example 3

The table shows experimental values of two quantities x and y which are known to be connected by an equation of the form $\frac{1}{y} = a\sqrt{x} + b$.

x	0.50	1.00	1.50	2.00	2.50	3.00
y	1.61	0.83	0.61	0.50	0.42	0.38

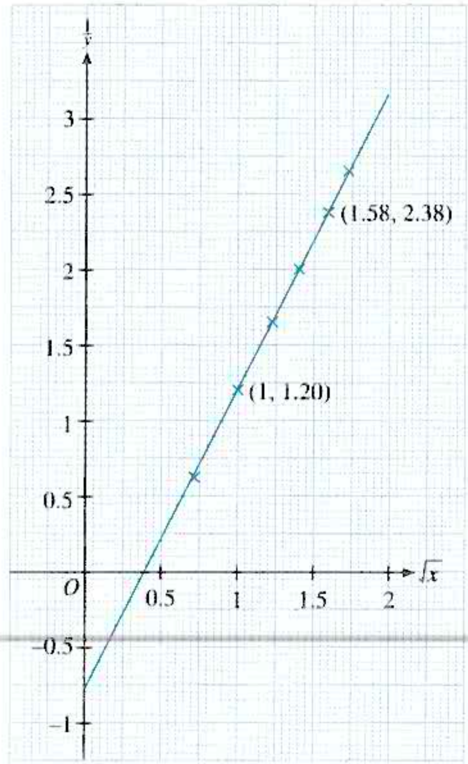
Plot $\frac{1}{y}$ against $\sqrt{x}$ and use the graph to estimate the values of a and b.

Solution:

The equation $\frac{1}{y} = a\sqrt{x} + b$ is of the form $Y = mX + c$ where $Y = \frac{1}{y}$ and $X = \sqrt{x}$. The table below shows corresponding values of $\sqrt{x}$ and $\frac{1}{y}$.

X	$\sqrt{x}$	0.71	1.00	1.22	1.41	1.58	1.73
Y	$\frac{1}{y}$	0.62	1.20	1.64	2.00	2.38	2.63

Plotting Y against X, a straight line is obtained as shown below.



$$\begin{aligned} a &= m \quad \textit{(gradient of the line)} \\ &\approx \frac{2.38 - 1.20}{1.58 - 1.00} \\ &\approx \frac{1.18}{0.58} \\ &\approx 2 \\ b &= c \left(\frac{1}{y}\textit{-intercept}\right) \\ &\approx -0.80 \end{aligned}$$

Therefore, the estimated values of a and b are **2** and **−0.8** respectively.

Example 4

Two variables x and y are known to be related by the equation

$$xy = a(x - y + b)$$

where a and b are constants. Some values of the two variables are shown in the following table.

x	0	1	2	3	4
y	2	1.64	1.50	1.43	1.38

Plot a graph of xy against $x - y$ and use it to estimate the value of a and of b.

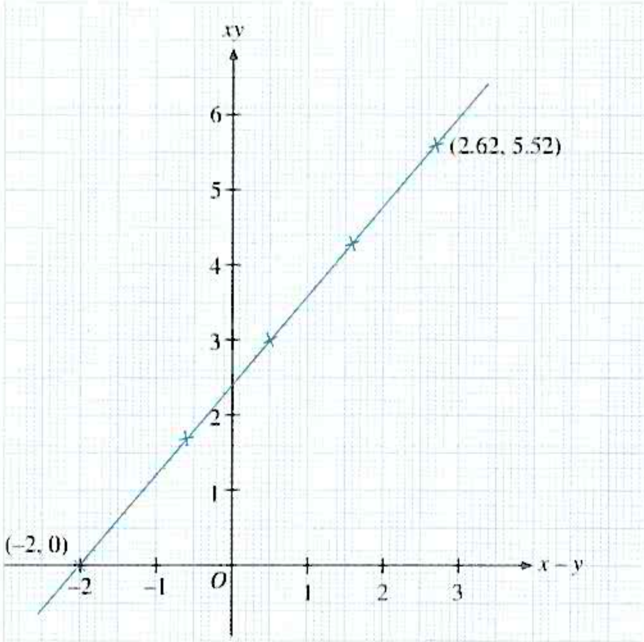
Solution:

The equation $\quad xy = a(x - y + b)$
or $\quad xy = a(x - y) + ab$
is of the form $Y = mX + c$ where $Y = xy$, $X = x - y$, $m = a$ and $c = ab$.

The table below shows some corresponding values of X and Y.

X	$x - y$	−2	−0.64	0.50	1.57	2.62
Y	xy	0	1.64	3.00	4.29	5.52

Plotting Y against X, a straight line graph is obtained as shown below.



$$a = m \quad \textit{(gradient)}$$
$$\approx \frac{5.52 - 0}{2.62 - (-2)}$$
$$\approx 1.19$$
$$ab = c \approx 2.40 \quad \textit{(xy-intercept)}$$
$$b \approx \frac{2.40}{a}$$
$$\approx 2.02$$

Therefore, the estimated values of a and b are **1.19** and **2.02** respectively.

In general, if two variables x and y are related by an equation with two unknown constants a and b, and this equation can be written in the form

$$Y = mX + c$$

where X and Y are each expression in terms of x and/or y, and m and c are constants, then we can estimate the values of a and b as follows:

(a) Take some experimental values of x and y and compute the corresponding values of X and Y.

(b) Use the computed values of X and Y to plot the points on a cartesian plane with X-axis and Y-axis.

(c) Draw a line passing through (or close to) all the plotted points.

(d) Obtain the estimated values of m and C using

$$m = \text{gradient of the line,}$$
$$c = Y\text{-intercept.}$$

(e) Express a and b in terms of m and c and obtain estimated values of a and b.

So far, we have dealt with equations involving variables x and y in the form $Y = mX + c$. However, there are equations which are not explicitly given in this form.

For example, $$y = \frac{1}{ax^2 + b}.$$

Taking the reciprocals of both sides, we have:

$$\frac{1}{y} = ax^2 + b$$

which is then of the form $Y = mX + c$ where $Y = \frac{1}{y}$ and $X = x^2$.

Example 5 Express the equation $y = ax + \frac{b}{x}$, where a and b are constants, in the form $Y = mX + c$ where X and Y are functions of x and/or y and m and c are constants.

Solution:

$$y = ax + \frac{b}{x}$$

Multiplying both sides by x so as to obtain a constant term b, we have

$$\mathbf{xy = ax^2 + b.}$$

Thus, $Y = xy$, $X = x^2$, $m = a$ and the constant term, $c = b$.
Alternatively, we divide both sides of the given equation by x and obtain

$$\frac{y}{x} = a + \frac{b}{x^2}$$

i.e. $$\frac{y}{x} = b\left(\frac{1}{x^2}\right) + a$$

Thus, $Y = \frac{y}{x}$, $X = \frac{1}{x^2}$, $m = b$ and the constant term, $c = a$.

Note: A general approach is to obtain a constant term, which is either a or b, as shown in the above example.

Example 6

Express the equation $y = \frac{x}{px + q}$, where p and q are constants, in the form $Y = mX + c$ where X and Y are each expressions in x and/or y.

Solution:

$$y = \frac{x}{px + q}$$

The equation can be rewritten as:

$$px + q = \frac{x}{y}$$

i.e. $$\frac{x}{y} = px + q$$

Thus $Y = \frac{x}{y}$, $X = x$, $m = p$ and $c = q$.

Alternatively, we have $$\frac{1}{y} = \frac{px + q}{x}$$

$$\frac{1}{y} = q\left(\frac{1}{x}\right) + p$$

Thus, $Y = \frac{1}{y}$, $X = \frac{1}{x}$, $m = q$ and $c = p$.

Example 7

The table shows experimental values of x and y which are known to be related by the equation $y = \frac{a}{x} + b\sqrt{x}$.

x	1	2	3	4	5	6
y	2.20	1.74	1.71	1.77	1.86	1.96

Explain how a straight line graph may be drawn to represent the given equation and draw it for the given data.
Use the graph to estimate the values of a and b.

Solution:

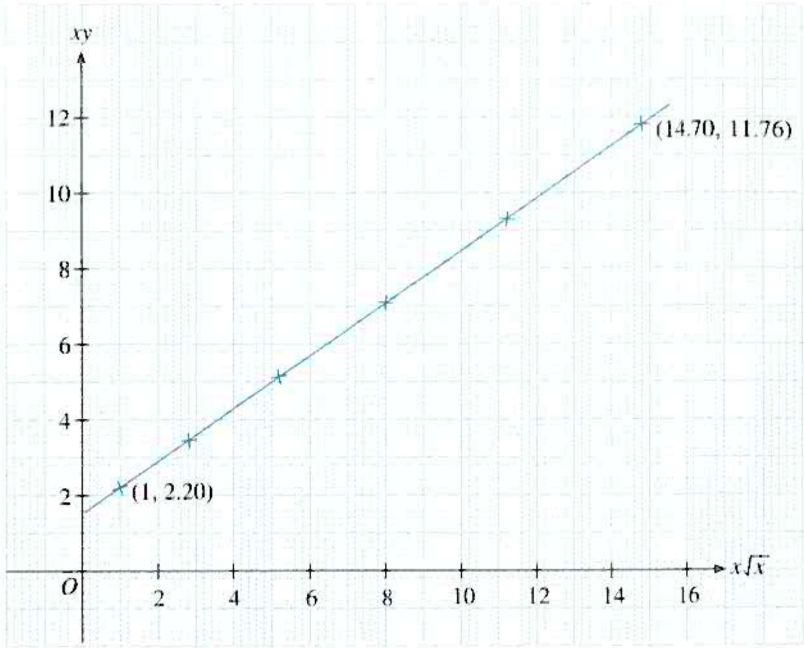
Multiplying the given equation throughout by x, we have

$$xy = bx\sqrt{x} + a$$

which is of the form $Y = mX + c$ where $Y = xy$, $X = x\sqrt{x}$, $m = b$ and $c = a$. So, X and Y are linearly related and the graph of xy against $x\sqrt{x}$ is a straight line. Then we have the following table:

X	$x\sqrt{x}$	1	2.83	5.20	8	11.18	14.70
Y	xy	2.20	3.48	5.13	7.08	9.30	11.76

The straight line graph is drawn as shown below.



Hence $b \approx \dfrac{11.76 - 2.20}{14.70 - 1}$ ***(gradient)***

≈ 0.70

and $a \approx 1.50$ ***(xy-intercept)***

The estimated values of a and b are **1.50** and **0.70** respectively.

Example 8 The population of a certain kind of bacteria in a test-tube is measured at hourly intervals.

Number of hours, x	0	1	2	3	4	5
Population, y	50	55	61	67	73	81

It is believed that y and x are related by an equation of the form $y = ab^x$ where a and b are constants. Express this equation in a form suitable for drawing a straight line graph. Draw this graph for the given data and use it to estimate the values of a and b. Give an expected population of the bacteria after 10 hours.

Solution:

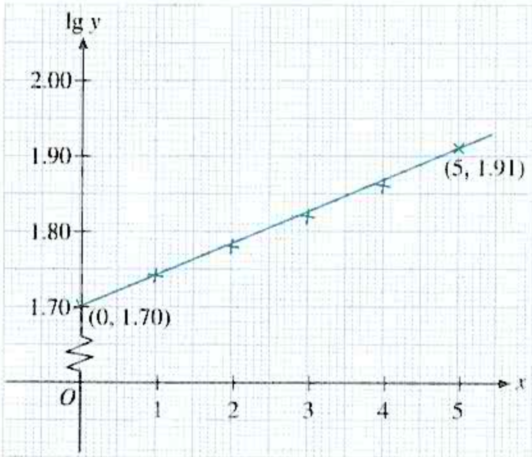
$y = ab^x \Rightarrow \lg y = \lg (ab^x)$
$= \lg a + \lg b^x$
$= \lg a + x \lg b$

Thus, $\lg y = (\lg b)x + \lg a$

which is of the form $Y = mX + c$ where $Y = \lg y$, $X = x$, $m = \lg b$ and $c = \lg a$.

To obtain a straight line graph, we plot $\lg y$ against x with the following table of calculated values.

x	0	1	2	3	4	5
$\lg y$	1.70	1.74	1.78	1.82	1.86	1.91



We have $\lg b \approx \dfrac{1.91 - 1.70}{5 - 0}$ *(gradient)*

≈ 0.042

$b \approx$ **1.1**

$\lg a \approx 1.7$ *(intercept on lg y-axis)*

$a \approx$ **50**

After 10 hours, $x = 10$.

Then $y \approx ab^x$

$\approx 50 \times (1.1)^{10}$

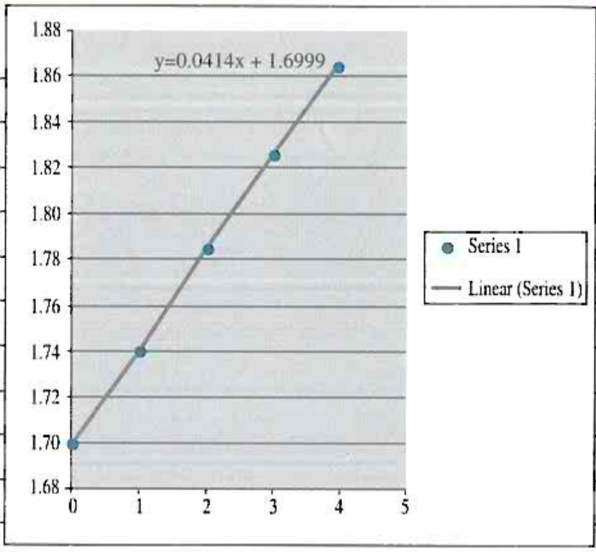
≈ 129.7

Therefore, the expected number of bacteria after 10 hours is **130**.

The following is an application of a spreadsheet program, Microsoft Excel, to compute and draw a linear graph from the data given in Example 8.

Use the spreadsheet program to study other examples and questions. You may want to calculate some estimated (predicted) values.

	A	B	C	D
1	x	y	$X=x$	$Y=LOG(y)$
2	0	50	0	1.70
3	1	55	1	1.74
4	2	61	2	1.79
5	3	67	3	1.83
6	4	73	4	1.86
7	5	81	5	1.91



Exercise 8.1

1. Each of figures (i) to (iv) shows part of a straight line graph obtained by plotting values of the variables indicated. For each case express y in terms of x.

(a)

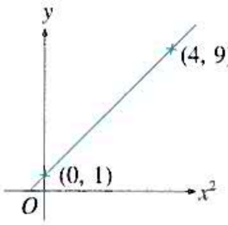


Fig. (i)

(b)

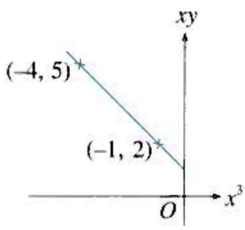


Fig. (ii)

(c)

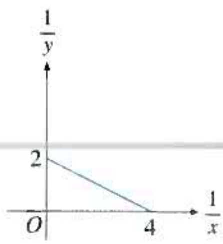


Fig. (iii)

(d)

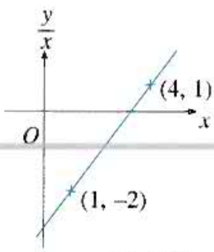


Fig. (iv)

2. Variables x and y are known to be related by an equation of the form $y = ax^2 + b$, where a and b are constants. Observed values of the two variables are shown in the following table.

x	1	2	3	4	5
y	6.2	5.6	4.6	3.2	1.4

By plotting y against x^2, draw a straight line graph and use it to estimate the value of a and of b.

3. The table shows experimental values of two variables x and y.

x	1	2	3	4	5
y	2.50	1.11	0.71	0.53	0.42

It is known that x and y are related by an equation of the form $\frac{1}{y} = ax + b$. By plotting $\frac{1}{y}$ against x, draw a straight line graph for the given data and use it to estimate the value of a and of b.

4. The pairs of values of x and y in the table below satisfy approximately the relationship $xy = h(x + k)$, where h and k are constants.

x	0.2	0.4	0.6	0.8	1.0
y	5.25	3.38	2.75	2.44	2.23

By plotting a graph of y against $\frac{1}{x}$, estimate the value of h and of k.

5. The data for x and y given in the table below are related by a law of the form $y = px^2 + x + q$, where p and q are constants.

x	1	2	3	4	5
y	41.5	38.0	31.5	22.0	9.5

By drawing a graph of $y - x$ against x^2, find estimates for p and q.

6. The following table gives values of N corresponding to some values of t.

t	1	2	3	4	5
N	1.33	1.01	0.92	0.89	0.87

Given that $\frac{a}{N} + \frac{b}{t} = 4$, plot a graph of $\frac{1}{N}$ against $\frac{1}{t}$ and use it to estimate the values of the constants a and b.

7. Express each of the following equations in a form suitable for drawing a straight line graph, stating the variables whose values are to be plotted. Explain how the corresponding unknown constants may be determined.

(a) $y = \dfrac{a}{b - x}$ (b) $y = p\sqrt{x} + qx$

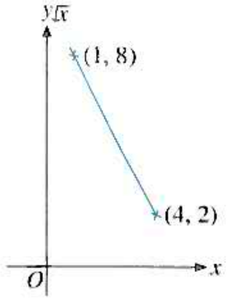
(c) $y = x^2 + ax + b$ (d) $y(y - a) = x - b$

(e) $mx^2 + ny^2 = x$ (f) $ay = x^b$

(g) $y = pq^{-x}$ (h) $e^y = px^2 - qx$

8. It is expected that the variables x and y are related by the equation $y = \lg(ax + b)$ where a and b are unknown constants. Given sets of experimental values of x and y, determine the variables whose values should be plotted in order to obtain a straight line graph, and explain how the graph may be used to determine the values of a and b.

9. The diagram shows part of a straight line graph drawn to represent the equation $y = \dfrac{h}{\sqrt{x}} + k\sqrt{x}$. Calculate the values of h and k.



10. When values of $\lg y$ are plotted against values of $\lg x$, a straight line is obtained passing through (0, 0.2) and (1.2, 3.4). Given that $y = ax^b$, find the values of a and b correct to 2 decimal places.

11. In an experiment, values of y are found for several values of x and the values are tabulated as shown below.

x	1	2	3	4
y	1.6	7.2	16.8	30.4

It is known that x and y are related by an equation of the form $y = ax^2 + bx$, where a and b are constants. Express this equation in a form suitable for drawing a straight line graph. Draw this graph for the given data and use it to find the values of a and b.

12. By drawing a suitable straight line graph, show that the corresponding values of x and y in the table below are approximately consistent with a law of the form $y = Cx + \dfrac{D}{x}$.

x	0.2	0.4	0.6	0.8
y	7.74	4.23	3.22	2.84

From your graph, estimate the values of C and D.

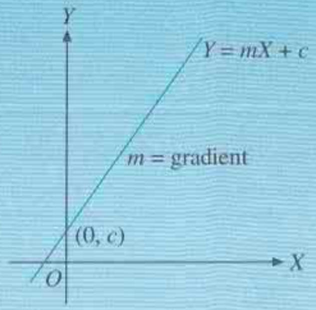
13. The table shows experimental values of two quantities, x and y, which are known to be connected by a law of the form $y = kb^x$.

x	1	2	3	4
y	30	75	190	470

Explain how a straight line graph may be drawn to represent the given equation and draw it for the given data. Use this graph to estimate the value of k and of b.

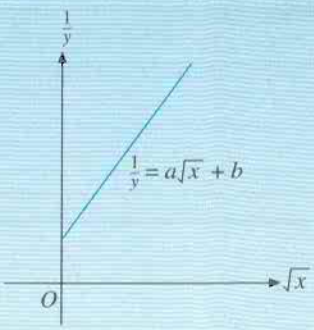
Important Notes

1. For two variables X and Y related by
$$Y = mX + c \quad \textit{(linear law)},$$
the graph is a straight line with
$$m = \text{gradient},$$
$$c = Y\text{-intercept}.$$
The values of m and c can be
(a) calculated if two points are given,
(b) estimated if experimental values of X and Y are used to plot the linear graph.

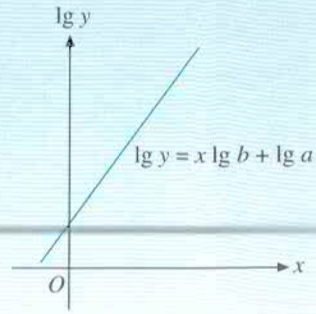


2. To apply the linear law for a non-linear equation in variables x and y, express the equation in the form
$$Y = mX + c$$
where X and Y are expressions in x and/or y.

(a) The expressions for X and Y are shown explicitly in the given equation in x and y.
For example, $\frac{1}{y} = a\sqrt{x} + b$
is of the form $Y = mX + c$
with $Y = \frac{1}{y}$, $X = \sqrt{x}$, $m = a$
and $c = b$.



(b) The expressions for X and Y are specified but not shown explicitly in the given equation in x and y.
For example, a linear graph with $\lg y$ against x is to be obtained from the equation
$$y = ab^x.$$
Since $\lg y = \lg (ab^x)$
$$= x \lg b + \lg a,$$
we have $Y = \lg y$, $X = x$,
$m = \lg b$ and $c = \lg a$.



Miscellaneous Examples

Example 9 Variables x and y are known to be related by an equation of the form $\frac{x+2}{a} + \frac{y^2}{b} = 1$, where a and b are constants.

Experimental values of x and y are shown in the following table. One of the values of y is subject to an abnormally large error.

x	1	2	3	4	5
y	2.65	3.00	3.32	3.71	3.87

Plot y^2 against $x + 2$ and use the graph to
(a) identify the abnormal reading and estimate its correct value,
(b) estimate the value of a and of b,
(c) estimate the value of x when $y = 2$.

Solution:

$$\frac{x+2}{a} + \frac{y^2}{b} = 1$$

Making y^2 the subject, we have

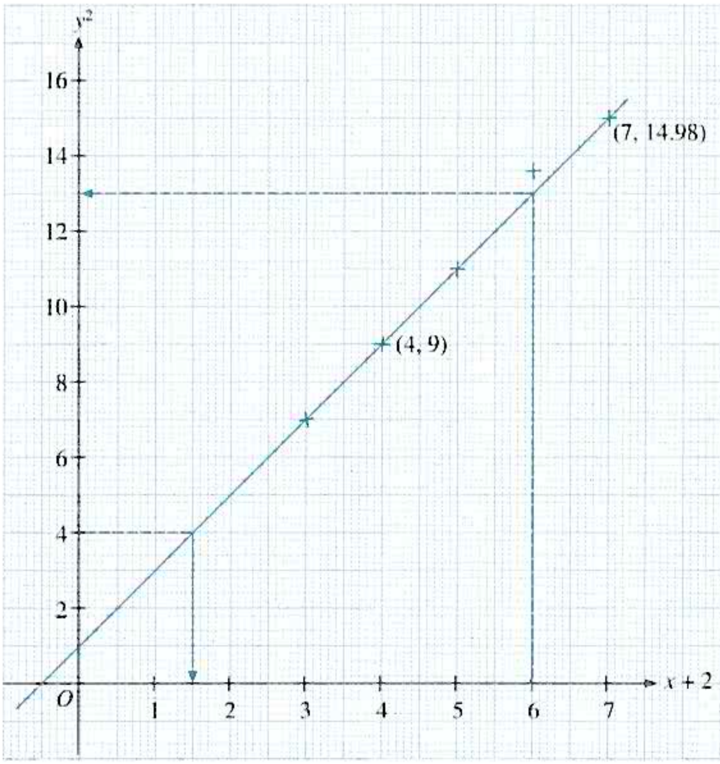
$$\frac{y^2}{b} = -\frac{x+2}{a} + 1$$

$$y^2 = -\frac{b}{a}(x + 2) + b$$

Plotting y^2 against $x + 2$ gives a straight line with gradient $= -\frac{b}{a}$ and y^2-intercept $= b$.

$x + 2$	3	4	5	6	7
y^2	7.02	9.00	11.02	13.76	14.98

Using these values, the graph is plotted as shown.



(a) From the graph, we observe that the abnormal reading for y is **3.71** at $x = 4$, i.e. $x + 2 = 6$.
Its correct value is given by $y^2 \approx 13$,
i.e. $y \approx$ **3.61**.

(b)
$$b = y^2\text{-intercept} \approx \mathbf{1}$$

$$-\frac{b}{a} = \text{gradient of the line} \approx \frac{14.98 - 9}{7 - 4} \approx 1.99$$

As $b \approx 1$, $-\frac{1}{a} \approx 2$
$a \approx$ **−0.5**

(c) When $y = 2$, $y^2 = 4$.
From the graph, $y^2 = 4 \Rightarrow x + 2 \approx 1.5$
$x \approx$ **−0.5**

Miscellaneous Exercise 8

1. Each of figures (i) and (ii) shows part of a straight line graph obtained by plotting values of the variables indicated, together with the coordinates of two points on the line. For each case express y as a function of x.

(a)

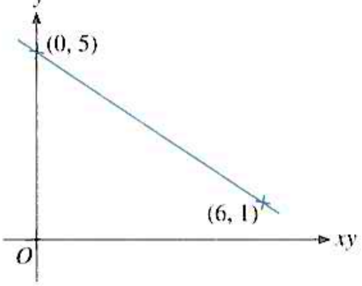


Fig. (i)

(b)

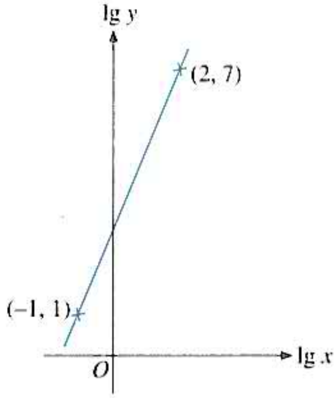
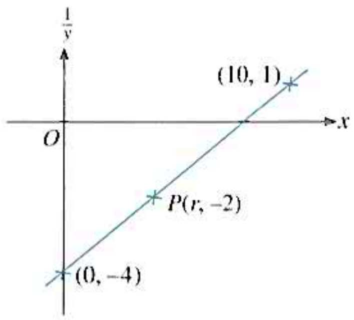


Fig. (ii)

2. The variables x and y are related by the equation $y = \frac{h}{2x + k}$. The diagram shows the graph of $\frac{1}{y}$ against x. Calculate the values of h and k. The point P lies on the line. Find the value of r.



3. Variables x and y are related by the equation $\frac{a}{x} + \frac{b}{y} = 2$, where a and b are constants. When the graph of $\frac{1}{y}$ against $\frac{1}{x}$ is drawn, a straight line is obtained. Given that the intercept on the $\frac{1}{y}$-axis is -0.5 and that the gradient of the line is 0.75, calculate the value of a and of b.

4. It is expected that the variables x and y are related by the equation $y = \sqrt{ax + b}$ where a and b are unknown constants. Given sets of experimental values of x and y, determine the variables whose values should be plotted in order to obtain a straight line graph, and explain how the graph may be used to determine the values of a and b.

5. The graph of x^2y against x is a straight line passing through $(-2, 1)$ and $(2, 7)$. Find

(a) the value of x when $y = \frac{10}{x^2}$,

(b) y in terms of x.

6. Variables x and y are related by an equation of the form $px^2 + qy^2 = 2$, where p and q are constants. Observed values of the two variables are shown below.

x	1	2	3	4	5
y	1.83	3.06	4.40	5.77	7.16

By plotting y^2 against x^2, draw a straight line graph and use it to estimate
(a) the value of p and of q,
(b) the positive value of x when $y = 5$.

7. The table shows experimental values of two variables x and y.

x	0.5	1.0	1.5	2.0
y	14.6	6.8	4.0	2.4

It is expected that x and y are related by an equation of the form $y = ax + \frac{b}{x}$ where a and b are constants. Express this equation in a form suitable for drawing a straight line graph.
Draw this graph for the given data and use it to estimate
(a) the value of a and of b,
(b) the value of y when $x = 1.7$.

8. The variables x and y are known to be connected by the equation $y = Ca^{-x}$. An experiment gave pairs of values of x and y as shown in the table. One of the values of y is subject to an abnormally large error.

x	1	2	3	4	5	6	7
y	56.20	29.90	25.10	8.91	6.31	3.35	1.78

Plot lg y against x and use the graph to
(a) identify the abnormal reading and estimate its correct value,
(b) estimate the value of C and of a,
(c) estimate the value of x when $y = 1$. *(C)*

9. Pairs of numerical values (x, y) are collected from an experiment and it is possible that either of the two following equations may be applicable to these data:
(a) $ax^2 + by^2 = 1$, where a and b are constants,
(b) $y = cx^d$, where c and d are constants.
In each case explain how you would use a graph to examine the validity of the equation. Explain how you would estimate the values of the constants if you found the equation to be approximately valid from your graph.

10. The table shows experimental values of two variables t and y.

t	1	2	3	4	5
y	12.2	7.0	4.0	2.3	1.3

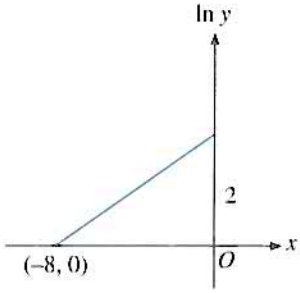
It is known that t and y are related by an equation of the form $y = Ae^{-bt}$. By plotting ln y against t, draw a straight line graph for the given data and use it to evaluate A and b.

11. It is assumed that x and y obey a law of the form $y = \ln(ax^2 + b)$. Pairs of values of x and y are recorded in an experiment and tabulated as follows:

x	0.2	0.4	0.6	0.8	1.0
y	0.322	0.482	0.703	0.948	1.194

By means of a straight line graph, verify that the law is valid.
Use your graph to estimate
(a) the values of a and b,
(b) the possible values of x if $y = \ln 3$,
(c) the value of y if $x = 0.1$.

12. The variables x and y are connected by the relation $y = k(ep)^x$, where k and p are constants. The graph of $\ln y$ against x is given. Find, to 3 significant figures, the values of k and p.



13. The period T of oscillation of a pendulum and its length l are related by a law of the form $T = al^b$.
Using the following experimental data, estimate
(a) a and b,
(b) the period of oscillation of a pendulum of length 0.9 m,
(c) the length of a pendulum whose period is 1 second.

l(metres)	0.4	0.6	0.8	1.0	1.2
T(seconds)	1.25	1.55	1.76	2.01	2.19

14. It is believed that variables x and y follow a law of the form $x + py = qxy$. In an experiment, values of y are found for certain values of x. These values are shown in the following table.

x	1	2	3	4	5
y	0.67	0.91	1.20	1.33	1.43

It is suspected that an unusually large error occurs in one of the values of y.

By plotting the graph of $\frac{x}{y}$ against x,
(a) identify the incorrect value of y and estimate its correct value,
(b) estimate the value of x when $5y = 4x$,
(c) estimate the values of p and q.

15. Variables x and y are known to be related by an equation of the form $axy - b = a(x^2 + bx)$, where a and b are constants. Observed values of the two variables are shown in the following table:

x	1	2	3	4
y	4	2.50	2.67	3.25

Plot $xy - x^2$ against x and use the graph to estimate

(a) the value of y when $x = 1.5$,

(b) the values of a and b.

16. The following values of x and y are believed to obey a law of the form $x^m y^n = 200$, where m and n are constants.

x	1	3	5	7
y	14.15	2.72	1.26	0.76

Rewrite the given law in a form suitable for drawing a straight line graph. Draw the graph and hence estimate

(a) the value of m and of n,

(b) the value of x when $y = 10$.

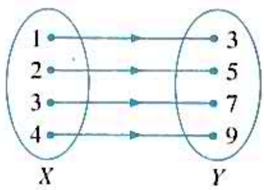
9 Functions

9.1 Introduction to Functions

Relations and Functions

The elements of the set $X = \{1, 2, 3, 4\}$ are associated with the elements of the set $Y = \{3, 5, 7, 9\}$ as depicted by the arrow diagram shown below. This association between X and Y is called a **relation** from X to Y.

The relation between the element 1 of X and the element 3 of Y is depicted by $1 \mapsto 3$. This indicates that 1 is the starting element and 3 is the ending element and thus we may say that 1 is mapped to 3. Similarly, we have $2 \mapsto 5$, $3 \mapsto 7$ and $4 \mapsto 9$. The relation satisfies the following condition:



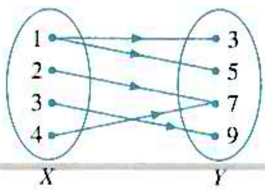
each element x of X is mapped to exactly one element y of Y

This relation is called a **function** or a **mapping**.

> **A function from X to Y is a relation which maps each element $x \in X$ to a unique element $y \in Y$.**

The diagram below shows another relation from X to Y.
We have:

$$\begin{aligned} 1 &\mapsto 3 \\ 1 &\mapsto 5 \\ 2 &\mapsto 7 \\ 3 &\mapsto 9 \\ 4 &\mapsto 7 \end{aligned}$$

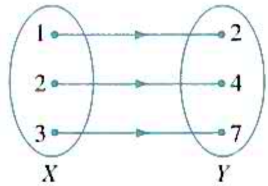


The element 1 is mapped to two elements of Y. Is this relation a function?

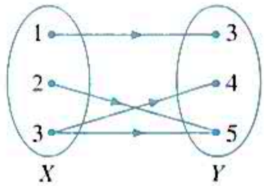
Example 1

Which of the following relations is not a function? State your reason.

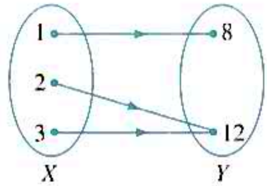
(a)



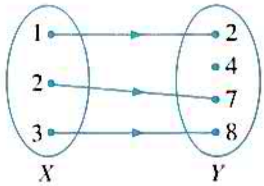
(b)



(c)



(d)



Solution:

(a) The relation is a function.
(b) The relation is not a function because $3 \mapsto 4$ and $3 \mapsto 5$.
(c) The relation is a function.
(d) The relation is a function.

Functions are generally denoted by the small letters f, g, h, … etc.

If f denotes the function from $X = \{1, 2, 3, 4\}$ to $Y = \{3, 5, 7, 9\}$ defined by:
$f : 1 \mapsto 3, \quad f : 2 \mapsto 5, \quad f : 3 \mapsto 7 \quad \text{and} \quad f : 4 \mapsto 9.$
The element 3 is called the **image** of 1. Similarly, 5, 7 and 9 are images of 2, 3 and 4 respectively.
Furthermore, any element x of X is related to an element y of Y by the **rule** $y = 2x + 1$. Therefore, the function can be simply defined by using the rule as

$$f : x \mapsto y, \quad \text{where } y = 2x + 1$$

or

$$f : x \mapsto 2x + 1.$$

When we write $f(x) = 2x + 1$, the image of 1 is $f(1) = 3$ whereas $f(x)$ is the image of x. For this function, each element of $X = \{1, 2, 3, 4\}$ must be a starting element and the set is called the **domain** of f. The set of all images $R = \{3, 5, 7, 9\}$ is called the **range** or **image set** of f.

Example 2

A function f is defined by $f : x \mapsto 2x^2 + 1$ with domain $X = \{-1, 2, 3\}$. Find the image set of f.

Solution:

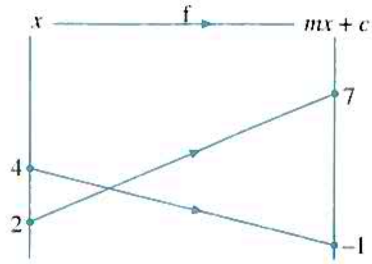
$f(x) = 2x^2 + 1$
The images of -1, 2 and 3 are:

$$f(-1) = 2(-1)^2 + 1 = 3$$
$$f(2) = 2(2)^2 + 1 = 9$$
$$f(3) = 2(3)^2 + 1 = 19$$

The image set is $R = \mathbf{\{3, 9, 19\}}$.

Example 3 The diagram shows part of the mapping $f : x \mapsto mx + c$. Find

(a) the value of m and of c,
(b) the image of 5 under f,
(c) the element whose image is 3.



Solution:

(a) $f(x) = mx + c$

$f : 2 \mapsto 7 \quad \Rightarrow \quad f(2) = 7$

i.e. $\quad 2m + c = 7$ ....(1)

$f : 4 \mapsto -1 \quad \Rightarrow \quad f(4) = -1$

i.e. $\quad 4m + c = -1$ ....(2)

Solving (1) and (2), we get: $m = -4$ and $c = 15$

(b) $f(x) = -4x + 15$

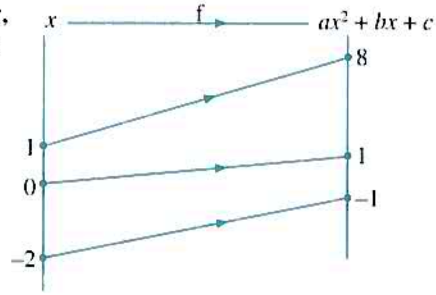
The image of 5 is $f(5) = -4 \times 5 + 15 = -5$.

(c) Let x be the element whose image is 3.

Then $f(x) = 3 \Rightarrow -4x + 15 = 3 \Rightarrow x = 3$

Example 4 The diagram shows part of the mapping f. Find

(a) the values of a, b and c,
(b) the possible values of k whose image is $2k$.



Solution:

(a) $f : 1 \mapsto 8 \quad \Rightarrow \quad f(1) = 8$

$a + b + c = 8$ ....(1)

$f : 0 \mapsto 1 \quad \Rightarrow \quad f(0) = 1$

$c = 1$ .... (2)

$f : -2 \mapsto -1 \quad \Rightarrow \quad f(-2) = -1$

$4a - 2b + c = -1$ .... (3)

Solving (1), (2) and (3), we get:

$a = 2$, $b = 5$ and $c = 1$

(b) From (a), $\quad f(x) = 2x^2 + 5x + 1$

$f : k \mapsto 2k \quad \Rightarrow \quad f(k) = 2k$

i.e. $\quad 2k^2 + 5k + 1 = 2k$

$2k^2 + 3k + 1 = 0$

$(2k + 1)(k + 1) = 0$

$k = -\frac{1}{2}$ or -1

Hence the possible values of k are $-\frac{1}{2}$ and -1.

Example 5 A function f is defined by $f : x \mapsto x^2 + 5x - 5$ for $x > 0$. Find the value of x which is unchanged by the mapping.

Solution: Since the image of x is x,

$$\begin{aligned} f(x) &= x \\ \text{i.e.} \quad x^2 + 5x - 5 &= x \\ x^2 + 4x - 5 &= 0 \\ (x + 5)(x - 1) &= 0 \\ x &= -5 \text{ or } 1 \end{aligned}$$

Since -5 is not in the domain, the possible value is **1**.

Expressions such as $\dfrac{1}{x-2}$ and $\dfrac{2x+1}{x-2}$ are not defined when $x = 2$. So the functions

$$f : x \mapsto \frac{1}{x-2} \quad \text{and} \quad g : x \mapsto \frac{2x+1}{x-2}$$

can be defined for all values of x except $x = 2$ and thus we write

$$f : x \mapsto \frac{1}{x-2}, x \neq 2 \quad \text{and} \quad g : x \mapsto \frac{2x+1}{x-2}, x \neq 2.$$

Hence the element 2 has no images under the functions f and g.

Example 6 A function f is defined by $f : x \mapsto 3x + \dfrac{5}{x}$, $x \neq 0$. Calculate
(a) the image of 5 under f,
(b) the possible values of x whose image is 8.

Solution: $f(x) = 3x + \dfrac{5}{x}$

(a) The image of 5 is $f(5) = 3 \times 5 + \dfrac{5}{5} = \mathbf{16}$

(b)

$$\begin{aligned} f(x) &= 8 \\ 3x + \frac{5}{x} &= 8 \\ 3x^2 + 5 &= 8x \\ 3x^2 - 8x + 5 &= 0 \\ (3x - 5)(x - 1) &= 0 \\ x &= \frac{5}{3} \text{ or } 1 \end{aligned}$$

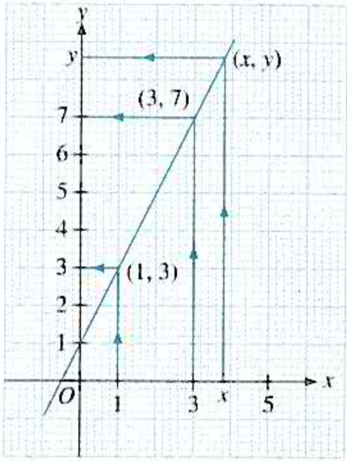
The possible values of x are $\mathbf{\frac{5}{3}}$ and **1**.

Note: In this example, the elements $\dfrac{5}{3}$ and 1 have the same image 8.

Graph of a Function

A useful representation of the function $f : x \mapsto 2x + 1$ is the Cartesian graph of the straight line $y = 2x + 1$.

The mappings $f : 1 \mapsto 3$ and $f : 3 \mapsto 7$ are shown with the corresponding points (1, 3) and (3, 7) in the diagram.

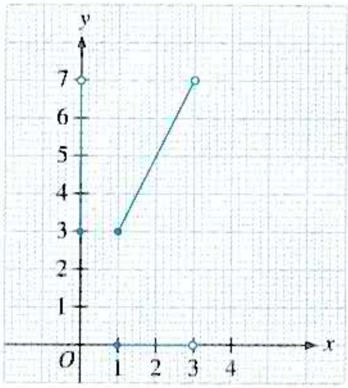


An immediate observation is that each point (x, y) on the line corresponds to a mapping

$$f : x \mapsto y$$

which maps a number x on the x-axis to a number y on the y-axis.

For a function $g : x \mapsto 2x + 1$ with domain $\{x : 1 \leqslant x < 3, x \in \mathbb{R}\}$, the graph is a line segment on the line as shown.



With $g(1) = 3$ and $g(3) = 7$, the end points are (1, 3) and (3, 7) and so the range of g is

$$\{y : 3 \leqslant y < 7, y \in \mathbb{R}\}$$

as shown.

f is a function $\Leftrightarrow$ each number x on the x-axis is mapped to exactly one number on the y-axis, i.e. any vertical line intersects the graph of f at exactly one point.

Example 7 A function is defined by $g : x \mapsto 3 - 2x$. With the aid of the graph of $y = 3 - 2x$, find

(a) the range of g corresponding to the domain
$$\{x : -1 \leqslant x \leqslant 2, x \in \mathbb{R}\},$$
(b) the domain of g corresponding to the range
$$\{y : -3 \leqslant y \leqslant 2, y \in \mathbb{R}\}.$$

Solution:

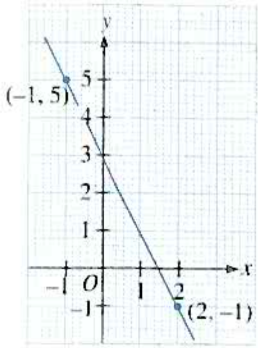
(a) $g(x) = 3 - 2x$

$g(-1) = 3 - 2(-1)$
$= 5$

$g(2) = 3 - 2(2)$
$= -1$

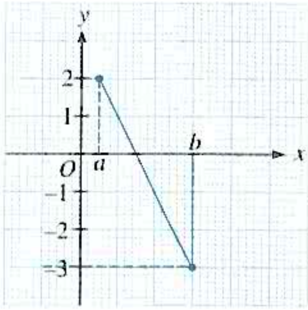
The graph of g is part of the line $y = 3 - 2x$ with end points $(-1, 5)$ and $(2, -1)$ as shown.

The range is $\{y : -1 \leqslant y \leqslant 5, y \in \mathbb{R}\}$.



(b) Refer to the diagram. Note that the corresponding domain is $\{x : a \leqslant x \leqslant b, x \in \mathbb{R}\}$ where $g(a) = 2$ and $g(b) = -3$.

$$g(a) = 2$$
$$3 - 2a = 2$$
$$a = \frac{1}{2}$$
$$g(b) = -3$$
$$3 - 2b = -3$$
$$b = 3$$



The domain is $\left\{x : \frac{1}{2} \leqslant x \leqslant 3, x \in \mathbb{R}\right\}$.

For simplicity, we shall mention domain and range as intervals in subsequent examples.

Example 8

A function h is defined by $h : x \mapsto x^2 - 4x + 5$. Find

(a) the range of h corresponding to the domain $1 \leqslant x < 4$,

(b) a domain of h corresponding to the range $2 \leqslant h(x) < 5$.

Solution:

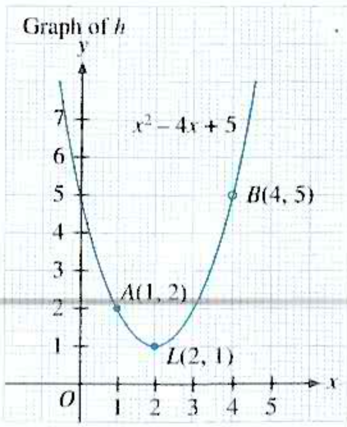
(a) $h(x) = x^2 - 4x + 5$
$= (x - 2)^2 + 1$

The graph of h has the lowest point $L(2, 1)$ as shown.

Since $h(1) = 2$ and $h(4) = 5$, we have the end points $A(1, 2)$ and $B(4, 5)$.

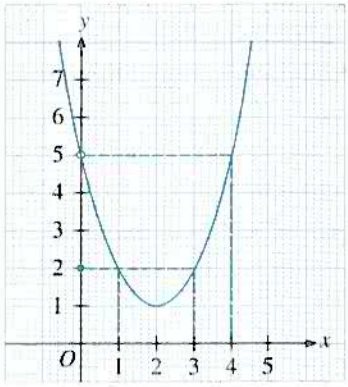
Noting that $x \neq 4$, $B(4, 5)$ is not a point on the graph of h.

From the graph, the range of h is $1 \leqslant h(x) < 5$.

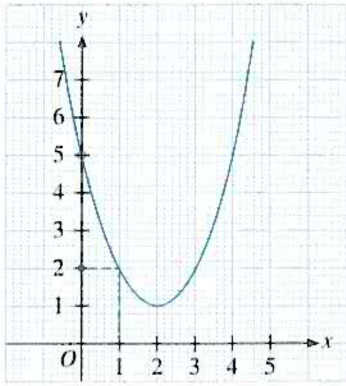


(b) For $h(x) = 2$, $x^2 - 4x + 5 = 2$
$$(x-1)(x-3) = 0$$
$$x = 1 \text{ or } 3$$
For $h(x) = 5$, $x^2 - 4x + 5 = 5$
$$x(x-4) = 0$$
$$x = 0 \text{ or } 4$$

Noting that $2 \leqslant h(x) < 5$, a corresponding domain is $3 \leqslant x < 4$ as shown in the diagram.



Note: Another corresponding domain is $0 < x \leqslant 1$ as shown.

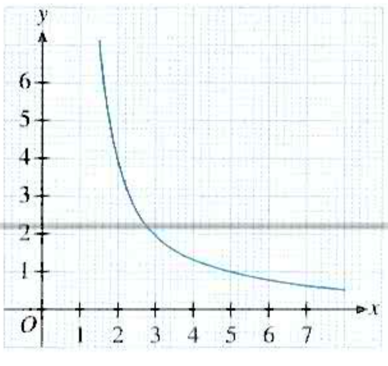


Example 9

The diagram shows the graph of the curve $y = \dfrac{4}{x-1}$ for $x > 1$. With the aid of the graph, find

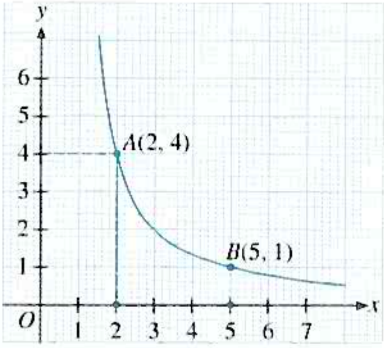
(a) the range of $f : x \mapsto \dfrac{4}{x-1}$ for the domain $2 < x < 5$,

(b) the domain of f defined by $f(x) = \dfrac{4}{x-1}$ corresponding to the range $1 \leqslant f(x) \leqslant 3$.



Solution: (a) $f(2) = \dfrac{4}{2-1} = 4$, $f(5) = \dfrac{4}{5-1} = 1$

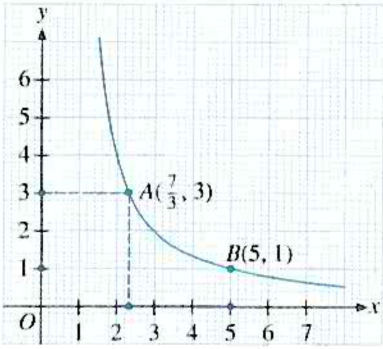
With $A(2, 4)$ and $B(5, 1)$ as two exclusive end points, the range of f is $1 < f(x) < 4$.



(b) From (a), we have $f(5) = 1$. For $f(x) = 3$, $\dfrac{4}{x-1} = 3$

$$x = \frac{7}{3}$$

i.e. $f\left(\frac{7}{3}\right) = 3$

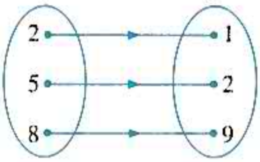


The domain of f is $\frac{7}{3} \leqslant x \leqslant 5$.

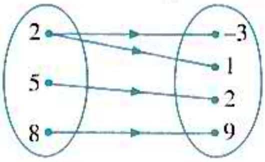
Exercise 9.1

1. Which of the following relations is not a function? State your reason.

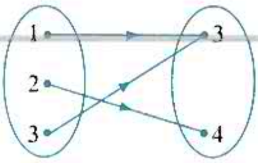
(a)



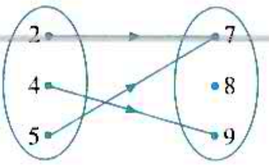
(b)



(c)

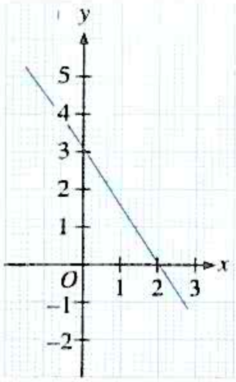


(d)

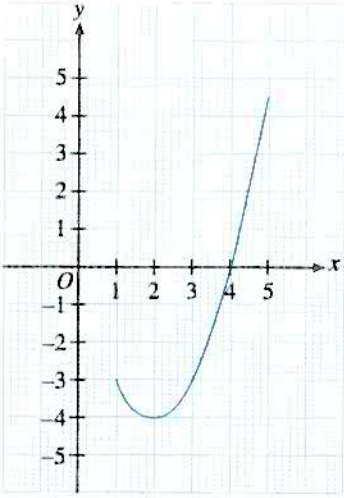


2. From the graph of each of the following relations, determine whether the relation is a function. State your reason.

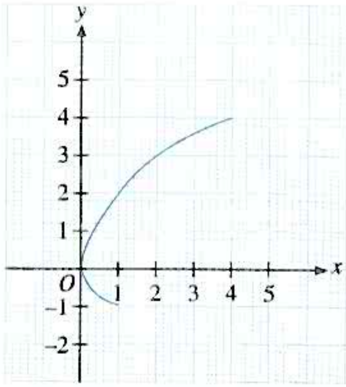
(a)



(b)



(c)



3. (a) A function f is defined by $f : x \mapsto 4x + 2$ for the domain $D = \{-1, 3, 5\}$. Find the image set of f.

   (b) A function f is defined by $f : x \mapsto \dfrac{a}{x - a}$, $x \neq a$. Given that $f(2) = 3$, find the value of a and of $f(3)$.

4. A function g is defined by $g : x \mapsto 3 + \dfrac{6}{x - 1}$, $x \neq 1$.

   (a) Find the images of -2, $\dfrac{1}{2}$ and $\dfrac{5}{4}$.

   (b) Given that the image of a is $2a$, find the possible values of a.

5. A function f is defined by $f : x \mapsto ax + b$. The images of 1 and 5 are -2 and 10 respectively.

   (a) Calculate the value of a and of b.

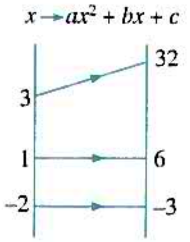
   (b) If the domain of f is $X = \{1, 3, 5\}$, find the range of f.

6. The arrow diagram shows part of the function $f : x \mapsto ax^2 + bx + c$. Find

   (a) the values of a, b and c,

   (b) the positive number x whose image is 2,

   (c) the set of numbers whose image under f is 2.



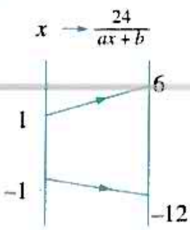
7. The arrow diagram shows part of the mapping

   $f : x \mapsto \dfrac{24}{ax + b}$, $x \neq -\dfrac{b}{a}$, $a \neq 0$.

   (a) Find the value of a and of b.

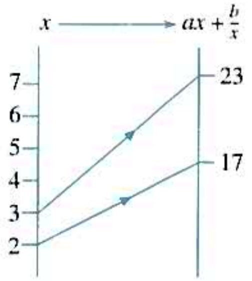
   (b) Find the element that has an image of 8 under this mapping.

   (c) Find the two values of x for which $f(x) = x$.



8. Given the function $f : x \mapsto \dfrac{2}{ax + b}$, $x \neq -\dfrac{b}{a}$ such that $f(0) = -2$ and $f(2) = 2$, find
   (a) the value of a and of b,
   (b) the values of x for which $f(x) = x$.
   Show that $f(p) + f(-p) = 2f(p^2)$.

9. The arrow diagram represents part of the mapping $f : x \mapsto ax + \dfrac{b}{x}$, $x \neq 0$. Find
   (a) the value of a and of b,
   (b) the image of 1 under this mapping.



10. For each of the following functions
   (a) $f : x \mapsto 5x - 1$, **(b)** $f : x \mapsto 2x + 1$,
   (c) $f : x \mapsto 5 - x$, **(d)** $f : x \mapsto 7 - 2x$,
   where $x \in \mathbb{R}$, find
   (i) the range corresponding to a domain of $1 \leqslant x \leqslant 3$,
   (ii) the domain corresponding to a range of $-1 \leqslant f(x) \leqslant 5$.

11. The function f is defined by $f : x \mapsto x^2 - 2x$ for the domain $-2 \leqslant x \leqslant 0$.
   (a) Find the range of f.
   (b) State another domain for which f has the same range.

12. **(a)** For a function f defined by $f : x \mapsto 4x^2 - 3$, find a domain of x corresponding to the range $1 < f(x) < 13$.
   (b) Given that $g : x \mapsto (x + 1)^2$, find the range of g corresponding to the domain
   (i) $0 \leqslant x \leqslant 2$, **(ii)** $-2 \leqslant x \leqslant 2$.

13. A function f is defined by $f : x \mapsto 2(x - 1)^2 + 1$.
   (a) Find the range corresponding to the domain $0 \leqslant x \leqslant 3$.
   (b) Find a domain of x corresponding to the range $3 \leqslant f(x) \leqslant 5$.

14. Two functions f and g are defined by $f : x \mapsto \dfrac{1}{x + 1}$, $x \neq -1$, and $g : x \mapsto \dfrac{x}{x - 2}$, $x \neq 2$. Find the values of x for which $f(x) = 10g(x)$.

15. Two functions f and g are defined by $f : x \mapsto \dfrac{3}{ax - b}$, $x \neq \dfrac{b}{a}$ and $g : x \mapsto bx + a$, where a and b are positive constants. Given that $f(1) = g(1)$ and $f(2)g(2) = 4$, find the value of a and of b.

16. A function g is defined by $g : x \mapsto 3x - 1$. Find, in terms of a, the expressions
   (a) $g(a + 1)$, **(b)** $g(3a - 1)$,
   (c) $g(2a^2 - 1)$, **(d)** $g\left(\dfrac{a}{a + 1}\right)$.

17. Two functions are defined by $f : x \mapsto ax + 1$ and $g : x \mapsto \dfrac{4b}{x - 1}$, $x \neq 1$, where a and b are constants. Given that $f(a) = g(b)$ and $f\left(\dfrac{1}{a}\right) = g\left(\dfrac{1}{b}\right)$, find the possible values of a and b.

9.2 Composite Functions

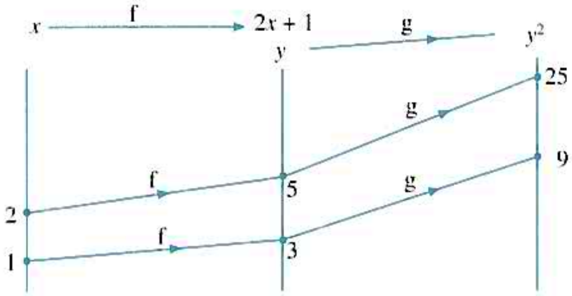
Let f and g be two functions defined by

$$f : x \mapsto 2x + 1$$

and

$$g : x \mapsto x^2.$$

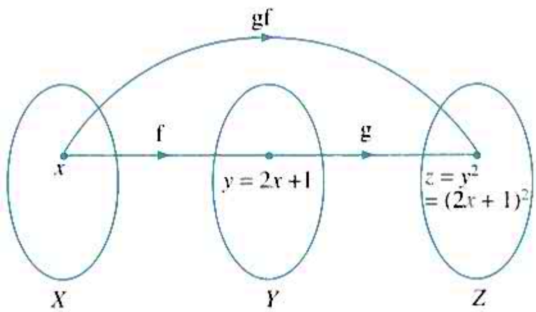
The diagram shows part of the mapping under f followed by g.



We have:

$$1 \overset{f}{\mapsto} 2 \times 1 + 1 = 3 \overset{g}{\mapsto} 3^2 = 9$$
$$2 \overset{f}{\mapsto} 2 \times 2 + 1 = 5 \overset{g}{\mapsto} 5^2 = 25$$
$$x \overset{f}{\mapsto} 2x + 1 = y \overset{g}{\mapsto} y^2 = (2x + 1)^2$$

Following the pattern, we obtain the arrow diagram as shown.



As we can see, there is a direct mapping from an element $x \in X$ to an element $z \in Z$ defined by $x \mapsto (2x + 1)^2$.

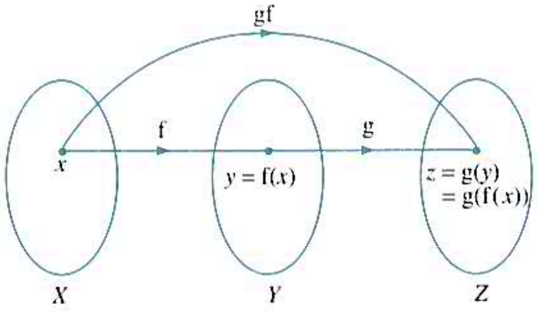
Noting that the image of x is $g(f(x)) = g(2x + 1) = (2x + 1)^2$, the new function is called a **composite function** of g and f and is denoted by gf. So we write

$$gf : x \mapsto (2x + 1)^2$$

and

$$gf(x) = (2x + 1)^2.$$

In general, we have $gf(x) = g(f(x))$ as shown:



For the above functions, we define the composite function gf where

$$\begin{aligned} gf(x) &= g(f(x)) \\ &= g(2x + 1) \\ &= (2x + 1)^2 \end{aligned}$$

or $$gf : x \mapsto (2x + 1)^2.$$

Similarly, we can define another composite function fg where

$$\begin{aligned} fg(x) &= f(g(x)) \\ &= f(x^2) \\ &= 2x^2 + 1 \end{aligned}$$

or $$fg : x \mapsto 2x^2 + 1.$$

Obviously, the two composite functions gf and fg are different functions.

With $ff(x) = f(f(x)) = f(2x + 1) = 2(2x + 1) + 1 = 4x + 3$, we can define another composite function $ff : x \mapsto 4x + 3$

or simply $f^2 : x \mapsto 4x + 3$.

Similarly, $gg(x) = g(g(x)) = g(x^2) = x^4$,

i.e. $gg : x \mapsto x^4$

or $g^2 : x \mapsto x^4$.

Example 10 Two functions f and g are defined by $f : x \mapsto 2x + 1$ and $g : x \mapsto x^2 - 1$.

(a) Express in similar form the functions gf, fg and f^2.

(b) Find the values of x for which $fg(x) = 7$.

(c) Find the values of x for which $gf(x) = g(x)$.

Solution:

(a) $f(x) = 2x + 1$ and $g(x) = x^2 - 1$.

For gf, $gf(x) = g(f(x)) = g(2x + 1)$

$$\begin{aligned} &= (2x + 1)^2 - 1 \\ &= 4x^2 + 4x \end{aligned}$$

Hence $\mathbf{gf : x \mapsto 4x^2 + 4x}$.

For fg, $fg(x) = f(g(x)) = f(x^2 - 1)$
$= 2(x^2 - 1) + 1$
$= 2x^2 - 1$

Hence $\mathbf{fg : x \mapsto 2x^2 - 1}$.

For f^2, $ff(x) = f(f(x)) = f(2x + 1)$
$= 2(2x + 1) + 1$
$= 4x + 3$

Hence $\mathbf{f^2 : x \mapsto 4x + 3}$.

(b)

$$\begin{aligned} fg(x) &= 7 \\ 2x^2 - 1 &= 7 \\ x^2 &= 4 \end{aligned}$$

$$x = -2 \quad \text{or} \quad x = 2$$

The values of x for which $fg(x) = 7$ are $\mathbf{-2}$ and $\mathbf{2}$.

(c)

$$\begin{aligned} gf(x) &= g(x) \\ 4x^2 + 4x &= x^2 - 1 \\ 3x^2 + 4x + 1 &= 0 \\ (3x + 1)(x + 1) &= 0 \end{aligned}$$

$$x = -\frac{1}{3} \quad \text{or} \quad -1$$

The values of x for which $gf(x) = g(x)$ are $\mathbf{-\frac{1}{3}}$ and $\mathbf{-1}$.

Example 11

A function f is defined by $f : x \mapsto \dfrac{2x}{x - 1}$, $x \neq 1$.

(a) Obtain expressions for f^2 and f^3.

(b) State the values of x for which the functions f^2 and f^3 are not defined.

Solution:

(a) $f(x) = \dfrac{2x}{x - 1}$, $x \neq 1$

$$\begin{aligned} f^2(x) &= f(f(x)) \\ &= f\left(\frac{2x}{x-1}\right), \ x \neq 1 \\ &= \frac{2\left(\frac{2x}{x-1}\right)}{\left(\frac{2x}{x-1}\right) - 1} \\ &= \frac{\frac{4x}{x-1}}{\frac{2x - (x-1)}{x-1}} \\ &= \mathbf{\frac{4x}{x+1}, \ x \neq -1} \end{aligned}$$

$$\begin{aligned} f^3(x) &= ff^2(x) \\ &= f(f^2(x)) \\ &= f\left(\frac{4x}{x+1}\right), x \neq 1, -1 \\ &= \frac{2\left(\frac{4x}{x+1}\right)}{\left(\frac{4x}{x+1}\right) - 1} \\ &= \frac{\frac{8x}{x+1}}{\frac{4x - (x+1)}{x+1}} \\ &= \frac{8x}{3x-1}, x \neq \frac{1}{3} \end{aligned}$$

(b) f^2 is not defined for $x = 1$ and $x = -1$.

f^3 is not defined for $x = 1$, $x = -1$ and $x = \frac{1}{3}$.

Exercise 9.2

1. For each of the following pairs of functions, obtain expressions in the same form for gf and fg.

(a) $f : x \mapsto 3x$, $\quad g : x \mapsto 3 - 2x$

(b) $f : x \mapsto 2x + 1$, $\quad g : x \mapsto 2 - x^2$

(c) $f : x \mapsto x - 4$, $\quad g : x \mapsto \frac{2}{x}, x \neq 0$

(d) $f : x \mapsto 1 + 2x$, $\quad g : x \mapsto \frac{x}{x-1}, x \neq 1$

2. For each of the following functions, obtain expressions in the same form for f^2 and f^3.

(a) $f : x \mapsto 2x + 3$

(b) $f : x \mapsto \frac{3x}{x-1}, x \neq 1$

(c) $f : x \mapsto \frac{x}{x-1}, x \neq 1$

(d) $f : x \mapsto \frac{3}{2x-1}, x \neq \frac{1}{2}$

3. Two functions f and g are defined by $f : x \mapsto 3x + 4$ and $g : x \mapsto x^2 + 6$. Using this notation, obtain expressions for fg and gf. Find the values of x for which

(a) f = g, $\quad$ **(b)** fg = gf.

4. **(a)** Given that $f : x \mapsto 3x + 2$ for the domain $-1 \leqslant x \leqslant 2$, find f^2 in similar form and state the range of f^2.

(b) Given that $f : x \mapsto x^2$ for the domain $-1 \leqslant x \leqslant 2$, find f^2 in similar form and state the range of f^2.

5. Two functions f and g are defined by $f : x \mapsto 2x + p$ and $g : x \mapsto 3x + q$. If fg = gf for all values of x, find the relation between p and q. Given that $p = 3$, find the values of x for which
   (a) $f^2 = g$, **(b)** $g^2 = f$.

6. Two functions are defined by $f : x \mapsto 2x - 1$ and $g : x \mapsto \frac{x}{x+1}$, $x \neq 1$. Obtain expressions in the same form for fg and gf. Find the value of x for which fg = gf.

7. A function f is defined by $f : x \mapsto x + 1$. Another function g is defined such that $gf : x \mapsto x^2 + 2x + 5$. Express $gf(x)$ in the form $a(x + 1)^2 + b$ and hence write down the expression for $g(x)$.

8. Given that $f : x \mapsto ax + b$ and $f^3 : x \mapsto 27x + 26$, find
   (a) the value of a and of b, **(b)** an expression for f^4.

9. Functions f and g are defined by $f : x \mapsto \frac{x}{x-1}$, $x \neq 1$, and $g : x \mapsto ax + b$, where a and b are constants.
   (a) Obtain an expression for f^2.
   (b) Given that $g(3) = 8$ and $gf(2) = 5$, calculate the value of a and of b.

10. Functions f and g are defined by $f : x \mapsto \frac{2}{x-3}$, $x \neq 3$, and $g : x \mapsto ax^2 + b$, where a and b are constants.
    (a) Given that $g(2) = 5$ and $gf(1) = -1$, calculate the value of a and of b.
    (b) Obtain an expression for gf.

11. Two functions are defined by $f : x \mapsto \frac{1}{x+1}$, $x \neq -1$, $x \neq -\frac{1}{2}$, and $g : x \mapsto \frac{x}{x-2}$, $x \neq 2$, $x \neq 1$. Obtain expressions in similar form for fg and gf. Find the values of x for which $fg(x) + gf(x) = 0$.

12. The function f is defined by $f : x \mapsto \frac{x+1}{x-1}$, $x \neq 1$. Show that $f^2(x) = x$, $x \neq 1$, and hence express $f^5(x)$ and $f^{10}(x)$ in their simplest forms.

13. Two functions are defined by $f : x \mapsto x - 2$ and $g : x \mapsto \frac{2}{x}$, $x \neq 0$. Express in similar form,
    (a) f^2, **(b)** f^5, **(c)** g^2, **(d)** g^5.

14. A function $f : x \mapsto \frac{b}{x-a}$, $a > 0$, is such that $f(b) = b$ and $f(2a) = 2a$. Find the value of a and of b. If $ff(x) = x$, show that $x^2 - x - 2 = 0$.

15. A function f is defined by $f : x \mapsto \frac{2}{x} + k$, $x \neq 0$, where k is a constant. Given that $f^2(2) = \frac{2}{3}f(1)$, calculate the possible values of k.

*16. A function f is defined by $f : x \mapsto \frac{x}{x+1}$, $x \neq -1$. Prove that $f^2(x) = \frac{x}{2x+1}$, $x \neq -1$, $x \neq -\frac{1}{2}$, and obtain a similar expression for $f^3(x)$. Hence suggest a possible expression for $f^n(x)$.

9.3 Inverse Functions

One-one Functions

The diagrams below show two given functions f and g respectively.

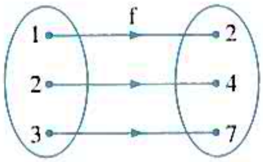


Fig. (i)

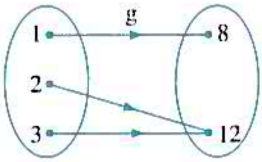


Fig. (ii)

For the function f in Fig. (i), each element y of Y is the image of only one element x of X and so the function f is called a **one-one function**.

For the function g in Fig. (ii), the element 12 of Y is the image of two elements, 2 and 3, of X and so the function f is not a one-one function.

Reversing the arrows, we have two relations depicted by the arrow diagrams as shown.

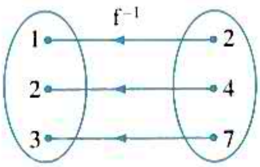


Fig. (iii)

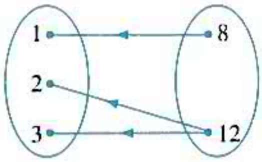


Fig. (iv)

From the reverse arrow diagram of a one-one function f, the relation depicted in Fig. (iii) is a function. This function is called the inverse of f and is denoted by f^{-1}. But the reverse mapping of the non one-one function g depicted in Fig. (iv) is not a function.

The association between the one-one function and its inverse f^{-1} is clearly shown as follows:

Under f		Under f^{-1}
$f(1) = 2$	$\Leftrightarrow$	$f^{-1}(2) = 1$
$f(2) = 4$	$\Leftrightarrow$	$f^{-1}(4) = 2$
$f(3) = 7$	$\Leftrightarrow$	$f^{-1}(7) = 3$
$f(a) = b$	$\Leftrightarrow$	$f^{-1}(b) = a$

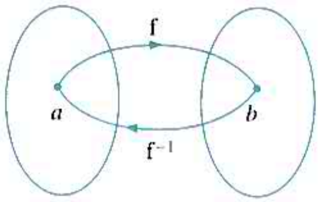
If f is a one-one function:

(1) f has an inverse function f^{-1}

(2) b is the image of a under f $\Leftrightarrow$ a is the image of b under f^{-1}.

i.e. $b = \mathrm{f}(a) \Leftrightarrow a = \mathrm{f}^{-1}(b)$

The arrow diagram shows that
$b = \mathrm{f}(a) \quad \Leftrightarrow \quad a = \mathrm{f}^{-1}(b)$.



Example 12 A function f is defined by $\mathrm{f} : x$ a $\dfrac{x}{x-2}$, $x \neq 2$. Find

(a) f(5), **(b)** $\mathrm{f}^{-1}(2)$.

Solution:

(a) $\mathrm{f}(x) = \dfrac{x}{x-2}, x \neq 2$

$$\mathrm{f}(5) = \frac{5}{5-2} = \frac{5}{3} = \mathbf{1\tfrac{2}{3}}$$

(b) Let $a = \mathrm{f}^{-1}(2)$

Then $\mathrm{f}(a) = 2$

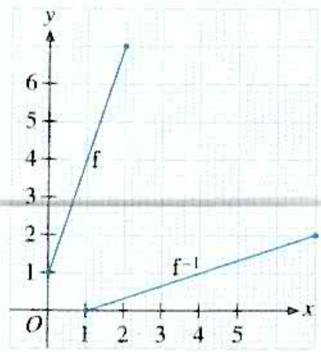
$$\frac{a}{a-2} = 2$$
$$a = 2a - 4$$
$$a = 4$$
$$\therefore \mathrm{f}^{-1}(2) = \mathbf{4}$$

Example 13 Let f be a function defined by $\mathrm{f} : x$ a $3x + 1$ for $0 \leqslant x \leqslant 2$.

(a) Sketch the graph of f and state the range of f.

(b) Show that f is a one-one function.

(c) Find f^{-1} in similar form and sketch the graph of f^{-1}.

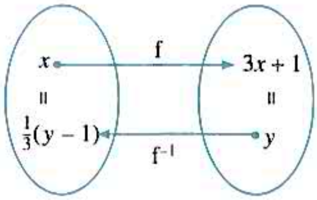


Solution:

(a) From the graph of f with end points at (0, 1) and (2, 7), the range of f is
$\boldsymbol{R = \{y : 1 \leqslant y \leqslant 7\}}$.

(b) For each $y \in R$, y is the image of only one $x \in X$ as shown. This shows that f is a one-one function.

(c) Let y be the image of x under f. Then



$$y = 3x + 1$$
$$x = \frac{1}{3}(y - 1)$$

The inverse function is
$$f^{-1} : y \mapsto x$$
where $x = \frac{1}{3}(y - 1)$

or $f^{-1} : y \mapsto (y - 1)$.

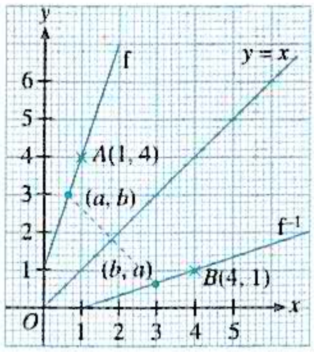
This process is shown in the diagram.
Replacing y by x for f^{-1}, we have

$$\mathbf{f^{-1} : x \mapsto \frac{1}{3}(x - 1)}$$

The graph of f^{-1} is the line segment $y = \frac{1}{3}(x - 1)$ for $1 \leqslant x \leqslant 7$ as shown.

Graph of an Inverse Function

From the above example, the point $B(4, 1)$ on the graph of f^{-1} is the reflection of $A(1, 4)$ on the graph of f in the line $y = x$. In fact, any point (b, a) on the group of f^{-1} is the reflection of (a, b) on the graph of f in the line $y = x$. It is clear that the graph of f^{-1} is the reflection of the graph of f in the line $y = x$ as shown.

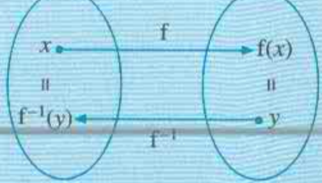


In general,

(a) $f : x \mapsto y \iff f^{-1} : y \mapsto x$

$y = f(x) \iff x = f^{-1}(y)$

(b) **Any point (b, a) on the graph of f^{-1} is the reflection of the point (a, b) on the graph of f in the line $y = x$. Geometrically, the graph of f^{-1} is the reflection of the graph of f in the line $y = x$.**



Example 14 A function f is defined by $f : x \mapsto \frac{x+1}{x-2}$, $x \neq 2$. Find in similar form f^{-1} and state the value of x for which f^{-1} is not defined.

Solution:

$$y = f(x) \quad \Leftrightarrow \quad x = f^{-1}(y)$$

Let $y = f(x)$. Then $y = \frac{x+1}{x-2}$, $x \neq 2$

$$xy - 2y = x + 1$$
$$x(y - 1) = 2y + 1$$
$$x = \frac{2y+1}{y-1},\ y \neq 1$$

Since $x = f^{-1}(y)$, $f^{-1}(y) = \frac{2y+1}{y-1}$, $y \neq 1$

Replacing y by x, $f^{-1}(x) = \frac{2x+1}{x-1}$, $x \neq 1$

and so $\mathbf{f^{-1} : x \mapsto \frac{2x+1}{x-1}}$ **$, x \neq 1$.**

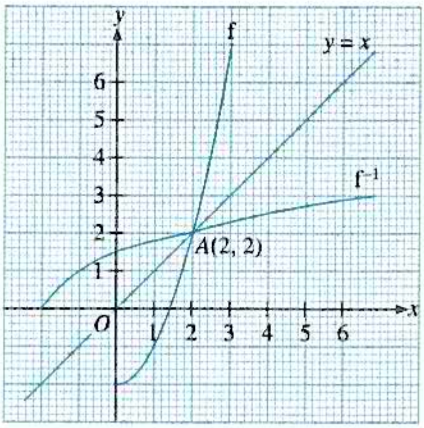
Example 15 A function f is defined by $f : x \mapsto x^2 - 2$ for the domain $0 \leqslant x \leqslant 3$.

(a) Sketch the graphs of f and f^{-1}.

(b) Find the value of x for which $f(x) = f^{-1}(x)$.

Solution:

(a) The graph of f is part of the quadratic curve $y = x^2 - 2$.
The graph of f^{-1} is the reflection of f in the line $y = x$ as shown.



(b) The graphs of f and f^{-1} meet at $A(2, 2)$ at which $f(x) = f^{-1}(x)$.
Since A lies on the line $y = x$ and the curve $y = x^2 - 2$,

$$x = x^2 - 2$$
$$x^2 - x - 2 = 0$$
$$(x + 1)(x - 2) = 0$$

Since $0 \leqslant x \leqslant 3$, $\quad x = \mathbf{2}$

Use a graph plotter to draw the graph of the function f given in Example 14. Is f a one-one function? Draw the graph of f^{-1}.

Inverse of a Composite Function

Example 16 Two functions f and g are defined by

$$f : x \mapsto 1 + \frac{3}{x},\ x \neq 0, \text{ and } g : x \mapsto 2x.$$

Express in similar form,

(a) f^{-1}, **(b)** g^{-1}, **(c)** gf,

(d) $(gf)^{-1}$, **(e)** $f^{-1}g^{-1}$.

Is $(gf)^{-1} = f^{-1}g^{-1}$?

Solution:

(a) Let $y = f(x)$.

Then $y = 1 + \frac{3}{x},\ x \neq 0$

$$\Rightarrow \quad x = \frac{3}{y-1},\ y \neq 1$$

Since $x = f^{-1}(y)$, $f^{-1}(y) = \frac{3}{y-1},\ y \neq 1$

i.e. $f^{-1}(x) = \frac{3}{x-1},\ x \neq 1$

$$\mathbf{f^{-1} : x \mapsto \frac{3}{x-1},\ x \neq 1}$$

(b) Let $y = g(x)$.

Then $y = 2x$, $x = \frac{y}{2}$

Since $x = g^{-1}(y)$, $g^{-1}(y) = \frac{y}{2}$

i.e. $g^{-1}(x) = \frac{x}{2}$

$$\mathbf{g^{-1} : x \mapsto \frac{x}{2}}$$

(c) $gf(x) = g(f(x))$

$$= g\left(1 + \frac{3}{x}\right),\ x \neq 0$$

$$= 2\left(1 + \frac{3}{x}\right)$$

$$= 2 + \frac{6}{x},\ x \neq 0$$

$$\mathbf{gf : x \mapsto 2 + \frac{6}{x},\ x \neq 0}$$

(d) Let $y = \mathrm{gf}(x)$

$$y = 2 + \frac{6}{x},\ x \neq 0$$

$$\Rightarrow \quad x = \frac{6}{y - 2},\ y \neq 2$$

Since $\quad x = (\mathrm{gf})^{-1}(y)$,

$$(\mathrm{gf})^{-1}(y) = \frac{6}{y - 2},\ y \neq 2$$

i.e. $(\mathrm{gf})^{-1}(x) = \dfrac{6}{x - 2},\ x \neq 2$

$$\mathbf{(gf)^{-1} : x \mapsto \frac{6}{x - 2},\ x \neq 2}$$

(e) $\mathrm{f}^{-1}\mathrm{g}^{-1}(x) = \mathrm{f}^{-1}(\mathrm{g}^{-1}(x))$

$$= \mathrm{f}^{-1}\left(\frac{x}{2}\right)$$

$$= \frac{3}{\left(\frac{x}{2}\right) - 1}$$

$$= \frac{6}{x - 2},\ x \neq 2$$

$$\therefore\ \mathbf{f^{-1}g^{-1} : x \mapsto \frac{6}{x - 2},\ x \neq 2}$$

From (d) and (e), we have $\mathbf{(gf)^{-1} = f^{-1}g^{-1}}$.

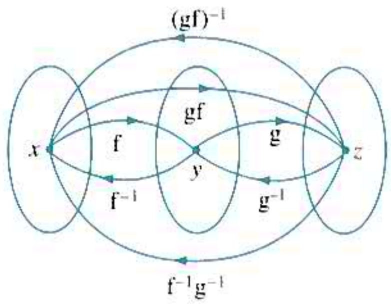
The diagram shows that, for any two one-one functions f and g,

$$(\mathrm{gf})^{-1} : z \mapsto x$$

and

$$\mathrm{f}^{-1}\mathrm{g}^{-1} : z \mapsto x$$

and hence $(\mathrm{gf})^{-1}$ and $\mathrm{f}^{-1}\mathrm{g}^{-1}$ are the same function, i.e. $(\mathrm{gf})^{-1} = \mathrm{f}^{-1}\mathrm{g}^{-1}$.

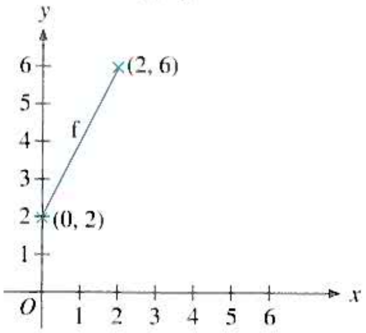


$(\mathrm{gf})^{-1} = \mathrm{f}^{-1}\mathrm{g}^{-1}$ holds for any two one-one functions.

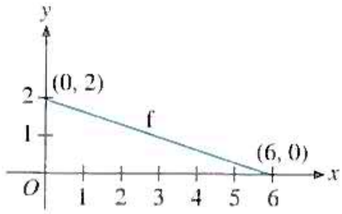
Exercise 9.3

1. Each of the following diagrams shows the graph of a one-one function f. In each case, sketch the graph of f^{-1}.

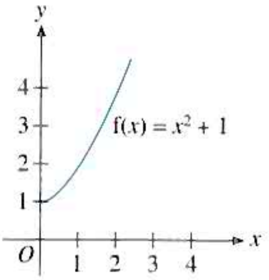
(a)



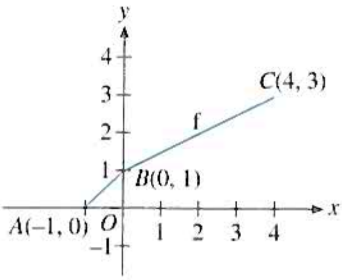
(b)



(c)



(d)



2. A one-one function f is defined by $\mathrm{f} : x \mapsto x^2 - 2x$ for $1 < x < 3$.
   (a) Sketch the graph of f and state the range of f.
   (b) Sketch the graph of f^{-1} on the same diagram and state the domain of f^{-1}.

3. Find f^{-1} in similar form for each of the following one-one functions.

   (a) $\mathrm{f} : x \mapsto 3x - 2$ **(b)** $\mathrm{f} : x \mapsto \dfrac{3}{x-1}, x \neq 1$

   (c) $\mathrm{f} : x \mapsto \dfrac{2x}{x-2}, x \neq 2$ **(d)** $\mathrm{f} : x \mapsto \dfrac{2x+3}{2x-1}, x \neq \dfrac{1}{2}$

4. A function f is defined by $\mathrm{f} : x \mapsto \dfrac{4x-9}{x-2}, x \neq 2$.
   (a) Find $\mathrm{f}^{-1}(-1)$ and $\mathrm{f}^{-1}(1)$.
   (b) Find the value of x for which $4\mathrm{f}^{-1}(x) = x$.

5. A function f is defined by $\mathrm{f} : x \mapsto \dfrac{2x+2}{x-1}, x \neq 1$.
   (a) Find $\mathrm{f}^{-1}(3)$.
   (b) Given that $\mathrm{f}^{-1}(p) = kp$, express k in terms of p.

6. Given the function $\mathrm{f} : x \mapsto 6x - \dfrac{2}{x}, x > 0$, find the value of
   (a) $\mathrm{f}^{-1}(-1)$, **(b)** $\mathrm{f}^{-1}(-4)$.
   For what value of x is $\mathrm{f}(x) = 1$?

7. For the function defined by $f : x \mapsto \dfrac{a}{x-1} + b$, state the value of x for which f is not defined. Given that $f(2) = 3$ and $f(3) = 2$,
   (a) find the value of a and of b,
   (b) show that $ff(x) = x$,
   (c) find $f^{-1}(x)$.

8. The functions f and g are defined by $f : x \mapsto 2x + 3$ and $g : x \mapsto \dfrac{1}{x}$, $x \neq 0$. Write down in similar form, expressions for fg, gf, f^{-1}, g^{-1}. Find the value of x for which $fg(x) = g^{-1}(x)$.

9. A function is defined by $f : x \mapsto \dfrac{2x+1}{x-1}$ for all values of x except $x = 1$. Express the function f^{-1} in similar form and state the value of x for which f^{-1} is not defined. Find the values of
   (a) $f^{-1}(3)$, **(b)** x for which $f(x) = 4f^{-1}(x)$.

10. Functions f and g are defined on the set of real numbers by $f : x \mapsto \dfrac{3}{x+2}$, $x \neq k$, and $g : x \mapsto 2x + 1$.
   (a) State the value of k.
   (b) Express fg in similar form and state the value of x for which fg is not defined.
   (c) Find the value of p for which $f^{-1}(p) = g(9)$.

11. A function f is defined by $f : x \mapsto \dfrac{3x}{x-3}$ for all values of x except $x = 3$. Show that $ff(x) = x$ for all values of x except $x = 3$. Find $f(5)$ and $f^{-1}(5)$.

12. A function f is defined by $f : x \mapsto \dfrac{x+1}{x-1}$, $x \neq 1$. If $f^2(2) = 3f^{-1}(a)$, find the value of a.

13. A function f is defined by $f : x \mapsto \dfrac{a}{x-1}$, $x \neq 1$.
   (a) Find f^{-1} in similar form.
   (b) If $f(a) + f^{-1}(2a) = 1$, find the value of a.

14. Two functions f and g are defined by $f : x \mapsto \dfrac{2x}{x-1}$, $x \neq 1$, and $g : x \mapsto \dfrac{x+a}{x}$, $x \neq 0$. Find f^{-1} in similar form. Given that $gf^{-1}(3) = 4$, calculate the value of a.

15. Functions f and g are defined by $f : x \mapsto \dfrac{3x-1}{x-2}$, $x \neq 2$, and $g : x \mapsto \dfrac{2x-1}{x-3}$, $x \neq 3$. Show that $fg : x \mapsto x$. Evaluate $f^{-1}(5)$, $g^{-1}(4)$ and $ffg(4)$.

16. A function f is defined by $f : x \mapsto \dfrac{3-x}{2x}$, $x \neq 0$.
   (a) Draw the graph of f.
   (b) Find the values of x for which $f(x) = f^{-1}(x)$.

17. Two functions f and g are defined by $f : x \mapsto \frac{x-1}{x+1}$, $x \neq -1$, and $g : x \mapsto mx + c$, where m and c are constants.

(a) Find an expression for f^{-1}.

(b) Given that $g^{-1}(3) = f^{-1}(2)$ and that $f^{-1}g(4) = 1$, find the value of m and of c.

18. A function f is defined by $f : x \mapsto \frac{3}{x} + 2$ for $1 \leqslant x \leqslant 5$.

(a) Draw the graph of f.

(b) Find the value of x for which $f(x) = f^{-1}(x)$.

(c) Draw, on the same diagram, the graph of f^{-1}.

19. Given the functions $f : x \mapsto 2x + 1$ and $g : x \mapsto 5 - x$, express the following functions in similar form.

(a) ff (b) g^{-1} (c) fg

(d) $(fg)^{-1}$ (e) f^{-1} (f) $g^{-1}f^{-1}$

20. A function is defined by $f : x \mapsto \frac{1}{2x+1}$, $x \neq -\frac{1}{2}, -\frac{3}{2}$.

(a) Write in similar form expressions for f^2 and f^{-1}.

(b) Show that $(f^2)^{-1}(x) = (f^{-1})^2(x)$.

21. The functions f and g are defined by $f : x \mapsto 2x + 3$ and $g : x \mapsto 3x - 2$, where $x \in \mathbb{R}$. Write down, in similar form, fg, f^{-1}, g^{-1}, and show that $g^{-1}f^{-1} = (fg)^{-1}$.

22. Two functions are defined by $f : x \mapsto 3x + 1$, $x \in \mathbb{R}$ and $g : x \mapsto \frac{2}{x}$ (for $x \neq 0$).

Find in similar form

(a) fg and $(fg)^{-1}$, (b) f^{-1}, g^{-1} and $g^{-1}f^{-1}$.

23. Given the functions $f : x \mapsto 2x - 3$, $x \in \mathbb{R}$ and $g : x \mapsto \frac{1}{x-1}$ (for $x \neq 1$), find in similar form

(a) fg and $(fg)^{-1}$, (b) f^{-1}, g^{-1} and $g^{-1}f^{-1}$.

Is $(fg)^{-1} = g^{-1}f^{-1}$?

24. Two functions f and g are defined by $f : x \mapsto 5 - 2x$, $x \in \mathbb{R}$ and $g : x \mapsto \frac{x-1}{x+1}$, $x \neq -1$. Find in similar form, expressions for

(a) f^{-1} and g^{-1}, (b) $g^{-1}f^{-1}$.

Hence write in similar form, an expression for $(fg)^{-1}$.

9.4 Absolute Valued Functions

The numbers 5 and –5 have the same numerical value 5. The absolute value of a number x, denoted by $|x|$, is the numerical value of x.

Thus, we write $|5| = 5$ and $|-5| = 5$.

Example 17 Given that $f(x) = |3x - 2|$, find
(a) the value of $f(-1)$ and $f(2)$,
(b) the values of x for which $f(x) = 8$,
(c) the value of x for which $f(x) = x$.

Solution:
(a) $f(-1) = |3 \times (-1) - 2| = |-5| = 5$
$f(2) = |3 \times 2 - 2| = |4| = 4$

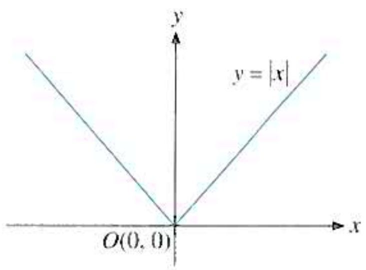
(b) $f(x) = 8 \Leftrightarrow |3x - 2| = 8$
$\Leftrightarrow 3x - 2 = -8$ or $3x - 2 = 8$
$\Leftrightarrow 3x = -6$ or $3x = 10$
$\Leftrightarrow x = -2$ or $x = \frac{10}{3}$

(c) $f(x) = x \Leftrightarrow |3x - 2| = x$
$\Leftrightarrow 3x - 2 = -x$ or $3x - 2 = x$
$\Leftrightarrow 4x = 2$ or $2x = 2$
$\Leftrightarrow x = \frac{1}{2}$ or $x = 1$

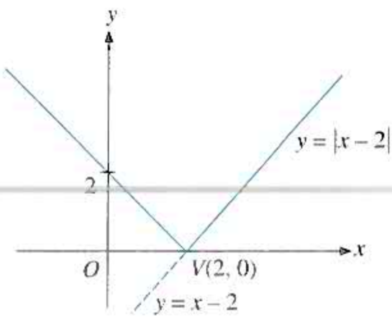
For the absolute valued expression $|x|$, we have:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

The function defined by $f(x) = |x|$ is called the **absolute valued function** and it has a V-shaped graph with the vertex at (0, 0) as shown.

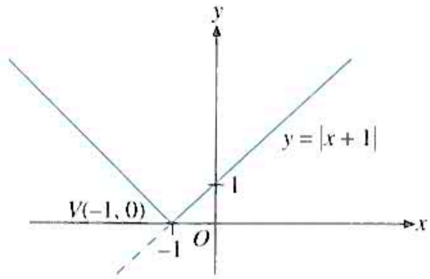


Similarly, the function $f : x \mapsto |x - 2|$ is a V-shaped graph with vertex at $V(2, 0)$ where $x - 2 = 0$.
The graph of $y = |x - 2|$ can be obtained as follows:
(a) Draw the line $y = x - 2$.
(b) Reflect in the x-axis the part of the line below the x-axis.

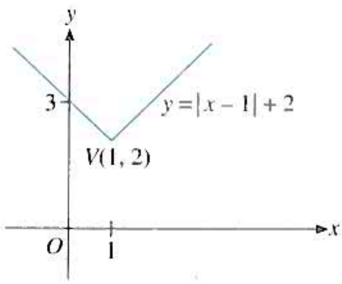


Example 18 Sketch the graphs of the following functions.
(a) $f : x \mapsto |x + 1|$ **(b)** $g : x \mapsto |x - 1| + 2$

Solution: **(a)** The graph of f is V-shaped with vertex at $V(-1, 0)$ as shown below.

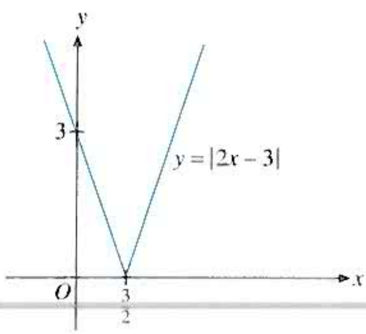


(b) The graph of g is V-shaped with vertex at $V(1, 2)$ as shown below.



Note: In (a), V is the point where $|x + 1| = 0$, i.e. $x + 1 = 0$.
In (b), V is the point where $|x - 1| = 0$, i.e. $x - 1 = 0$.

Further, the graph of a function defined by $f : x \mapsto |2x - 3|$ is also V-shaped with vertex at V where $2x - 3 = 0$, i.e. $x = \frac{3}{2}$ as shown below.



Note that the line that slopes upwards is of gradient 2 and the other that slopes downwards is of gradient -2.

In general, the graph of a function defined by $f : x \mapsto |mx + c|$ is a V-shaped graph with vertex at V where $mx + c = 0$ and the line that slopes upwards is of gradient $|m|$.

Example 19

A function f is defined by $f : x \mapsto |2x - 1|$ for the domain $-1 \leqslant x \leqslant 3$.

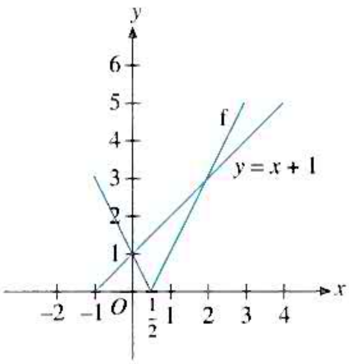
(a) Sketch the graph of f and state the range of f.

(b) Find the range of values of x for which $f(x) \leqslant 3$.

(c) Sketch the graph $y = x + 1$ and hence obtain the values of x for which $|2x - 1| = x + 1$.

Solution

(a) The graph of f is V-shaped with vertex $V\left(\frac{1}{2}, 0\right)$ as shown below.



Range of f = $\{y : 0 \leqslant y \leqslant 5, y \in \mathbb{R}\}$

(b) $f(x) = 3 \quad \Rightarrow \quad |2x - 1| = 3$

$2x - 1 = -3 \quad \text{or} \quad 2x - 1 = 3$

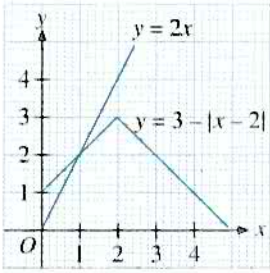
$x = -1 \quad \text{or} \quad x = 2$

From the graph, we see that $-1 \leqslant x \leqslant 2 \Rightarrow 0 \leqslant f(x) \leqslant 3$.
Hence the required range is $-1 \leqslant x \leqslant 2$.

(c) The straight line $y = x + 1$ cuts the graph of f at (0, 1) and (2, 3). The values of x for which $|2x - 1| = x + 1$ are **0** and **2**.

Example 20 Using graph paper, draw on the same diagram, the graphs of $y = 3 - |x - 2|$ and $y = 2x$, for $0 < x < 5$.
Find the value of x for which $3 - |x - 2| = 2x$.

Solution: The graphs of $y = 3 - |x - 2|$ and $y = 2x$ are as shown.



The graphs intersect at $x = 1$ and hence the required value of x is **1**.

Using a graph plotter, draw graphs of $y = |ax - b|$ and $y = mx + c$ for various values of a, b, m and c. Find graphically the solutions of the equation $|ax - b| = mx + c$.

Exercise 9.4

1. **(a)** A function f is defined by $f : x \mapsto |3x - 1|$ where $x \in \mathbb{R}$. Find f(–2), f(2) and the values of x for which $f(x) = 3$.
   (b) A function f is defined by $f : x \mapsto |x^2 - 2x|$ where $x \in \mathbb{R}$. Find f(1) and f(3).

2. Solve the following equations.
   (a) $|2x - 3| = x$ **(b)** $|x + 1| = 2x - 3$
   (c) $|2x + 4| = x^2 + 1$ **(d)** $|x + 1| = 2x - 5$

3. For each of the following functions, sketch the graph and state the range of the function.
   (a) $f : x \mapsto |x - 3|$, for $0 < x \leqslant 3$ **(b)** $f : x \mapsto |x + 2|$, for $-3 \leqslant x \leqslant 3$
   (c) $f : x \mapsto |2x - 1|$, for $-1 < x \leqslant 3$ **(d)** $f : x \mapsto |3x - 5|$, for $0 < x \leqslant 3$
   (e) $f : x \mapsto |x - 1| + 1$, for $-1 < x \leqslant 3$ **(f)** $f : x \mapsto |2x - 5| - 2$, for $0 < x \leqslant 5$

4. For each of the following functions, sketch the graph and state the range of the function.
   (a) $f : x \mapsto 3 - |x|$, for $-2 < x \leqslant 2$ **(b)** $f : x \mapsto -|x - 2|$, for $-2 \leqslant x \leqslant 4$
   (c) $f : x \mapsto |2 - x^2|$, for $-2 < x \leqslant 1$ **(d)** $f : x \mapsto 5 - |2x - 3|$, for $-1 < x \leqslant 2$

5. A function f is defined by $f : x \mapsto |x^2 - 2x|$ for the domain $1 \leqslant x \leqslant 3$.
   (a) Find f(1) and f(3).
   (b) Sketch the graph of f and determine whether f is a one-one function.

6. **(a)** Find the range of the function $f : x \mapsto |x + 1|$ for the domain $-2 \leqslant x \leqslant 2$. Find also the value of x for which $f(x) = 2x$.
   (b) A function f is defined by $f : x \mapsto |2x + 3| + 1$ with domain $-4.5 \leqslant x < 2$. Find the corresponding range. Find the values of x for which $f(x) = f(-3x)$.

7. **(a)** Sketch the graph of the function $f : x \mapsto |3x - 5| - 2$ and find the range of values of x for which $f(x) \leqslant 2$.
   (b) Sketch the graph of the function $g : x \mapsto |x - 1| - 3$ and find the range of values of x for which $g(x) > 0$.

8. **(a)** Sketch the graph of $y = -|x - 2|$ for $-2 \leqslant x \leqslant 4$. Find the corresponding range.
   (b) Sketch the graph of $y = 2 - |2x - 1|$ for $-2 \leqslant x \leqslant 5$. Find the range of values of x for which $y \leqslant -5$.

9. Sketch the graphs of $y = 2x - 1$ and $y = |x - 2|$ on the same diagram. Solve the simultaneous equations $y = 2x - 1$ and $y = |x - 2|$.

10. Sketch the graphs of $y = -2x + 3$ and $y = |2x - 1|$ on the same diagram. Solve the simultaneous equations $y = -2x + 3$ and $y = |2x - 1|$.

11. Using graph paper, draw, on the same diagram, the graphs of $y = 2 + |x - 3|$ and $y = 2x$ for $0 < x < 7$.
    Find the values of x for which $2 + |x - 3| = 2x$.

*12. A function f is defined by $f : x \mapsto |x^2 - 3x|$. Find the values of x for which $f(x) = x$.

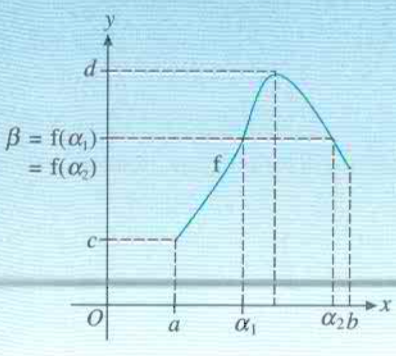
*13. A function f is defined by $f : x \mapsto |x - a| + b$. Given that $f(3) = 3$ and $f(-1) = 3$, find the value of a and of b.

Important Notes

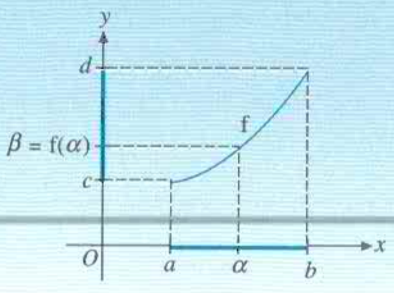
1. A function (or mapping) f maps each element x of X (domain) to exactly one element y of Y (range).

   A function f is called a one-one function if each element y is the image of only one element x of X.

   A useful representation of a function f is the cartesian graph of $y = f(x)$.



f is not a one-one function



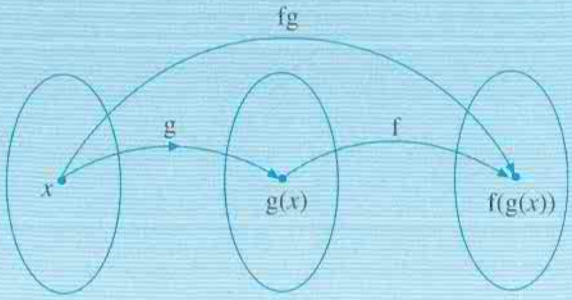
f is a one-one function

(a) Any vertical line cuts the graph of f at only one point.
(b) The value $\beta = \mathrm{f}(\alpha)$ is the image of α under f.
(c) Corresponding to the domain $a \leqslant x \leqslant b$, the range is $c \leqslant \mathrm{f}(x) \leqslant d$.

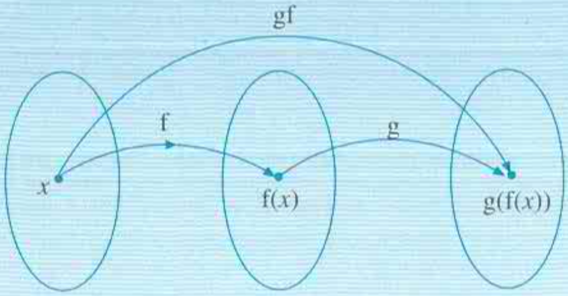
2. Composite functions

For two functions f and g, we have the composite functions:

(a) fg where $\mathrm{fg}(x) = \mathrm{f}(\mathrm{g}(x))$



(b) gf where $\mathrm{gf}(x) = \mathrm{g}(\mathrm{f}(x))$



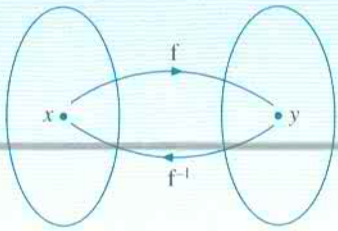
(c) f^2 where $\mathrm{f}^2(x) = \mathrm{ff}(x) = \mathrm{f}(\mathrm{f}(x))$

In general, $\mathrm{fg} \neq \mathrm{gf}$.

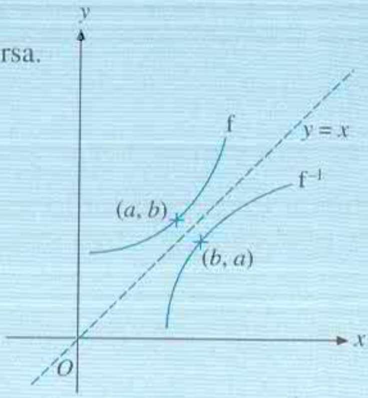
3. Inverse functions

A one-one function has an inverse. For the one-one function $\mathrm{f} : x \mapsto y$, the inverse function is $\mathrm{f}^{-1} : y \mapsto x$.

That is $\mathrm{f} : x \mapsto y \;\Leftrightarrow\; \mathrm{f}^{-1} : y \mapsto x$ or $y = \mathrm{f}(x) \;\Leftrightarrow\; x = \mathrm{f}^{-1}(y)$.

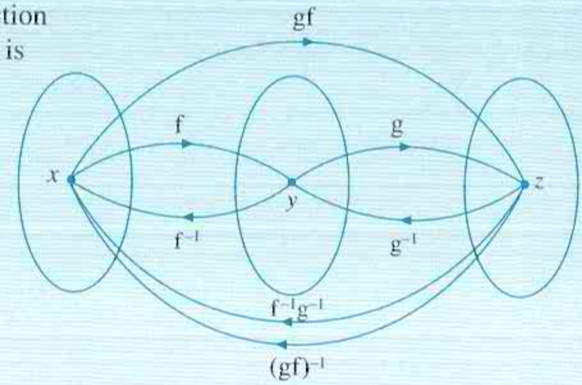


(a) The graph of f^{-1} is the reflection of the graph of f in the line $y = x$ and vice versa.



(b) $f^{-1}f(x) = ff^{-1}(x) = x$

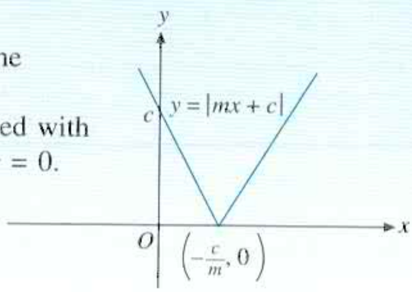
(c) For the composite function gf the inverse function is $(gf)^{-1} = f^{-1}g^{-1}$.



4. Absolute valued functions

(a) The function $f : x \mapsto |x|$ is called the absolute valued function.

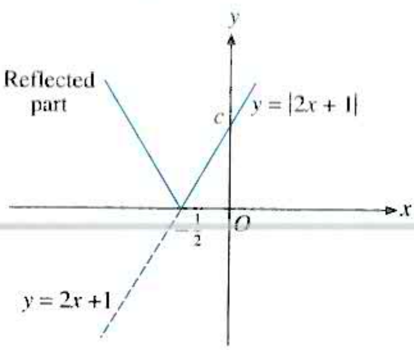
(b) The graph of $y = |mx + c|$ is V-shaped with its vertex at the point where $mx + c = 0$.



(c) Sketch the graph of $y = |f(x)|$ as follows:

Step 1: Sketch the graph of $y = f(x)$.

Step 2: Reflect in the x-axis that part of the graph below the x-axis.



Miscellaneous Examples

Example 21 Functions f and g are defined by

$$f : x \mapsto ax + 2,\ a \neq 0$$

and $$g : x \mapsto \frac{bx}{x+1},\ x \neq -1,\ b > 0.$$

(a) Find in similar form, g^{-1} and fg^{-1}.

(b) Given that $fg^{-1}(-1) = g\left(\frac{1}{b}\right)$, find a in terms of b.

If $fg^{-1}(2) = 12$, find the value of a and of b.

Solution:

(a) We have $f(x) = ax + 2$ and $g(x) = \frac{bx}{x+1},\ x \neq -1$.

Let y be the image of x under g.

Then
$$\begin{aligned} y &= \frac{bx}{x+1},\ x \neq -1 \\ y(x+1) &= bx \\ y &= bx - xy \\ &= x(b-y) \\ x &= \frac{y}{b-y},\ y \neq b \end{aligned}$$

So, $$g^{-1} : y \mapsto \frac{y}{b-y},\ y \neq b$$

or $$\mathbf{g^{-1} : x \mapsto \frac{x}{b-x},\ x \neq b.}$$

$$fg^{-1}(x) = f\left(\frac{x}{b-x}\right) = \frac{ax}{b-x} + 2,\ x \neq b$$

So, $\mathbf{fg^{-1} : x \mapsto \frac{ax}{b-x} + 2,\ x \neq b}$

(b)
$$fg^{-1}(-1) = g\left(\frac{1}{b}\right)$$

$$\Rightarrow \quad -\frac{a}{b+1} + 2 = \frac{b\left(\frac{1}{b}\right)}{\frac{1}{b}+1}$$

$$\Rightarrow \quad \frac{-a + 2(b+1)}{b+1} = \frac{b}{1+b}$$

$$\Rightarrow \quad -a + 2(b+1) = b$$

$$\Rightarrow \quad \mathbf{a = b + 2} \quad \text{.............................(1)}$$

$$fg^{-1}(2) = 12$$

$$\Rightarrow \quad \frac{2a}{b-2} + 2 = 12$$

$$\Rightarrow \quad a = 5b - 10 \quad \text{.........................(2)}$$

Solving (1) and (2), we have $a = \mathbf{5}$, $b = \mathbf{3}$.

Example 22 A function g is defined by $y = |x^2 - 4|$ for the domain $-4 \leqslant x \leqslant 4$.

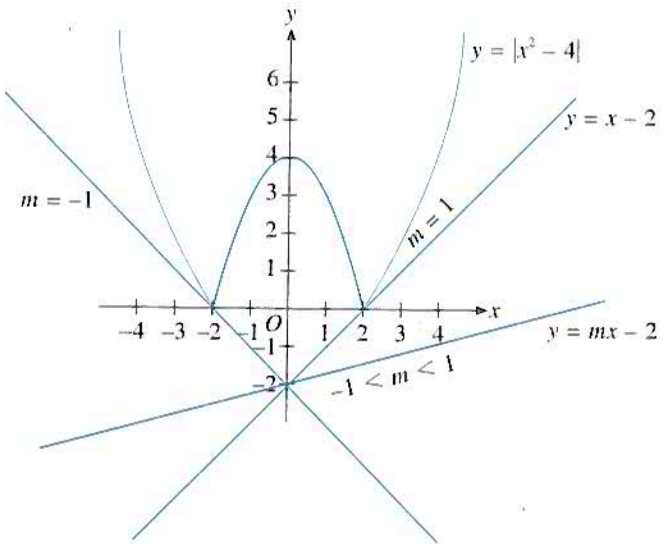
(a) Sketch the graph of g and find the range of g.

(b) Sketch the line $y = x - 2$ on the same diagram and find the root of the equation $|x^2 - 4| = x - 2$.

(c) Find the range of values of m for which the equation $|x^2 - 4| = mx - 2$ has no roots.

Solution:

(a) We first sketch the graph of $y = x^2 - 4$ for $-4 \leqslant x \leqslant 4$ and reflect in the x-axis the portion below the x-axis.
When $x = 4$, $y = |4^2 - 4| = 12$.
The range is $\{y : 0 \leqslant y \leqslant 12, y \in \mathbb{R}\}$.

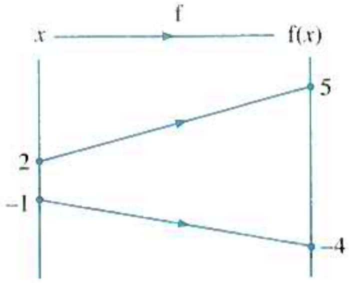


(b) The line $y = x - 2$, meets the curve $y = |x^2 - 4|$ at (2, 0). The solution of the equation $|x^2 - 4| = x - 2$ is **2**.

(c) $|x^2 - 4| = mx - 2$ has no roots
$\Rightarrow$ the straight line $y = mx - 2$ does not cut the curve $y = |x^2 - 4|$,
$\Rightarrow$ gradient of the straight line $y = mx - 2$, must satisfy the condition $\mathbf{-1 < m < 1}$ as shown.

Miscellaneous Exercise 9

1. The figure shows part of the mapping
$$f : x \mapsto px + q,\ x \in \mathbb{R}$$
Find
(a) the value of p and of q,
(b) the image of 3 under f,
(c) the element whose image is 8.



2. Given that $f : x \mapsto 2 + \frac{a}{x},\ x \neq 0$, and that $f^{-1}\left(\frac{1}{2}\right) = -2$, find the value of a. Find also the elements which are unchanged under f.

3. Given the functions $f : x \mapsto 2 - x$ and $g : x \mapsto \frac{3}{x}$, where $x \in \mathbb{R},\ x \neq 0$, find in similar form,
(a) f^{-1}, (b) ff, (c) gg, (d) fg, (e) gfg.

4. Functions f and g are defined by
$$f : x \mapsto 4x - 3,$$
$$g : x \mapsto 2 - \frac{5}{x},\ x \neq 0,$$
where $x \in \mathbb{R}$.
Find an expression for the function
(a) ff, (b) gf, (c) f^{-1}, (d) g^{-1}, (e) $(fg)^{-1}$.

5. Functions f and g are defined by
$$f : x \mapsto \frac{x - 3}{2},\ x \in \mathbb{R},$$
$$g : x \mapsto \frac{3 - x}{x + 1},\ x \neq -1.$$
(a) Find $f^{-1}(2)$, $gf(-1)$, $fgf(2)$.
(b) Express in similar form, the function
(i) g^{-1}, (ii) fg, (iii) gfg.

6. A function f is defined by $f : x \mapsto \frac{x}{1 - x},\ x \neq 1$.
(a) Show that $f^2 : x \mapsto \frac{x}{1 - 2x}$, for $x \neq \frac{1}{2}, 1$.
(b) Obtain f^3 in a similar form.

7. A function h is defined by $h : x \mapsto \frac{x + 3}{x - 3}$ (for $x \neq 3$).
(a) Show that $h(3 + p) + h(3 - p) = 2$ where p is a positive number.
(b) Find the positive number q such that $h(q) = q - 1$.

8. Functions f and g are defined by

$$f : x \mapsto 4x - 3,$$
$$g : x \mapsto px + q, \text{ where } p > 0.$$

Find the values of p and q for which $gg(x) = f(x)$ for all values of x.

9. Functions f and g are defined by

$$f : x \mapsto ax + b, x \in \mathbb{R}, a \text{ and } b \text{ are constants},$$
$$g : x \mapsto \frac{3}{x-2}, x \neq 2.$$

Find expressions for fg and gf. Given that $fg(5) = 5$ and $g^{-1}(2) = f(2)$, calculate the value of a and of b.

10. The functions f and g are defined by

$$f : x \mapsto \frac{3x+2}{x}, x \neq 0,$$
$$g : x \mapsto \frac{1}{x}, x \neq 0.$$

Show that

(a) $f(x) = 2g(x) + 3$,

(b) $f^2(x) = 2gf(x) + 3$.

11. Sketch the graph and find the range of each of the following functions for the domain $-2 \leqslant x \leqslant 3$.

(a) $f : x \mapsto |2x + 1|$

(b) $g : x \mapsto x^2 - x - 2$

(c) $h : x \mapsto |x^2 - x|$

(d) $k : x \mapsto 3 - |x^2 - 2|$

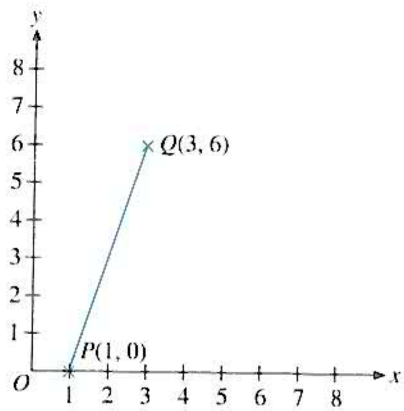
12. For each of the following functions, state the range that corresponds to the given domain.

(a) $f(x) = |2x - 9|$ for the domain $0 \leqslant x \leqslant 6$

(b) $g(x) = (x - 2)^2 - 3$ for the domain $0 \leqslant x \leqslant 5$

(c) $h(x) = |2 - 3(x - 1)^2|$ for the domain $-1 < x < 2$

13. The diagram shows the graph of $y = f(x)$. The scale is the same on each axis. Copy the diagram and on it sketch the graph of $y = f^{-1}(x)$, indicating the point on $y = f^{-1}(x)$ corresponding to the points P and Q. State the relationship between the graphs of $y = f(x)$ and $y = f^{-1}(x)$. Hence calculate the element whose images under f and f^{-1} are the same.



14. Two functions f and g are defined by $f : x \mapsto \frac{1}{3}(1 - x)$ and $g : x \mapsto |x + 1|$, where $x \in \mathbb{R}$.

Sketch the graphs of f and g on the same diagram. Solve the simultaneous equations:

$$3y = 1 - x,$$
$$y = |x + 1|.$$

15. Functions f and g are defined by $f : x \mapsto \dfrac{x+2}{x-2}$, $x \neq 2$, and $g : x \mapsto mx + c$, where $x \in \mathbb{R}$.

(a) Obtain an expression for f^{-1}.

(b) Given that $g^{-1}(2) = f(3)$ and $gf^{-1}(2) = 5$, find the value of m and of c.

16. A function f is defined by $f : x \mapsto |x^2 - a| + b$ for the domain $-2 \leqslant x \leqslant 3$. Given that $f(-2) = b$ and $f(3) = 8$, find the value of a and of b. Sketch the graph of f and state the corresponding range.

17. **(a)** Using graph paper, draw, on the same diagram, the graphs of

$$y = 2 - |x - 2|,$$

$$y = \frac{1}{2}x + 2,$$

for $-1 \leqslant x \leqslant 5$.

How many pairs of values, (x, y), satisfy both equations?

(b) On graph paper, using the same scale on each axis, draw the graph of

$h : x \mapsto \dfrac{2x+2}{x+2}$ for the domain $-1 \leqslant x \leqslant 3$.

(i) By drawing the appropriate straight line on the graph, obtain a solution of the equation $h(x) = h^{-1}(x)$.

(ii) State the domain of $h^{-1}(x)$.

(iii) Using the same axes as for $h(x)$, draw, on the same diagram, the graph of $h^{-1}(x)$. *(C)*

Revision Exercise 5

1. **(a)** $\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 2 & -2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 27 & 10 \\ 16 & 6 \end{pmatrix}$.
   (i) Evaluate $2\mathbf{B} - 3\mathbf{A}$.
   (ii) Find $\mathbf{A}^{-1}$, the inverse of $\mathbf{A}$.
   (iii) Given that $\mathbf{AX} = \mathbf{B}$, find the matrix $\mathbf{X}$.

   (b) Find the inverse of the matrix $\begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}$. Hence solve the simultaneous equations $-3x + 2y = 14$ and $x - 2y = -10$.

2. The vertices of a triangle are $A(1, 1)$, $B(5, -1)$ and $C(0, 9)$. Find
   (a) the equation of the perpendicular bisector of AB,
   (b) the coordinates of F, the foot of the perpendicular from C to the perpendicular bisector of AB,
   (c) the area of the triangle ABF and hence, deduce the area of the triangle ABC.

3. In the rectangle $ABCD$, lettered anticlockwise, A and B are the points $(-1, 2)$ and $(2, -1)$ respectively. Given that C lies on the line $x + y = 15$, find
   (a) the equation of BC,
   (b) the coordinates of C,
   (c) the coordinates of D,
   (d) the area of the rectangle $ABCD$.

4. Experimental values of x and y are given in the following table.

x	0.25	0.5	0.75	1	1.25	1.5
y	148.11	33.89	12.88	5.50	2.08	0.22

   It is known that x and y are related by $x^2y = a - bx^2$. Plot y against $\frac{1}{x^2}$ and use the graph to estimate the values of a and b.

5. It is known that the variables x and y are related by the equation $y = \frac{x + p}{x + q}$, where p and q are unknown constants. Express this equation in a form suitable for drawing a straight line graph and state which variable should be used for each axis. Explain how the value of p and of q could be determined from this graph.

6. **(a)** Given that $f : x \mapsto \frac{x + 1}{x - a}$, (for $x \neq a$) and that $2f(4) - 5 = 0$, find
   (i) the value of a,
   (ii) $f^{-1}(-2)$.

   (b) Two functions are defined by $f : x \mapsto 3x + 1$, $g : x \mapsto \frac{2}{1 - x}$, $x \neq 1$.
   (i) Solve the equation $f(2x + 3) = x^2 - 6$.
   (ii) Find in similar form, gf and gg.

7. Sketch the graph of $y = |3x - 7|$. Find
   (a) the range of y corresponding to the domain $0 \leq x \leq 4$,
   (b) the range of values of x for which $y > 4$.

Revision Exercise 6

1. (a) Given that $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, find $\mathbf{A}^2$, $\mathbf{A}^3$ and $\mathbf{A}^4$. Hence deduce $\mathbf{A}^{23}$.

   (b) The determinant of the matrix $\begin{pmatrix} a+1 & -2 \\ 2a-1 & a \end{pmatrix}$ is 4.
      (i) Form an equation in a and show that it reduces to $a^2 + 5a - 6 = 0$.
      (ii) Solve this equation.
      (iii) For the smaller value of a, find the inverse of the given matrix.

2. Find the coordinates of the foot of the perpendicular from $A(-1, 3)$ to the line $3x - 2y = 4$ and hence, calculate the perpendicular distance from A to the line. If the point B is the reflection of A in the line, find the coordinates of B.

3. Given that $A(1, 0)$, $B(5, 2)$ and $C(3, 4)$ are the vertices of a triangle, find
   (a) the equations of the perpendicular bisectors of AC and BC,
   (b) the coordinates of D, the point of intersection of the two bisectors,
   (c) the coordinates of E, the midpoint of AB,
   (d) the gradient of DE and show that DE is perpendicular to AB.

4. (a) The function f is defined by $f : x \mapsto a + \frac{b}{x}$, where a and b are constants. Given that $f(3) = 2$ and $f^{-1}(7) = -2$, find the value of a, b and the image of the element -3 under f.

   (b) Given the function $g^{-1} : x \mapsto \frac{2x-1}{x+3}$, $x \neq -3$, find g in similar form.

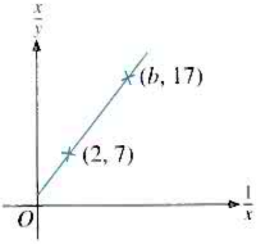
5. Two functions f and g are defined by $f : x \mapsto 2x + 1$, $g : x \mapsto (x-1)^2 + 1$. Find, in similar forms, gf and fg. Another function h is defined by $h : x \mapsto |gf(x) - 2fg(x)|$. Find an expression for the function h and sketch its graph.

6. The table shows experimental values of two variables x and y.

x	1	2	3	4	5
y	6.57	10.79	17.75	29.16	47.92

   It is known that $y = ab^{\frac{x}{2}}$. Express this equation in a form suitable for drawing a straight line graph. Draw this graph for the given data and use it to estimate
   (a) the value of a and of b,
   (b) the value of x when $y = 25$.

7. The diagram shows a straight line graph representing the curve $x^2 - 2xy + ay = 0$, together with the coordinates of two points on the line. Calculate the value of a and of b.



Revision Exercise 7

1. **(a)** Given that $\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 5 & -2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix}$, find

   (i) $\mathbf{BA}$, **(ii)** $\mathbf{BA} - 3\mathbf{A}$.

   If $\mathbf{B}^2\mathbf{A} = \begin{pmatrix} 14 & a & b \\ -13 & -16 & c \end{pmatrix}$, find the values of a, b and c.

   (b) Use a matrix method to solve the simultaneous equations $5x - 2y = 17$ and $-3x + 2y = -11$.

2. Given three points $A(-1, 2)$, $B(2, 1)$ and $C(h, k)$ such that the gradient of BC is twice the gradient of AC, show that $(h - 5)(k - 3) = 6$. If C lies on the line $x - y = 1$, find the coordinates of C.

3. Given the functions $\mathrm{f} : x \mapsto 2x + 3$ and $\mathrm{g} : x \mapsto \frac{2}{x}$, find in similar form

   (a) fg, **(b)** gf.

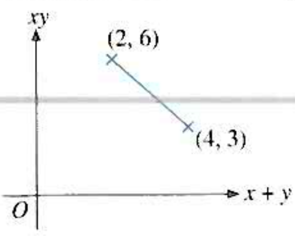
   Show that there are no real values of x which have the same image under fg and gf.

4. Function f and g are defined by $\mathrm{f} : x \mapsto \frac{4x}{x - 3}$, $x \neq 3$; $\mathrm{g} : x \mapsto |x - 2|$.

   (a) Evaluate gf(1) and $\mathrm{f}^{-1}(8)$.
   (b) Find ff in similar form.
   (c) Sketch the graph of $y = \mathrm{gg}(x)$. State also the range of values of k for which the equation $\mathrm{gg}(x) = k$ has only two real solutions.

5. Function f is defined by $\mathrm{f} : x \mapsto (x - 1)^2$, $1 \leqslant x \leqslant 3$. Sketch, on the same diagram, the graphs of f and f^{-1}. State the domain of f^{-1}.

6. The diagram shows part of a straight line graph obtained by plotting xy against $x + y$, together with the coordinates of two of the points on the line. Express y in terms of x.



7. Experimental values of x and y are tabulated below.

x	1	2	3	4
y	0.47	2.03	2.87	3.45

If $y = \ln(ax^2 + b)$, where $x > 0$, plot e^y against x^2 and use the graph to estimate
(a) the value of a and of b,
(b) the value of x when $e^y = 5$,
(c) the value of y if $x = 2.5$.

Revision Exercise 8

1. Two shops sell the same type of diskettes and inkjet paper. A box of 10 diskettes costs \$3.70 at Shop A while a ream of inkjet paper costs \$6.50 at Shop B. Peter and Mary plan to buy the following quantities:

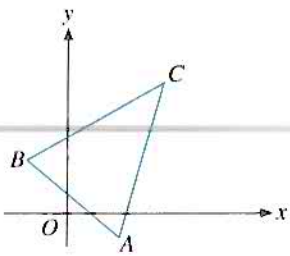
	Diskettes (in boxes)	Inkjet paper (in reams)
Peter	6	2
Mary	4	3

Peter needs to pay \$35.80 for his supply regardless of whether he buys from Shop A or Shop B.

It is given that $\mathbf{P} = \begin{pmatrix} 6 & 2 \\ 4 & 3 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 3.70 & a \\ b & 6.50 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} 35.80 & 35.80 \\ 35.20 & 34.70 \end{pmatrix}$.

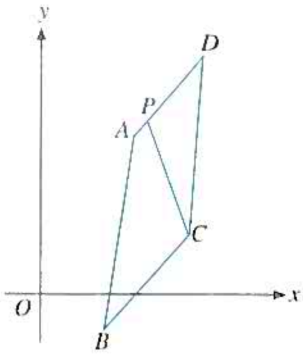
(a) Write down an equation connecting $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$.
(b) What is the significance of the elements in the second row of the matrix $\mathbf{R}$?
(c) Find $\mathbf{P}^{-1}$.
(d) Find the values of a and b and explain the significance of these values.

2. The figure shows a triangle ABC with $A(2, -1)$ and $B(-1, 1)$. The gradients of AC and BC are $2m$ and m respectively. Find
(a) the equation of AC and of BC in terms of m,
(b) the coordinates of C in terms of m,
(c) the coordinates of C if $B\hat{A}C = 90°$.



3. $ABCD$ is a parallelogram where $A(3, 5)$, $B(2, -1)$ and $C(5, 2)$. Find
   (a) the coordinates of D,
   (b) the equation of AD.

   If the point P lies on AD such that the area of the trapezium $PABC$ is 9 square units, find the coordinates of P.



4. Two variables x and y are related by $x^n y = A$. Experimental values of y are obtained for various values of x and tabulated as shown below.

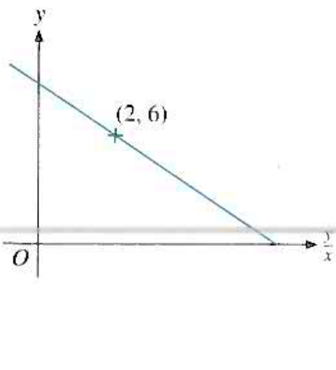
x	0.2	0.4	0.6	0.8	1.0
y	32.50	4.06	1.20	0.51	0.11

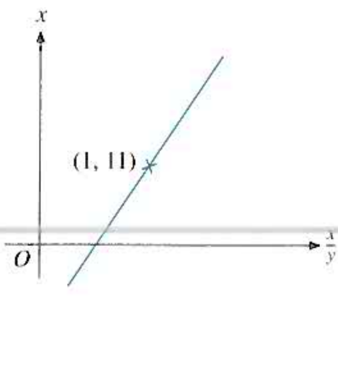
   One of the readings is subject to an abnormally large error. Plot a suitable straight line graph for the given data and use the graph to
   (a) identify the abnormal reading and estimate its correct value,
   (b) estimate the value of A and of n.

5. If the function f defined by $f(x) = \dfrac{ax}{2x - 3}$, $x \neq \dfrac{3}{2}$, satisfies $ff(x) = x$ for all real values of x except $\dfrac{3}{2}$, find the value of a.

6. Sketch the graph of $y = |x^2 - 5x|$. Find the range of values of x for which $|x^2 - 5x| < 6$.

7. Two variables x and y are related by a certain equation. This equation may be expressed in two forms suitable for drawing straight line graphs. The two graphs are shown, with the variables plotted at each axis and the coordinates of a point on each line. Find the equation relating x and y.





10 Trigonometric Functions

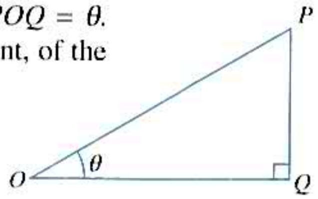
In this chapter, we shall see how the definitions of the cosine, sine and tangent ratios for acute angles can be extended to any angle and how these ratios are in turn related to those of acute angles. Some properties of these trigonometric functions will then be studied using their graphs followed by a discussion on how to sketch simple trigonometric curves. Finally, we shall introduce three more trigonometric functions.

Further discussions on trigonometric functions and their properties are covered in chapter 11.

10.1 Trigonometric Ratios and General Angles

Trigonometric Ratios of Acute Angles

The diagram shows the right-angled triangle OPQ with $\angle POQ = \theta$. The three trigonometric ratios, namely cosine, sine and tangent, of the acute angle θ are defined respectively as follows:

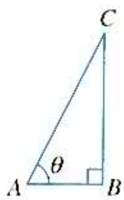


$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{OQ}{OP}$$

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{PQ}{OP}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{PQ}{OQ}$$

Example 1 Given the right-angled triangle ABC and that $\tan\theta = 2$, find $\sin\theta$ and $\cos\theta$.



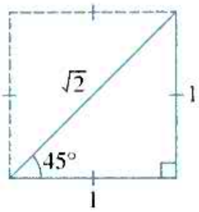
Solution: Since $\tan\theta = \frac{2}{1}$, let $BC = 2$ units and $AB = 1$ unit.

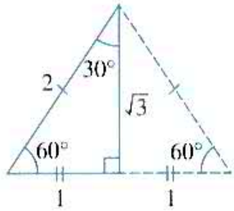
By Pythagoras' Theorem, $AC = \sqrt{5}$ units.

$\therefore\ \sin\theta = \frac{BC}{AC} = \frac{2}{\sqrt{5}}$ and $\cos\theta = \frac{1}{\sqrt{5}}$

Trigonometric Ratios of Some Special Angles

The trigonometric ratios of angles measuring 30°, 45° and 60° can be obtained using the right-angled triangles formed by taking 'half' of a square and an equilateral triangle. (See diagrams below).





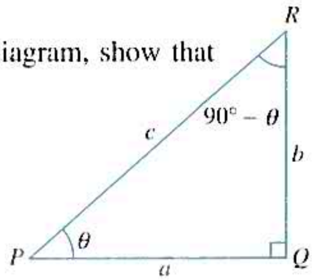
$\cos 45° = \frac{1}{\sqrt{2}}$ $\qquad$ $\cos 60° = \frac{1}{2}$ $\qquad$ $\cos 30° = \frac{\sqrt{3}}{2}$

$\sin 45° = \frac{1}{\sqrt{2}}$ $\qquad$ $\sin 60° = \frac{\sqrt{3}}{2}$ $\qquad$ $\sin 30° = \frac{1}{2}$

$\tan 45° = 1$ $\qquad$ $\tan 60° = \sqrt{3}$ $\qquad$ $\tan 30° = \frac{1}{\sqrt{3}}$

Trigonometric Ratios of Complementary Angles

In the triangle PQR in the diagram, $\angle P$ and $\angle R$ are complementary angles as $\angle P + \angle R = 90°$.



Example 2 Using the right-angled triangle in the diagram, show that

$$\sin(90° - \theta) = \cos\theta.$$

Hence deduce the value of

$$\frac{\cos 20°}{\cos 20° + \sin 70°}.$$

Solution: We have $\cos\theta = \frac{a}{c}$ and $\sin(90° - \theta) = \frac{a}{c}$.

$\therefore \sin(90° - \theta) = \cos\theta$

Since $\sin 70° = \sin(90° - 20°) = \cos 20°$,

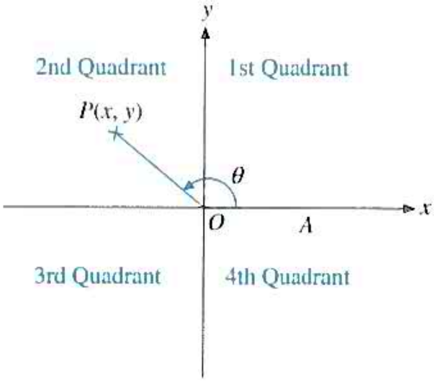
$$\frac{\cos 20°}{\cos 20° + \sin 70°} = \frac{\cos 20°}{2\cos 20°} = \frac{1}{2}.$$

Note: From the diagram we see that for complementary angles θ and $90° - \theta$:

$$\mathbf{\sin(90° - \theta) = \cos\theta}$$
$$\mathbf{\cos(90° - \theta) = \sin\theta}$$
$$\mathbf{\tan(90° - \theta) = \frac{1}{\tan\theta}}$$

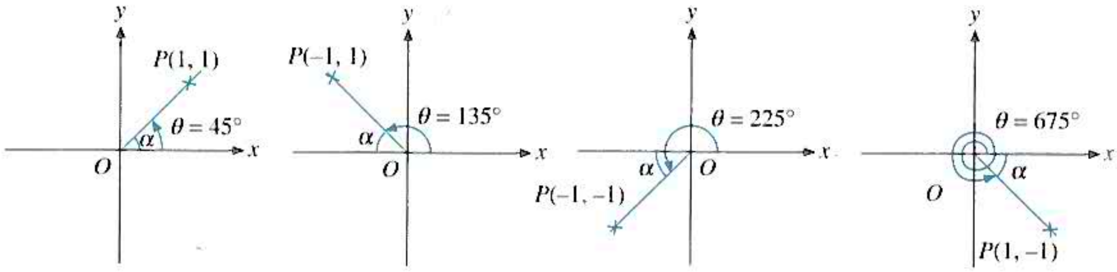
General Angles

So far our discussion has been on acute angles. Let us now consider angles in a cartesian plane. First, observe that the x- and y-axes divide the plane into the 1st, 2nd, 3rd and 4th quadrants in an **anticlockwise** sense as shown in the diagram.

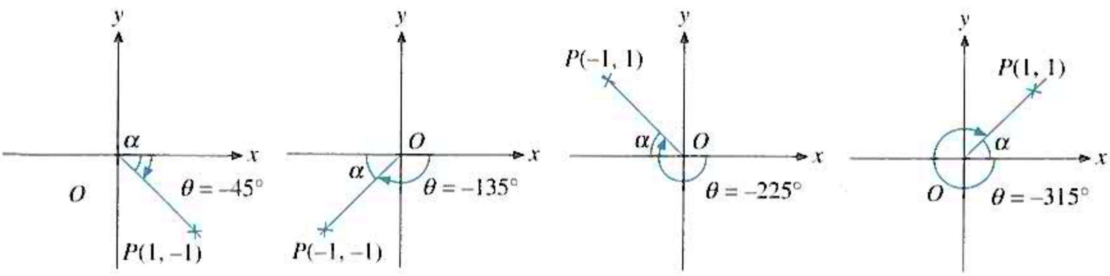


An initial line segment OA on the positive x-axis is rotated in the anticlockwise sense until it reaches OP. The angle θ is measured from the **positive** x-axis and it is said to be in the quadrant where OP lies.

The following four diagrams show some values of θ in the four quadrants, with an acute angle $\alpha = 45°$. Observe that all the angles θ are in the **anticlockwise** sense as indicated by the arrowheads and the values of θ are said to be **positive**.



In the following four diagrams, $\alpha = 45°$. Observe that the angles θ are **negative** in value because they are in the **clockwise** sense as indicated by the arrowheads.

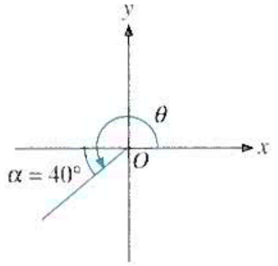


In the above eight diagrams shown, α is the positive acute angle between OP and the x-axis. The angle α is known as the **basic angle**, **associated acute angle** or **reference angle**.

Example 3 Given that $0° < \theta < 360°$ and θ has a basic angle of $40°$, find θ if it is in the
(a) 3rd quadrant,
(b) 4th quadrant.

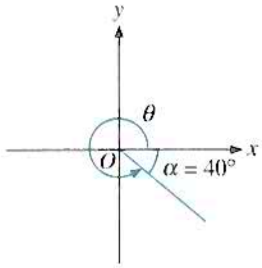
Solution:

(a)



$$\theta = 180° + \alpha$$
$$= \mathbf{220°}$$

(b)

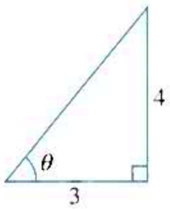


$$\theta = 360° - \alpha$$
$$= \mathbf{320°}$$

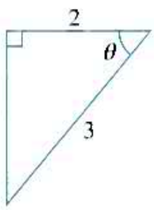
Exercise 10.1

1. For each of the following triangles, find the values of $\cos\theta$, $\sin\theta$ and $\tan\theta$.

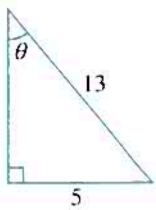
(a)



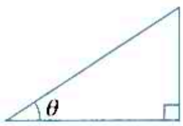
(b)



(c)



2. For the given right-angled triangle below, complete the table of trigonometric ratios on the right.



	$\cos\theta$	$\sin\theta$	$\tan\theta$
(a)	$\frac{8}{17}$		
(b)			$\frac{1}{2}$
(c)		$\frac{7}{25}$	
(d)			3

3. Given that $\sin\theta = \frac{1}{2}$, find the value of $\sin\theta\cos(90° - \theta)$.

4. Given that $\tan A = 2$, find the value of $2\tan A + \tan(90° - A)$.

5. Without using a calculator, evaluate

(a) $\dfrac{\sin 45°}{\cos 30° + \sin 60°}$,

(b) $\tan 45° + \tan 30° \tan 60°$.

6. Without using a calculator, find the value of

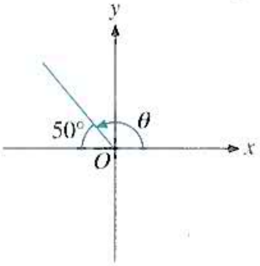
(a) $\dfrac{\sin 65^\circ}{\cos 25^\circ}$, (b) $\tan 75^\circ \tan 15^\circ$.

7. State the quadrant of the angle θ and find the value of the basic angle α if

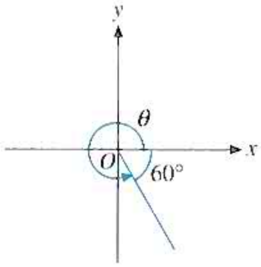
(a) $\theta = 250^\circ$, (b) $\theta = 390^\circ$, (c) $\theta = -60^\circ$, (d) $\theta = -100^\circ$.

8. For each of the following diagrams, find the value of the angle θ.

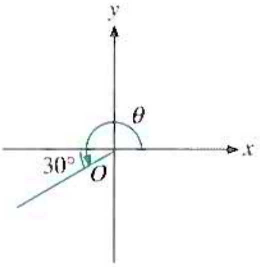
(a)



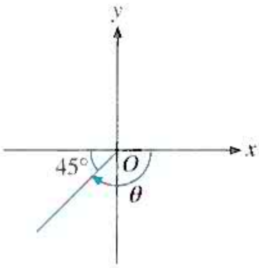
(b)



(c)



(d)



9. Find all the angles between 0° and 360° which make a basic angle of α with the x-axis if

(a) $\alpha = 20^\circ$, (b) $\alpha = 70^\circ$, (c) $\alpha = 35^\circ$.

10. Find all the angles between -180° and 180° with basic angle $\alpha = 10^\circ$.

*11. Given $-360^\circ < A < 720^\circ$ and that A has a basic angle $\alpha = 80^\circ$, find all the values of A if A is in the

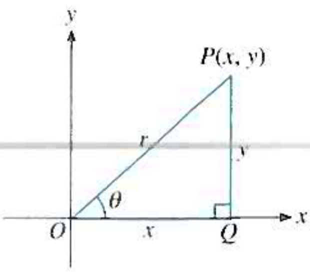
(a) 2nd quadrant, (b) 3rd quadrant, (c) 1st quadrant.

10.2 Trigonometric Ratios of Any Angles

We are now ready to introduce trigonometric ratios for any angle. We begin with the case where θ is acute by drawing a right-angled triangle as shown below.

Observe that θ is in the 1st quadrant, $OQ = x$, $PQ = y$ and $OP = r = \sqrt{x^2 + y^2}$.
Using the triangle OPQ, we have:

$$\cos\theta = \frac{OQ}{OP},\ \sin\theta = \frac{PQ}{OP},\ \tan\theta = \frac{PQ}{OQ}$$



In terms of x, y and r, we have:

$$\cos\theta = \frac{x}{r},\ \sin\theta = \frac{y}{r},\ \tan\theta = \frac{y}{x}$$

Thus, by defining the ratios in terms of the coordinates (x, y) and the length r, the definitions of trigonometric ratios are extended to any angle.

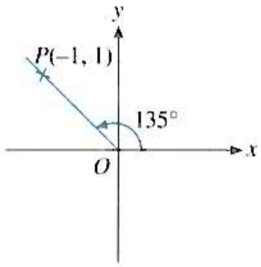
Example 4 Find the values of $\cos\theta$, $\sin\theta$ and $\tan\theta$ when $\theta = 135°$.

Solution: When $\theta = 135°$, P has coordinates $(-1, 1)$.
So, $x = -1$, $y = 1$
and $r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$$\therefore\ \cos 135° = \frac{x}{r} = -\frac{1}{\sqrt{2}}$$

$$\sin 135° = \frac{y}{r} = \frac{1}{\sqrt{2}}$$

$$\tan 135° = \frac{y}{x} = -1$$

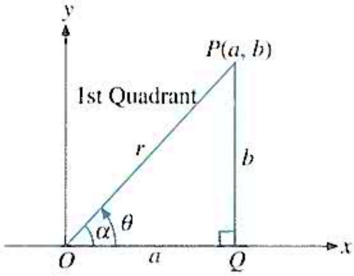


Signs of Trigonometric Ratios in the Four Quadrants

Observe from the above example that the signs of the trigonometric ratios depend on the signs of x and y and hence on the quadrant in which the angle θ lies.

We shall now examine the trigonometric ratios of angles in the four quadrants. Take note of the signs of the trigonometric ratios and how these ratios are related to the positive ratios of the basic angle α.

(a) In the diagram on the right, θ is in the 1st quadrant, $\alpha = \theta$ and P has coordinates (a, b), where $a > 0$ and $b > 0$.

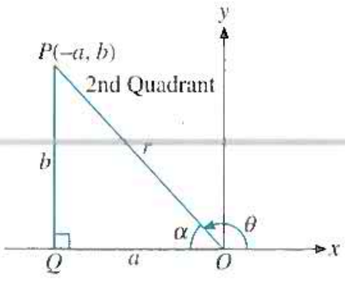


$$\therefore\ \cos\theta = \frac{x}{r} = \frac{a}{r} = \cos\alpha$$

$$\sin\theta = \frac{y}{r} = \frac{b}{r} = \sin\alpha$$

$$\tan\theta = \frac{y}{x} = \frac{b}{a} = \tan\alpha$$

(b) In the diagram on the right, θ is in the 2nd quadrant, $\alpha = 180° - \theta$ and P has coordinates $(-a, b)$. So,



$$\cos\theta = \frac{-a}{r},\ \sin\theta = \frac{b}{r},\ \tan\theta = \frac{b}{-a}$$

From $\triangle OPQ$,

$$\cos\alpha = \frac{a}{r},\ \sin\alpha = \frac{b}{r},\ \tan\alpha = \frac{b}{a}$$

So, $\cos\theta = -\cos\alpha$, $\sin\theta = \sin\alpha$,

$\tan\theta = -\tan\alpha$

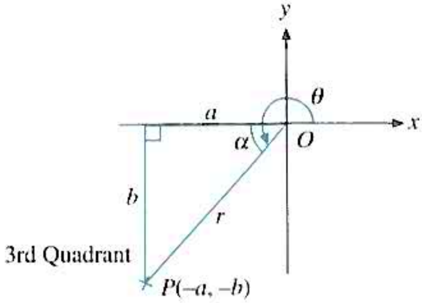
Using these results for Example 4, $\theta = 135°$ and $\alpha = 180° - \theta = 45°$. So,

$$\cos 135° = -\cos 45° = -\frac{1}{\sqrt{2}},$$

$$\sin 135° = \sin 45° = \frac{1}{\sqrt{2}},$$

$$\tan 135° = -\tan 45° = -1, \text{ as before.}$$

(c) In the diagram on the right, θ is in the 3rd quadrant, $\alpha = \theta - 180°$ and P has coordinates $(-a, -b)$. So,

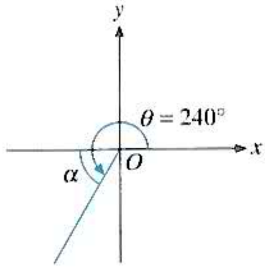


$$\cos\theta = \frac{-a}{r} = -\cos\alpha,$$

$$\sin\theta = \frac{-b}{r} = -\sin\alpha,$$

$$\tan\theta = \frac{-b}{-a} = \tan\alpha.$$

For example, when $\theta = 240°$, $\alpha = \theta - 180° = 60°$.

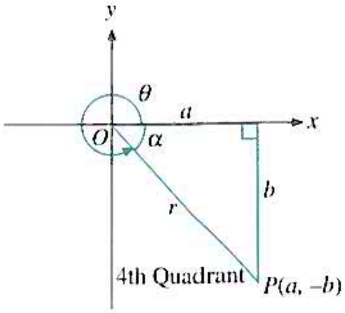


$$\therefore\ \cos 240° = -\cos 60° = -\frac{1}{2},$$

$$\sin 240° = -\sin 60° = -\frac{\sqrt{3}}{2},$$

$$\tan 240° = \tan 60° = \sqrt{3}.$$

(d) In the diagram on the right, θ is in the 4th quadrant, $\alpha = 360° - \theta$ and P has coordinates $(a, -b)$. So,

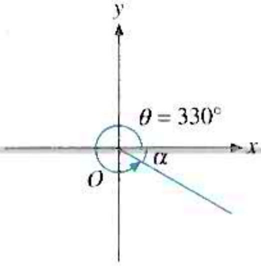


$$\cos\theta = \frac{a}{r} = \cos\alpha,$$

$$\sin\theta = \frac{-b}{r} = -\sin\alpha,$$

$$\tan\theta = \frac{-b}{a} = -\tan\alpha.$$

For example, when $\theta = 330°$,
$\alpha = 360° - \theta = 30°$.



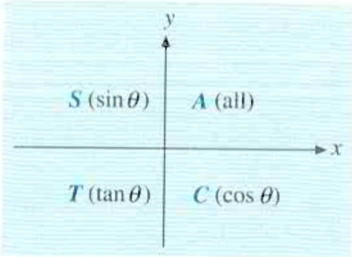
$$\therefore\ \cos 330° = \cos 30° = \frac{\sqrt{3}}{2},$$

$$\sin 330° = -\sin 30° = -\frac{1}{2},$$

$$\tan 330° = -\tan 30° = -\frac{1}{\sqrt{3}}.$$

Notice that in the 1st quadrant, all three ratios are positive,
in the 2nd quadrant, only $\sin\theta$ is positive,
in the 3rd quadrant, only $\tan\theta$ is positive,
in the 4th quadrant, only $\cos\theta$ is positive.

These results are summarised below using the initial letters of the **positive** ratios in the four quadrants.



With the aid of the diagram above, we can easily write down the trigonometric ratios of θ in terms of those of the basic angle α as follows:

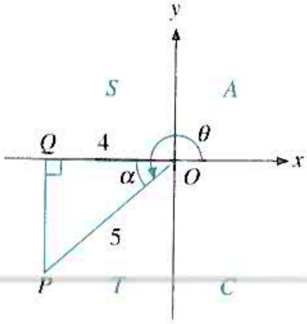
$\sin\theta = \sin\alpha$,	if θ is in the 1st or 2nd quadrant
$\sin\theta = -\sin\alpha$,	if θ is in the 3rd or 4th quadrant
$\cos\theta = \cos\alpha$,	if θ is in the 1st or 4th quadrant
$\cos\theta = -\cos\alpha$,	if θ is in the 2nd or 3rd quadrant
$\tan\theta = \tan\alpha$,	if θ is in the 1st or 3rd quadrant
$\tan\theta = -\tan\alpha$,	if θ is in the 2nd or 4th quadrant

Example 5 Given that $\cos\theta = -\frac{4}{5}$ and $180° < \theta < 270°$, evaluate $\tan\theta$ and $\sin\theta$.

Solution: $180° < \theta < 270° \Rightarrow \theta$ is in the 3rd quadrant and so

$$\cos\theta = -\cos\alpha = -\frac{4}{5}$$

$$\Rightarrow \cos\alpha = \frac{4}{5}$$



Let $OQ = 4$ units and $OP = 5$ units.

Then $PQ^2 = \sqrt{5^2 - 4^2}$

$PQ = 3$

$\therefore\ \tan\theta = \tan\alpha = \frac{3}{4}$

$\sin\theta = -\sin\alpha = -\frac{3}{5}$

Example 6 Given that $\tan\theta = -\frac{1}{2}$ and that $\tan\theta$ and $\sin\theta$ have opposite signs, find the value of $\cos\theta$ and of $\sin\theta$.

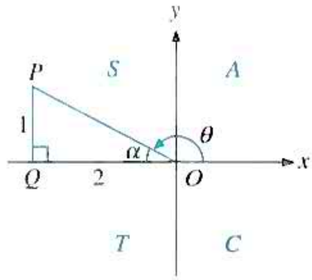
Solution: Since $\tan\theta$ and $\sin\theta$ have opposite signs,

$\tan\theta < 0 \Rightarrow \sin\theta > 0.$

θ is in the 2nd quadrant and so

$$\tan\theta = -\tan\alpha = -\frac{1}{2}$$

$$\Rightarrow \tan\alpha = \frac{1}{2}$$



Let $PQ = 1$ unit and $OQ = 2$ units.

By Pythagoras' Theorem, $OP = \sqrt{5}$ units.

$$\therefore \cos\theta = -\cos\alpha = -\frac{2}{\sqrt{5}}$$

$$\sin\theta = \sin\alpha = \frac{1}{\sqrt{5}}$$

Example 7 Express the trigonometric ratios of $-70°$ in terms of the ratios of its basic angle.

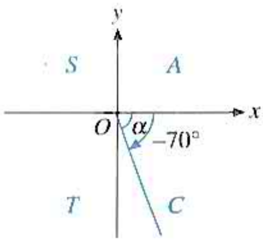
Solution: $\theta = -70°$ is in the 4th quadrant

$\alpha = 70°$

$\therefore \cos(-70°) = \cos\alpha = \cos 70°$

$\sin(-70°) = -\sin\alpha = -\sin 70°$

$\tan(-70°) = -\tan\alpha = -\tan 70°$



The above results are particular cases of the following more general results:

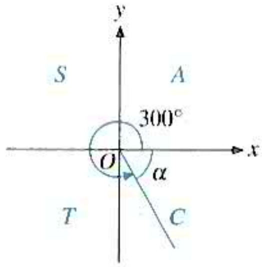
For any angle θ,

$$\cos(-\theta) = \cos\theta$$
$$\sin(-\theta) = -\sin\theta$$
$$\tan(-\theta) = -\tan\theta$$

Thus, $\cos(-150°) = \cos 150°$, $\sin(-40°) = -\sin 40°$ and $\tan(-300°) = -\tan 300°$.

Example 8 Without using a calculator, evaluate the trigonometric ratios of 300°. Hence deduce the trigonometric ratios of 660°.

Solution: $\theta = 300°$ is in the 4th quadrant and $\alpha = 360° - 300° = 60°$.

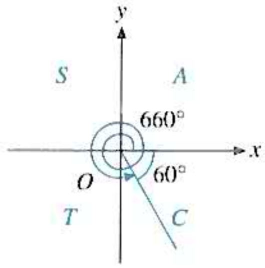


$$\therefore \cos 300° = \cos \alpha = \cos 60° = \frac{1}{2}$$

$$\sin 300° = -\sin \alpha = -\sin 60° = -\frac{\sqrt{3}}{2}$$

$$\tan 300° = -\tan \alpha = -\tan 60° = -\sqrt{3}$$

Note that both the angles, 300° and 660° (= 300° + 360°) are in the 4th quadrant and they have the same basic angle $\alpha = 60°$.



$$\therefore \cos 660° = \cos 300° = \frac{1}{2}$$

$$\sin 660° = \sin 300° = -\frac{\sqrt{3}}{2}$$

$$\tan 660° = \tan 300° = -\sqrt{3}$$

Basic Trigonometric Equations

Basic trigonometric equations of the forms $\sin \theta = k$, $\cos \theta = k$ and $\tan \theta = k$, where k is a constant, are solved as follows:

(a) By considering the sign of k, identify the possible quadrants in which θ lies.

(b) Find the basic angle, α. Remember that α is acute and $\sin \alpha$, $\cos \alpha$ and $\tan \alpha$ are all positive.

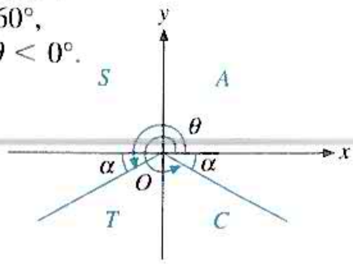
(c) Find all the values of θ in the required interval. Remember that θ must be measured from the **positive** x-axis.

Example 9 Find all the values of θ such that

(a) $\sin \theta = -0.5$, where $0° < \theta < 360°$,

(b) $\tan \theta = -1$, where $0° < \theta < 360°$,

(c) $\cos \theta = 0.46$, where $-360° < \theta < 0°$.



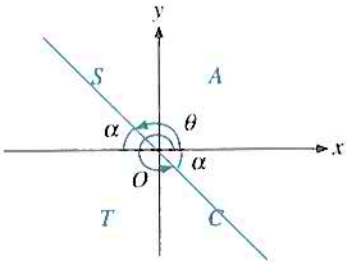
Solution: **(a)** $\sin \theta = -0.5$,

$0° < \theta < 360°$

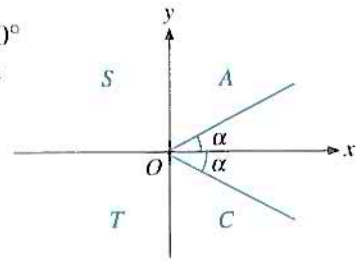
$\sin \theta < 0 \Rightarrow \theta$ is in the 3rd or 4th quadrant.

For the acute angle α,
$\sin \alpha > 0$
and so $\sin \alpha = 0.5$
$\Rightarrow \quad \alpha = 30°$
$\therefore \quad \theta = 180° + \alpha, 360° - \alpha$
$= \mathbf{210°, 330°}$

(b) $\tan \theta = -1$,
$0° < \theta < 360°$
$\tan \theta < 0 \Rightarrow \theta$ is in the 2nd or 4th quadrant.
$\tan \alpha = 1 \Rightarrow \alpha = 45°$
$\therefore \quad \theta = 180° - \alpha, 360° - \alpha$
$= \mathbf{135°, 315°}$



(c) $\cos \theta = 0.46, -360° < \theta < 0°$
$\cos \theta > 0 \Rightarrow \theta$ is in the 1st or 4th quadrant.
$\cos \alpha = 0.46 \Rightarrow \alpha = 62.6°$
$\therefore \theta = -\alpha, -(360° - \alpha)$
$= \mathbf{-62.6°, -297.4°}$



Exercise 10.2

1. Without using a calculator, determine whether the following trigonometric ratios are positive or negative.
   (a) $\sin 230°$ **(b)** $\cos 140°$ **(c)** $\tan 215°$ **(d)** $\cos 350°$
   (e) $\tan 340°$ **(f)** $\sin 160°$ **(g)** $\cos (-60°)$ **(h)** $\tan (-155°)$

2. For each of the following conditions, determine the quadrant or possible quadrants in which θ must lie.
   (a) $\tan \theta > 0$
   (b) $\cos \theta > 0$ and $\sin \theta < 0$
   (c) $\cos \theta$ and $\tan \theta$ are of the same sign
   (d) $\sin \theta$ and $\tan \theta$ are of opposite signs

For questions 3 to 11, do not use a calculator.

3. Evaluate the following:
   (a) $\tan 300°$ **(b)** $\cos 330°$ **(c)** $\sin 150°$ **(d)** $\tan 315°$
   (e) $\sin 225°$ **(f)** $\cos 210°$ **(g)** $\tan (-120°)$ **(h)** $\sin 405°$

4. If θ is acute and $\cos \theta = \frac{4}{5}$, find the value of $\sin \theta$ and of $\tan \theta$.

5. Given that A is obtuse and that $\tan A = -\frac{1}{2}$, find the value of $\cos A$ and of $\sin A$.

6. Given that $\sin A = -\frac{5}{13}$ where $180° < A < 270°$, find the value of $\tan A$ and of $\cos(-A)$.

7. Given that $\sin A = \frac{2}{\sqrt{5}}$ and $90° < A < 180°$, find the value of $\cos A$ and of $\tan A$.

8. Given that $\cos A = \frac{1}{2}$ and that $\cos A$ and $\sin A$ have the same sign, find the value of $\sin(-A)$ and of $\tan A$.

9. Given that $\tan A = -\frac{5}{12}$ and that $\tan A$ and $\cos A$ have opposite signs, find the value of $\cos A$ and of $\cos(90° - A)$.

10. Given that $\cos A = \sqrt{\frac{2}{3}}$ where $180° < A < 360°$, find the value of

 (a) $\sin A$, **(b)** $\sin(90° - A)$, **(c)** $\tan(90° - A)$.

*11. Given that $\sin 20° = k$, express the following in terms of k.

 (a) $\sin 200°$ **(b)** $\cos 20°$ **(c)** $\tan(-20°)$ **(d)** $\sin 70°$

12. Find all the angles x where $0° < x < 360°$ such that

 (a) $\cos x = -0.71$, **(b)** $\tan x = 1.732$,

 (c) $\sin x = 0.866$, **(d)** $\tan x = -2$,

 (e) $10 \cos x - 3 = 0$, **(f)** $4(\tan x - 1) = 3(5 - 2 \tan x)$,

 (g) $2 \sin(-x) = 0.3$, **(h)** $2 \cos^2 x = 1$,

 (i) $3 \sin x + 2 = \tan 75°$, **(j)** $\frac{8 \cos x + 1}{2 - \cos x} = 3$.

13. Find all the angles between $-360°$ and $180°$ such that

 (a) $\sin x = -\frac{1}{2}$ **(b)** $\cos x = \frac{\sqrt{3}}{2}$

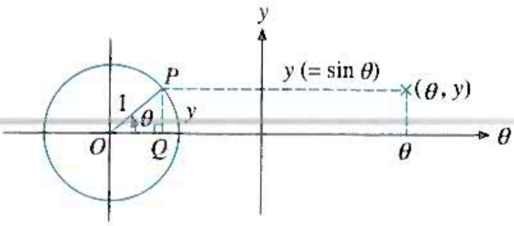
 (c) $\tan(-x) + 1 = 0$ **(d)** $\sqrt{2} \sin(90° - x) + 1 = 0$

14. Positive angles x and y are such that $x + 2y = 300°$ and $\tan y = 2 \cos 160°$. Find their values.

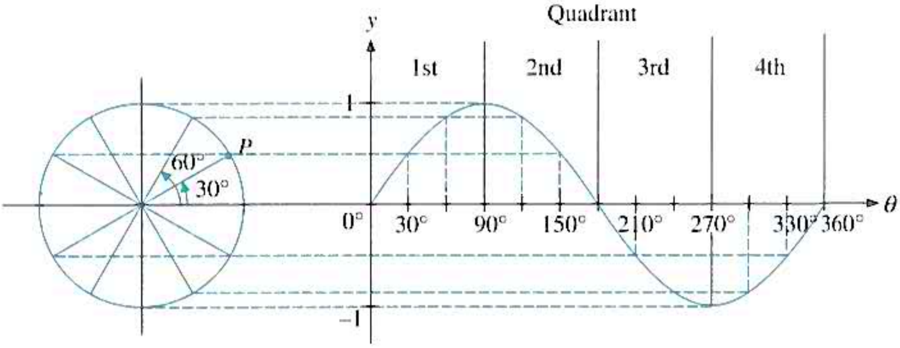
10.3 Graphs of the Sine, Cosine and Tangent Functions

(I) Graph of the Sine Function

The diagram shows a circle of radius 1 unit. From triangle OPQ, $\sin \theta = \frac{y}{1}$ and $y = \sin \theta$ defines a function known as the **sine function**. Observe that y is related to θ and the value of y can be plotted against the value of θ as shown.



By plotting the values of y against the corresponding values of θ, the graph of $y = \sin\theta$ for $0° \leqslant \theta \leqslant 360°$, is obtained as shown below.

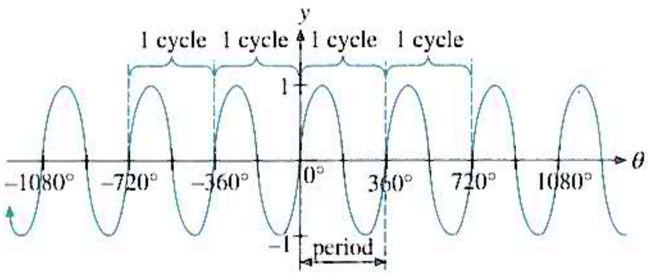


From the graph, observe the following:

(a) At the horizontal axis, $\sin\theta = 0$ when $\theta = 0°, 180°, 360°$.

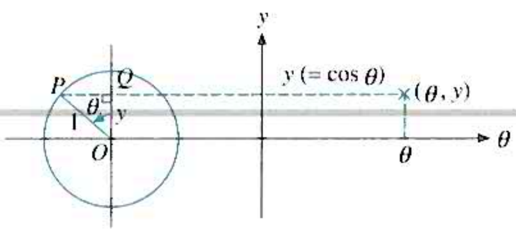
(b) The maximum value of $\sin\theta$ is 1 when $\theta = 90°$.
The minimum value of $\sin\theta$ is -1 when $\theta = 270°$.
Hence $-1 \leqslant \sin\theta \leqslant 1$.

(c) As P moves round the circle in either the clockwise or anticlockwise sense, the sine curve above repeats itself for every interval of $360°$ on either side. The length of the interval over which the sine curve repeats itself is called the **period**. Hence the sine function has a period of $360°$ as shown below.

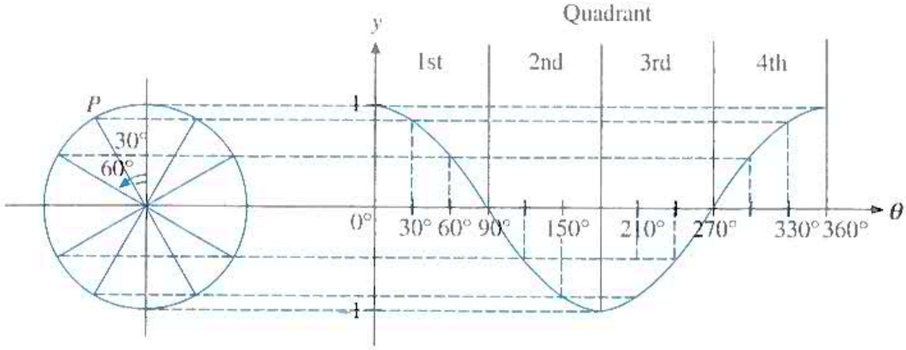


(II) Graph of the Cosine Function

The diagram shows a circle of radius 1 unit. From triangle OPQ, $\cos\theta = \frac{y}{1}$ and the **cosine function** $y = \cos\theta$ relates y to θ. The value of y can be plotted against the value of θ as shown.



By plotting the values of y against the corresponding values of θ, the graph of $y = \cos\,\theta$ for $0° \leqslant \theta \leqslant 360°$ is obtained as shown below.



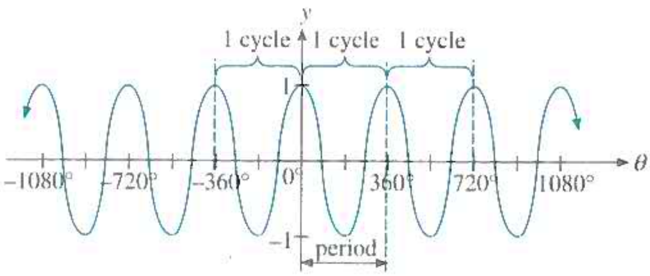
From the graph, observe the following:

(a) At the horizontal axis, $\cos\,\theta = 0$ when $\theta = 90°, 270°$.

(b) The maximum value of $\cos\,\theta$ is 1 when $\theta = 0°, 360°$.
The minimum value of $\cos\,\theta$ is -1 when $\theta = 180°$.
Hence $-1 \leqslant \cos\,\theta \leqslant 1$.

Note: There is no value of θ for which $\cos\,\theta < -1$ or $\cos\,\theta > 1$.

(c) As point P moves round the circle in either the clockwise or anticlockwise sense, the cosine curve repeats itself on either side. The cosine function also has a period of 360° as shown below.



Example 10 Given that $(\sin\,\theta - 1)(\cos\,\theta + 2) = 0$, find the value of θ where $0° \leqslant \theta \leqslant 360°$.

Solution

$(\sin\,\theta - 1)(\cos\,\theta + 2) = 0$

$\Rightarrow \sin\,\theta - 1 = 0 \quad$ or $\quad \cos\,\theta + 2 = 0$

$\sin\,\theta = 1 \quad$ or $\quad \cos\,\theta = -2$ (no solution)

$\theta = \mathbf{90°}$

Example 11 If $0° \leqslant \theta \leqslant 360°$, find the range of values of $3 - 2\cos\theta$. Hence state its minimum and maximum values.

Solution: For $0° \leqslant \theta \leqslant 360°$, $-1 \leqslant \cos\theta \leqslant 1$

Multiplying throughout by -2 (<0), the inequality signs are reversed.

$$-2(-1) \geqslant -2\cos\theta \geqslant -2(1)$$

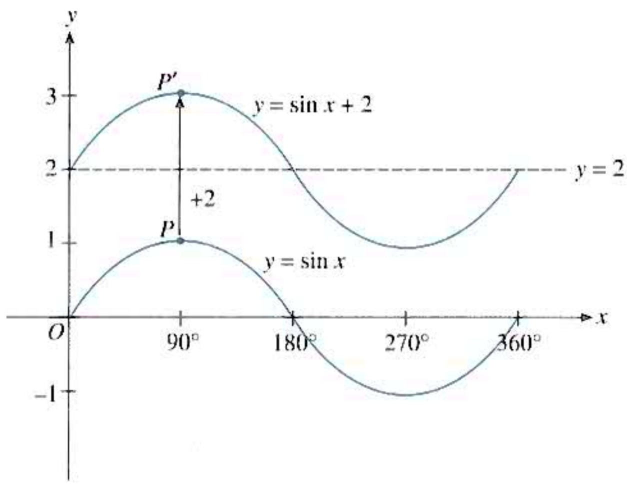
$$\Rightarrow 2 + 3 \geqslant -2\cos\theta + 3 \geqslant -2 + 3$$

$$\therefore \mathbf{1 \leqslant 3 - 2\cos\theta \leqslant 5}$$

Its minimum value is **1** and its maximum value is **5**.

Example 12 Sketch the graph of $y = \sin x + 2$ for $0° \leqslant x \leqslant 360°$.

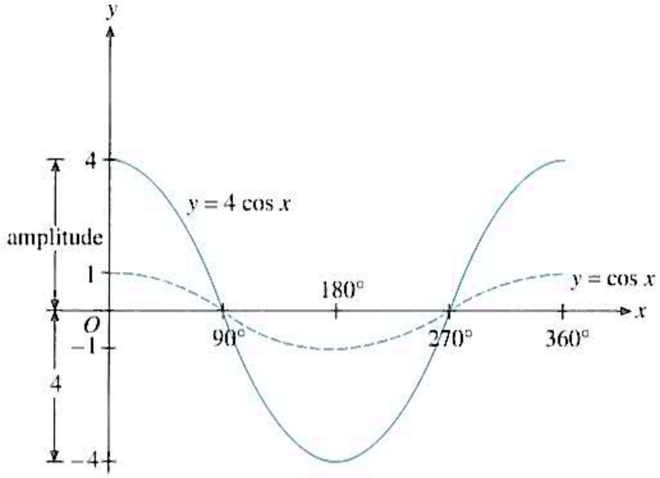
Solution: From the graph of $y = \sin x$, 2 is added to the value of y at each value of x. Hence the graph of $y = \sin x + 2$ is that of $y = \sin x$ shifted 2 units upwards.



Note: Observe that the line $y = \mathbf{2}$ passes through the 'middle' of the curve $y = \sin x + \mathbf{2}$. It is called the **axis of the curve**.

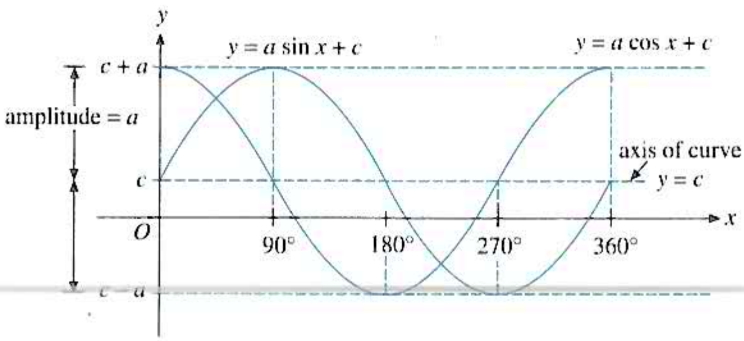
Example 13 Sketch the graph of $y = 4 \cos x$ for $0° \leqslant x \leqslant 360°$.

Solution: At each value of x, the value of y is obtained by multiplying $\cos x$ by 4. The graph of $y = 4 \cos x$ is sketched as follows:



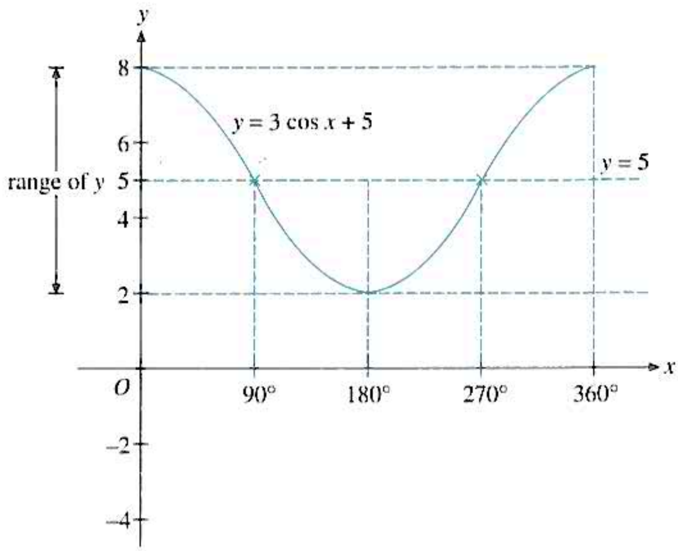
The distance of the maximum (or minimum) point from the axis of the curve is called the **amplitude**.

From Examples 12 and 13, we see that the curves $y = a \sin x + c$ and $y = a \cos x + c$, where $a > 0$, have axes $y = c$ and amplitude a. Hence their graphs are as follows:



Example 14 Sketch the graph of $y = 3\cos x + 5$ for the domain $0° \leq x \leq 360°$ and state the corresponding range of y.

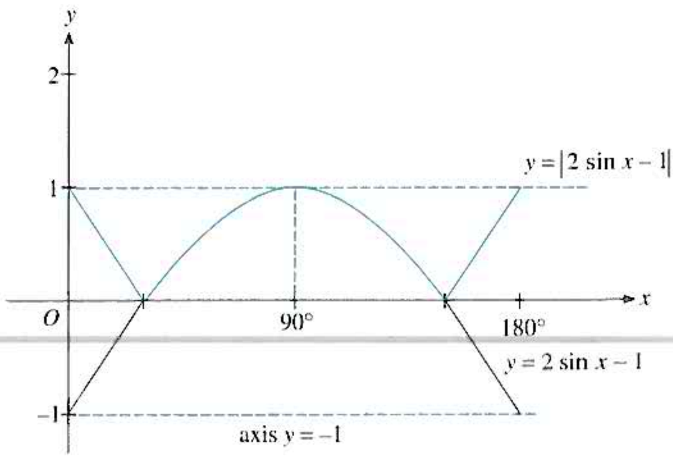
Solution: Observe that $a = 3$ and $c = 5$. So, the curve $y = 3\cos x + 5$ has axis $y = 5$ and amplitude 3 as sketched below:



From the graph, the range of y is $\{y: 2 \leq y \leq 8, y \in \mathbb{R}\}$.

Example 15 Sketch the graph of $y = |2\sin x - 1|$ for $0° \leq x \leq 180°$.

Solution: We first sketch the curve $y = 2\sin x - 1$ which has axis $y = -1$ and amplitude 2. The part of the curve below the x-axis is then reflected in the x-axis to obtain $y = |2\sin x - 1|$ as shown.

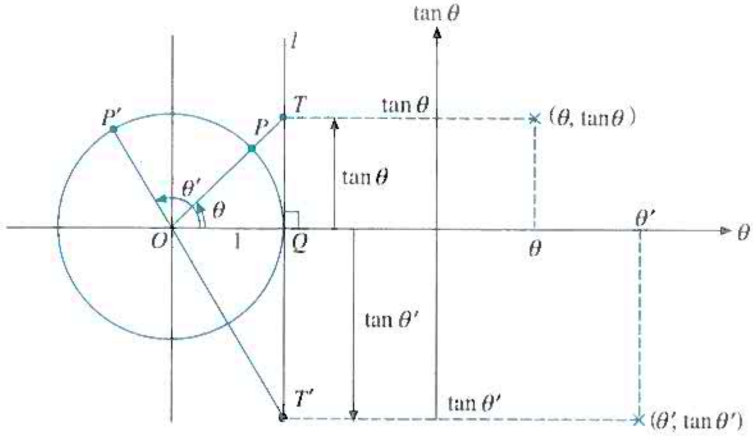


(III) Graph of the Tangent Function

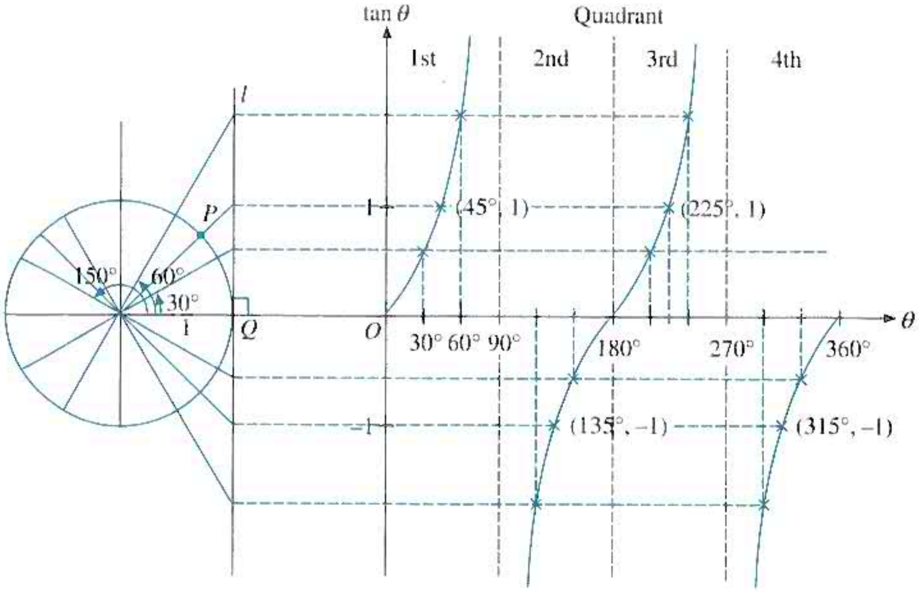
The line l is a tangent to the circle, radius 1 unit, at the point Q. OP is produced to meet l at T and from triangle OQT,

$$\frac{QT}{OQ} = \tan\theta \Rightarrow QT = \tan\theta.$$

For angles in the 2nd or 3rd quadrant such as θ', $P'O$ is produced to meet l at T' and so, $QT' = \tan\theta'$. The value of $\tan\theta$ can then be plotted against the value of θ as shown.



The graph of $y = \tan\theta$ for $0° \leqslant \theta \leqslant 360°$ is obtained as shown below.



From the graph, observe the following:

(a) The curve is **not** continuous; breaks occur at $\theta = 90°$ and $\theta = 270°$, where the function is undefined.

(b) The curve approaches the lines $\theta = 90°$ and $\theta = 270°$. Such lines are called **asymptotes**.

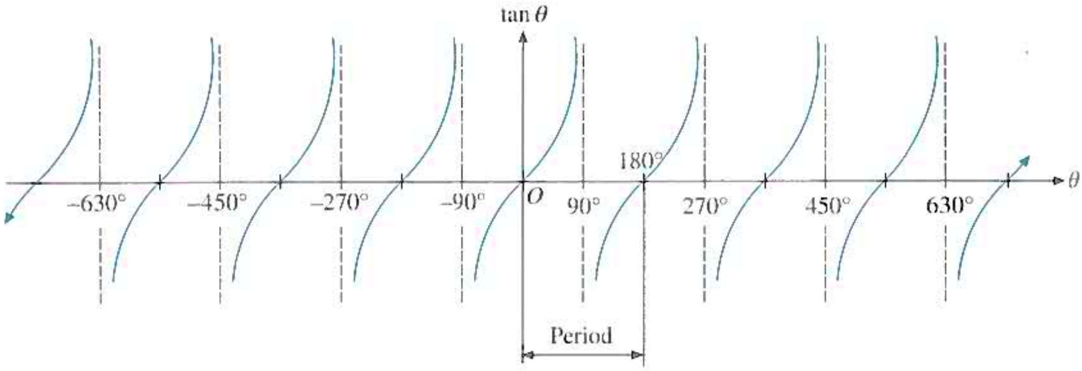
Observe also the following:

(a) At the horizontal axis, $\tan\theta = 0$ when $\theta = 0°, 180°, 360°$.

(b) Note that $\tan\theta = 1 \Rightarrow \theta = 45°, 225°$
and $\tan\theta = -1 \Rightarrow \theta = 135°, 315°$.

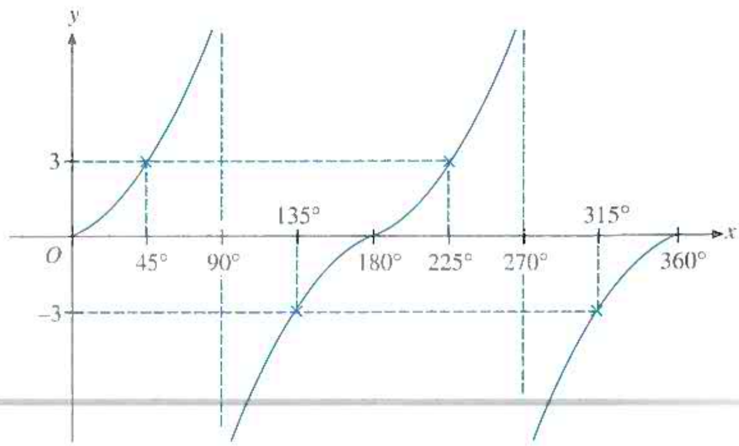
(c) Unlike $\sin\theta$ and $\cos\theta$, there are no maximum or minimum values for $\tan\theta$. So $\tan\theta$ can take any real value.

(d) As P moves round the circle in either the clockwise or anticlockwise sense, the curve repeats itself for every interval of 180°. Hence the tangent function has a period of 180° as shown below.



Example 16 Sketch the graph of $y = 3\tan x$ for $0° \leqslant x \leqslant 360°$ and state the corresponding range.

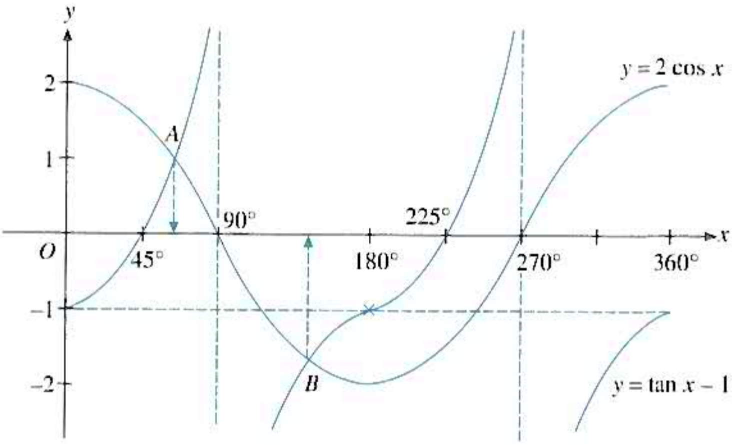
Solution: Each value of y is obtained by multiplying $\tan x$ by 3. When $x = 45°$, $y = 3\tan 45° = 3$. Note the values of x when $y = 3$ and $y = -3$.



$\therefore$ the corresponding range $= \mathbb{R}$

Example 17 On the same diagram sketch the graphs of $y = \tan x - 1$ and $y = 2 \cos x$ for the interval $0° \leqslant x \leqslant 360°$. State the number of solutions, in this interval, of the equation $\tan x = 2 \cos x + 1$.

Solution: The two graphs are sketched as shown.



$$\tan x = 2 \cos x + 1$$
$$\Rightarrow \quad \tan x - 1 = 2 \cos x$$

The required solutions are the values of x at which the graphs of $y = \tan x - 1$ and $y = 2 \cos x$ intersect. Since the graphs intersect at 2 points A and B, the equation has **2** solutions.

Exercise 10.3

1. Find the values of θ, where $0° \leqslant \theta \leqslant 360°$, such that
   (a) $(\sin \theta - 1)(\sin \theta + 1) = 0$, **(b)** $(\cos \theta - 1)(\cos \theta + 1) = 0$,
   (c) $\sin \theta(2 \cos \theta - 3) = 0$, **(d)** $\tan \theta(2 \cos \theta - 1) = 0$,
   (e) $\sin^2\theta(\tan \theta + 4) = 0$, **(f)** $(3 \sin \theta - 1)(\tan \theta + 1) = 0$.

2. If $0° \leqslant \theta \leqslant 360°$, find the minimum and maximum values of
   (a) $7 \sin \theta - 3$, **(b)** $5 \cos \theta + 2$, **(c)** $4 - 3 \sin \theta$.

3. Sketch, on separate diagrams, the following curves for the domain $0° \leqslant x \leqslant 360°$ and state the corresponding range of y.
   (a) $y = \sin x - 2$ **(b)** $y = 5 \cos x$ **(c)** $y = 4 \sin x - 2$
   (d) $y = 2 \cos x + 1$ **(e)** $y = 3(\cos x - 1)$ **(f)** $y = |3 \sin x - 2|$
   (g) $y = |5 \tan x|$ **(h)** $y = 2 \tan x - 3$ **(i)** $y = |3 \tan x + 2|$

4. Sketch, on separate diagrams, the following graphs.
   (a) $y = 4 \cos x - 3$ for $0° \leqslant x \leqslant 180°$
   (b) $y = 6 \tan x$ for $0° \leqslant x \leqslant 180°$
   (c) $y = 3 + 2 \sin x$ for $0° \leqslant x \leqslant 270°$
   (d) $y = \cos x - 1$ for $0° \leqslant x \leqslant 720°$
   (e) $y = 1 + 3 \sin x$ for $-360° \leqslant x \leqslant 720°$

5. On the same diagram, sketch the graphs of $y = |2 \sin x|$ and $y = |2 \sin x| + 1$ for $0° \leqslant x \leqslant 360°$.

6. Sketch the graph of $y = 3|\cos x| - 2$ for $0° \leqslant x \leqslant 360°$.

7. On the same diagram, sketch the graphs of $y = 3 \cos x$ and $y = 2 \sin x - 1$ for the interval $0° \leqslant x \leqslant 360°$. State the number of solutions, in this interval, of the equation $3 \cos x + 1 = 2 \sin x$.

8. Use a graphical method to determine how many solutions there are to the equation $|2 \tan x| = 1 + \sin x$ in the interval $0° \leqslant x \leqslant 360°$.

9. Sketch, on the same diagram, the graphs of $y = \tan x$ and $y = \cos x$, for the values of x from $0°$ and $360°$. Hence state
   (a) the number of roots of the equation $\tan x = \cos x$ in the range $0°$ to $360°$,
   (b) the range of values of x, between $0°$ and $270°$, for which $\tan x$ and $\cos x$ are both increasing as x increases. *(C)*

*10. Sketch, on the same diagram, the graphs of $y = 3 \cos x - 2$ and $y = 4 |\sin x|$ for the domain $0° \leqslant x \leqslant 360°$. Hence deduce the value of k for which the equation $3 \cos x - 2 = |4 \sin x| + k$ has 3 solutions in this domain.

11. **(a)** Use a graph plotter to obtain the graph of $y = \cos x \tan x$.
   This graph looks like another graph we know. Make a conjecture.
   (b) Obtain the graphs of $y = \sin^2 x$ and $y = \cos^2 x$.
   Next, obtain the graph of $y = \sin^2 x + \cos^2 x$ and you may be surprised by its shape.

   The above graphs illustrate 2 important results which we will prove in the next chapter.

10.4 Three More Trigonometric Functions

Three more trigonometric functions are defined by taking the **reciprocals** of the three trigonometric functions that we have learnt so far. We have:

the **sec**ant function, $\sec \theta = \dfrac{1}{\cos \theta}$, $\cos \theta \neq 0$

the **cosec**ant function, $\operatorname{cosec} \theta = \dfrac{1}{\sin \theta}$, $\sin \theta \neq 0$

the **cot**angent function, $\cot \theta = \dfrac{1}{\tan \theta}$, $\tan \theta \neq 0$.

Example 18 Given A is obtuse and $\cot A = -3$, find the exact value of
(a) $\sec A$, (b) $\operatorname{cosec} A$.

Solution:

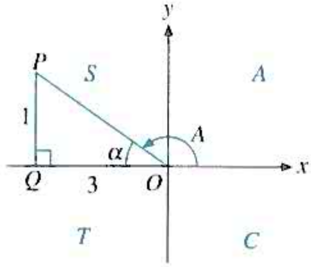
$\cot A = -3 \Rightarrow \dfrac{1}{\tan A} = -3 \Rightarrow \tan A = -\dfrac{1}{3}$.

A is in the 2nd quadrant and so $\tan A = -\tan\alpha = -\dfrac{1}{3}$.

$$\Rightarrow \tan\alpha = \frac{1}{3}$$

$OP = \sqrt{1^2 + 3^2} = \sqrt{10}$

(a) $\cos A = -\cos\alpha = -\dfrac{3}{\sqrt{10}}$



$\therefore \quad \sec A = \dfrac{1}{\cos A} = -\dfrac{\sqrt{10}}{3}$

(b) $\sin A = \sin\alpha = \dfrac{1}{\sqrt{10}}$

$\therefore \quad \operatorname{cosec} A = \dfrac{1}{\sin A} = \sqrt{10}$

Example 19 Solve the equation $\sec(-x) = 2$ for $0° \leqslant x \leqslant 360°$.

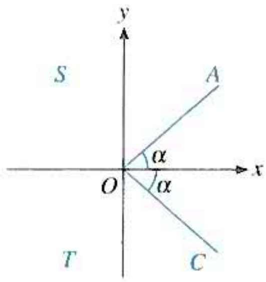
Solution:

$\sec(-x) = 2 \Rightarrow \dfrac{1}{\cos(-x)} = 2 \Rightarrow \cos(-x) = \dfrac{1}{2} \Rightarrow \cos x = \dfrac{1}{2}$

$\cos x > 0 \Rightarrow x$ is in the 1st or 4th quadrant.

$\cos\alpha = \dfrac{1}{2} \Rightarrow \alpha = 60°$

$\therefore \quad x = \alpha, 360° - \alpha$
$= 60°, 300°$



Exercise 10.4

1. Find the values of $\sec\theta$, $\operatorname{cosec}\theta$ and $\cot\theta$ given that
   (a) $\cos\theta = \frac{4}{5}$, θ is acute, (b) $\tan\theta = -\frac{3}{4}$, θ is reflex,
   (c) $\sin\theta = \frac{1}{2}$, θ is obtuse, (d) $\tan\theta = 3$, $90° < \theta < 270°$.

2. Given that $\tan\theta = -2$ and that $\tan\theta$ and $\cos\theta$ have opposite signs, find the value of
   (a) $\cot\theta$, (b) $\sin\theta$, (c) $\sec\theta$.

3. Given that $\sec\theta = 3$ and that θ is acute, find the value of
   (a) $\cos\theta$, (b) $\operatorname{cosec}\theta$, (c) $\tan(90° - \theta)$.

4. Given that $\cot\theta = -\frac{1}{\sqrt{2}}$ and that θ is obtuse, find the value of
   (a) $\tan(-\theta)$, (b) $\sec\theta$, (c) $\operatorname{cosec}(90° - \theta)$.

5. For each of the following conditions, state the possible quadrants in which θ must lie.
   (a) $\cot\theta < 0$
   (b) $\operatorname{cosec}\theta > 0$
   (c) $\sec\theta$ and $\operatorname{cosec}\theta$ have the same sign
   (d) $\cot\theta$ and $\sec\theta$ have opposite signs

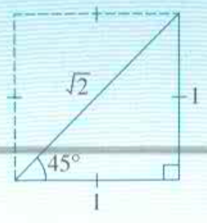
6. Solve the following equations for x between 0° and 360°.
   (a) $\operatorname{cosec} x = -2$ (b) $3\cot x = 4$
   (c) $\sec x = 5\sin 20°$ (d) $\cot(-x) = 2 + \cos 165°$
   (e) $\operatorname{cosec}^2 x = 1$ (f) $\sec x = 2\cos x$
   (g) $\cot^2 x = 3$ (h) $\sin x(\operatorname{cosec} x - \sqrt{2}) = 0$
   (i) $\cot x(\cos^2 x - 4) = 0$ (j) $\cot 50° \sec(-x) = 2$

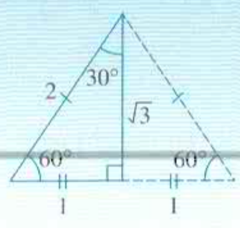
7. Without using a calculator, evaluate the following:
   (a) $\sec 60°$ (b) $\operatorname{cosec}(-30°)$ (c) $\cot 45°$
   (d) $\sec 150°$ (e) $\cot 330°$ (f) $\operatorname{cosec} 240°$

8. Sketch the graphs of $y = |\cos x|$ and $y = 3\sin x + 1$ on the same diagram for the domain $0° \leqslant x \leqslant 360°$. State the number of solutions in this domain of the equation.
   (a) $|\cos x| + 2 = 3(\sin x + 1)$ (b) $|\sec x(3\sin x + 1)| = 1$

Important Notes

1. **30°, 45°, 60°**
   Exact values of trigonometric ratios of 30°, 45° and 60° can be obtained from the following triangles.

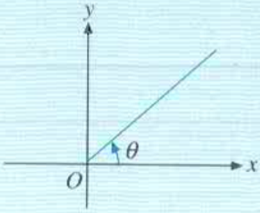




2. **General angles**

(a) For an angle θ measured from the positive x-axis:

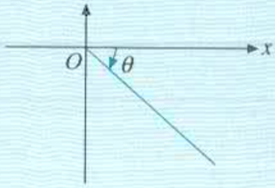
(i)



anticlockwise direction

$\theta > 0$

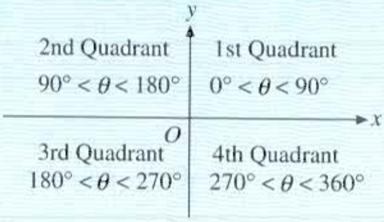
(ii)



clockwise direction

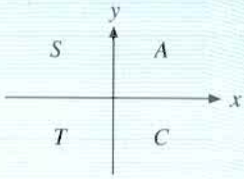
$\theta < 0$

(b) The quadrant of θ is determined by its value. For $0° < \theta < 360°$, we have:



3. **Trigonometric functions of any angle**

With the aid of the following diagram:



we have, for any angle θ with basic angle α:

$\sin\theta = \sin\alpha$,	if θ is in the 1st or 2nd quadrant
$\sin\theta = -\sin\alpha$,	if θ is in the 3rd or 4th quadrant
$\cos\theta = \cos\alpha$,	if θ is in the 1st or 4th quadrant
$\cos\theta = -\cos\alpha$,	if θ is in the 2nd or 3rd quadrant
$\tan\theta = \tan\alpha$,	if θ is in the 1st or 3rd quadrant
$\tan\theta = -\tan\alpha$,	if θ is in the 2nd or 4th quadrant

4. **Three more trigonometric functions are defined by:**

$$\sec\theta = \frac{1}{\cos\theta},\ \cos\theta \neq 0$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta},\ \sin\theta \neq 0$$

$$\cot\theta = \frac{1}{\tan\theta},\ \tan\theta \neq 0$$

5. **Useful relationships:**

(a) Negative angles:

$$\cos(-\theta) = \cos\theta$$
$$\sin(-\theta) = -\sin\theta$$
$$\tan(-\theta) = -\tan\theta$$

(b) Complementary angles: $\cos(90° - \theta) = \sin\theta$

$$\sin(90° - \theta) = \cos\theta$$

$$\tan(90° - \theta) = \frac{1}{\tan\theta},\ \tan\theta \neq 0$$

$$= \cot\theta$$

6. Graphs of trigonometric functions

For the interval $0° \leqslant \theta \leqslant 360°$:

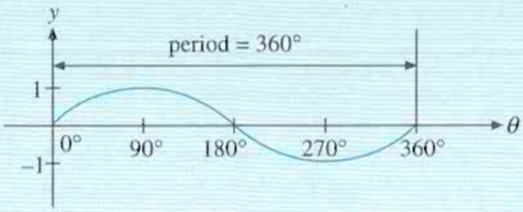
(a) $y = \sin\theta$

$-1 \leqslant \sin\theta \leqslant 1$

$\sin\theta = 0 \quad \Rightarrow \quad \theta = 0°, 180°, 360°$

$\sin\theta = 1 \quad \Rightarrow \quad \theta = 90°$

$\sin\theta = -1 \quad \Rightarrow \quad \theta = 270°$



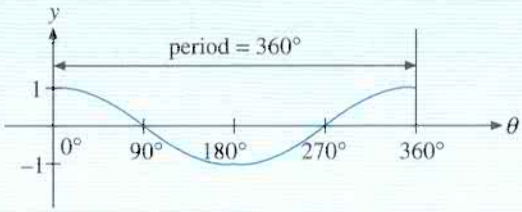
(b) $y = \cos\theta$

$-1 \leqslant \cos\theta \leqslant 1$

$\cos\theta = 0 \quad \Rightarrow \quad \theta = 90°, 270°$

$\cos\theta = 1 \quad \Rightarrow \quad \theta = 0°, 360°$

$\cos\theta = -1 \quad \Rightarrow \quad \theta = 180°$



(c) $y = \tan\theta$

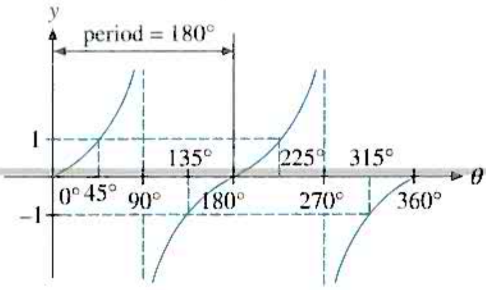
$\tan\theta$ can take any real value.

$\tan\theta = 0 \quad \Rightarrow \quad \theta = 0°, 180°, 360°$

$\tan\theta = 1 \quad \Rightarrow \quad \theta = 45°, 225°$

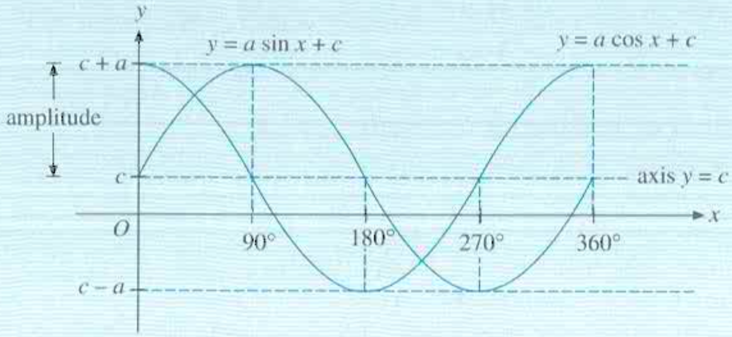
$\tan\theta = -1 \quad \Rightarrow \quad \theta = 135°, 315°$

$\tan\theta$ is undefined for $\theta = 90°, 270°$

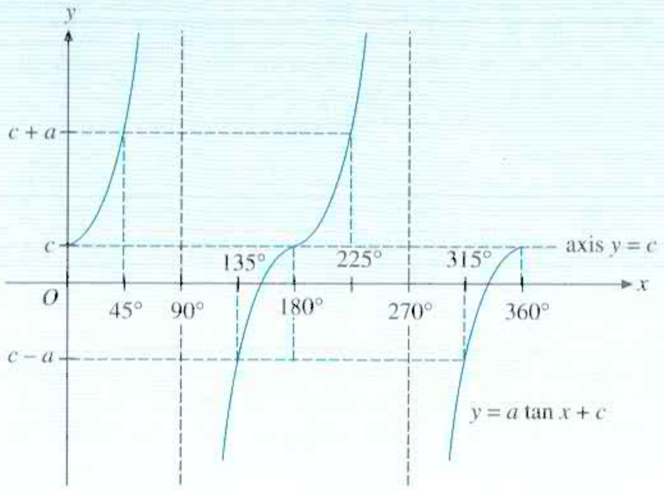


(d) $y = a \sin x + c$ and $y = a \cos x + c$, where $a > 0$

These curves have axis $y = c$ and amplitude a:



(e) $y = a \tan x + c,\ a > 0$



Take note of the values of x for which $y = c$, $y = c + a$ and $y = c - a$.

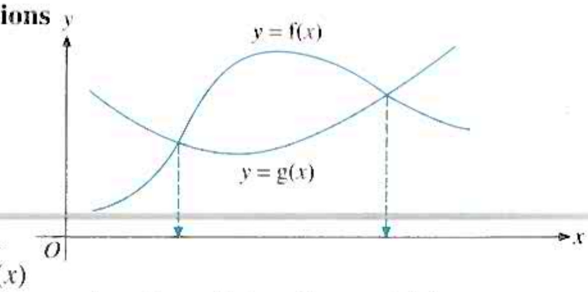
7. Solution of basic trigonometric equations of the form $\sin \theta = k$, etc.

Step 1: Identify the possible quadrants in which θ lies.

Step 2: Find the basic angle α.

Step 3: Find all the values of θ in the required interval. Remember that θ must be measured from the *positive* x-axis.

8. Graphical solution of equations



The solutions of the equation

$$f(x) = g(x)$$

are the values of x at which the graphs of $y = f(x)$ and $y = g(x)$ intersect.

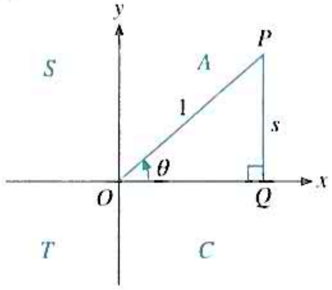
Miscellaneous Examples

Example 20 Given that $\sin\theta = s$ and θ is acute, find, in terms of s,
(a) $\tan\theta$, **(b)** $\sec\theta$.

Solution: θ is in the 1st quadrant and $\sin\theta = \frac{s}{1}$.

Let $PQ = s$ and $OP = 1$.

Then $OQ = \sqrt{1 - s^2}$.



(a) $\tan\theta = \frac{s}{\sqrt{1 - s^2}}$

(b) $\cos\theta = \sqrt{1 - s^2}$

$$\Rightarrow \sec\theta = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1 - s^2}}$$

Note: As θ is acute, $s > 0$ and the length $PQ = |s| = s$.

Example 21 Solve the equation $|5\cos x + 2| = 3$ for $0° \leqslant x \leqslant 360°$. Illustrate your answers by sketching on the same diagram the graphs of $y = |5\cos x + 2|$ and $y = 3$ for this interval.

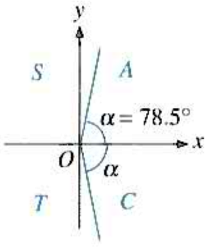
Solution: $|5\cos x + 2| = 3$

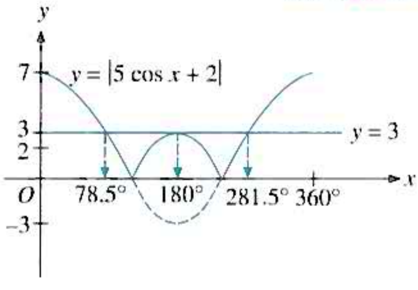
$\Rightarrow$ $5\cos x + 2 = -3$ or $5\cos x + 2 = 3$

$\cos x = -1$ $\qquad$ $\cos x = \frac{1}{5}$

$x = \mathbf{180°}$ $\qquad$ $x = \alpha, 360° - \alpha$

$= \mathbf{78.5°, 281.5°}$





Miscellaneous Exercise 10

1. Solve the following equations for $0° \leqslant x \leqslant 360°$.

(a) $5\cot x + 1 = \cos 70°$ **(b)** $(2\sin x + 1)(2\cos x + 5) = 0$

(c) $4\cos^2 x - 3 = \operatorname{cosec} 220°$ **(d)** $|2 - 3\tan x| = 0.5$

2. Without using a calculator, evaluate

(a) $\dfrac{\sin 120° + 2\cos 210°}{\sec 300°}$, (b) $\dfrac{\tan 120° + 4\sin 240°}{\cot 315° + 2\tan 225°}$.

3. Given that $\tan A = t$ and that A is acute, find in terms of t,

(a) $\sin A$, (b) $\sec A$,

(c) $\tan (90° - A)$, (d) $\tan (A - 90°)$.

4. Given that $\tan A = \frac{3}{4}$, $\cos B = -\frac{3}{5}$ and A and B are in the same quadrant, evaluate

(a) $\sin A$, (b) $\cot (-B)$, (c) $\dfrac{\tan B}{\cos A}$.

5. If $y = 7 - 3\cos x$, where $0° \leqslant x \leqslant 360°$, find the maximum and minimum values of y and the corresponding values of x.

6. Sketch, on separate diagrams, the following graphs for the given domains and state the corresponding range.

(a) $y = 5\sin x - 2$, $0° \leqslant x \leqslant 360°$

(b) $y = 1 + 2\cos x$, $0° \leqslant x \leqslant 720°$

(c) $y = 4\tan x - 3$, $0° \leqslant x \leqslant 180°$

(d) $y = 4\sin x - 1$, $0° \leqslant x \leqslant 180°$

(e) $y = \tan x - 1$, $-180° \leqslant x \leqslant 180°$

(f) $y = 6(\cos x + 1)$, $-360° \leqslant x \leqslant 180°$

7. On the same diagram, sketch the graphs of $y = 3|\sin x|$ and $y = |3\cos x| + 1$ for the interval $0° \leqslant x \leqslant 270°$. Hence find the number of solutions, in this interval, of the equation $|\sin x| - |\cos x| = \frac{1}{3}$.

8. Solve the equation $2\sec x = \cot 160°$ for values of x between $-180°$ and $180°$.

9. If $\sin A = -\frac{5}{13}$ and $-90° < A < 0°$, evaluate $\cos(-A)$ and $\cot(90° - A)$ without using a calculator.

*10. If $\sin 130° = k$, express $\sin 50°$ and $\tan 40°$ in terms of k.

11. Sketch, on the same axes, the graphs of $y = \tan x$ and $y = 2\cos x - 1$ for the interval $0° \leqslant x \leqslant 360°$ and hence find the number of solutions of the equation $(1 + \tan x)\sec x = 2$.

12. If $0° < \alpha < 360°$, $\tan \alpha < 0$ and $\operatorname{cosec} \alpha < 0$, find the range of values of $\frac{\alpha}{2}$.

*13. Solve the equation $(4 + 3\cot x)\tan x = 2$ for values of x between $-360°$ and $360°$.

14. The function $f(x) = a\sin x + b$, where $a > 0$, has a maximum value of 7 and a minimum value of -3. Find the value of a and of b. *(C)*

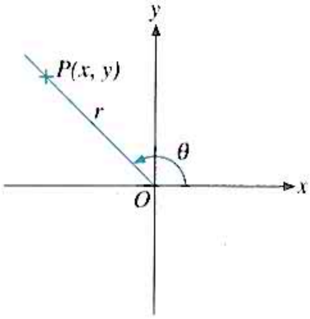
15. (a) Solve the equation $2\sin x + 1 = 0$ for $0° \leqslant x \leqslant 360°$.

(b) Sketch the graph of $y = 2\sin x + 1$ for the interval $0° \leqslant x \leqslant 360°$. Hence state the range of values of x in this interval which satisfies the inequality $2\sin x + 1 \geqslant 0$.

11 Simple Trigonometric Identities and Equations

11.1 Simple Identities

Basic Identities



Recall the definitions:

$$\cos \theta = \frac{x}{r} \quad \text{.......................} \quad (1)$$

$$\sin \theta = \frac{y}{r} \quad \text{.......................} \quad (2)$$

$$\tan \theta = \frac{y}{x},\ (x \neq 0) \quad \text{.......................} \quad (3)$$

where $r = \sqrt{x^2 + y^2}$.

Using the above definitions, we shall prove the following two important identities.

$$\mathbf{\tan \theta = \frac{\sin \theta}{\cos \theta},\ \cos \theta \neq 0}$$

$$\mathbf{\sin^2 \theta + \cos^2 \theta = 1}$$

Proof: If $\cos \theta \neq 0$, dividing (2) by (1) gives:

$$\frac{\sin \theta}{\cos \theta} = \frac{\frac{y}{r}}{\frac{x}{r}}$$

$$= \frac{y}{x}$$

$$= \tan \theta \quad \textit{(By (3))}$$

Also, $\sin^2 \theta + \cos^2 \theta = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2$

$$= \frac{x^2 + y^2}{r^2}$$

$$= \frac{r^2}{r^2} \quad \textit{(As } r = \sqrt{x^2 + y^2}\textit{)}$$

$$= 1$$

Example 1 Given that $\cos\theta = \frac{3}{5}$, find the possible values of

(a) $\sin\theta$, **(b)** $\tan\theta$.

Solution: Note that the quadrant in which θ lies is not given.

(a) Using $\sin^2\theta + \cos^2\theta = 1$, we have:

$$\begin{aligned}\sin^2\theta &= 1 - \cos^2\theta \\ &= 1 - \left(\frac{3}{5}\right)^2 \\ &= \frac{16}{25} \\ \sin\theta &= \pm\frac{4}{5}\end{aligned}$$

Hence the possible values of $\sin\theta$ are $\frac{4}{5}$ and $-\frac{4}{5}$.

(b) Using $\tan\theta = \frac{\sin\theta}{\cos\theta}$, we have:

$$\sin\theta = \frac{4}{5} \text{ and } \tan\theta = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{3}$$

or $$\sin\theta = -\frac{4}{5} \text{ and } \tan\theta = \frac{-\frac{4}{5}}{\frac{3}{5}} = -\frac{4}{3}$$

Hence the possible values of $\tan\theta$ are $\frac{4}{3}$ and $-\frac{4}{3}$.

Example 2 Given that $x = 2\sin\theta$ and $y = \cos\theta + 1$, show that

$$x^2 + 4(y - 1)^2 = 4.$$

Solution:

$$x = 2\sin\theta \quad\Rightarrow\quad \sin\theta = \frac{x}{2}$$

$$y = \cos\theta + 1 \quad\Rightarrow\quad \cos\theta = y - 1$$

$$\begin{aligned}\sin^2\theta + \cos^2\theta &= 1 \\ \left(\frac{x}{2}\right)^2 + (y-1)^2 &= 1 \\ \frac{x^2}{4} + (y-1)^2 &= 1 \\ x^2 + 4(y-1)^2 &= 4\end{aligned}$$

Recall that:

$$\sec\theta = \frac{1}{\cos\theta}, \quad \operatorname{cosec}\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

So, $\cot\theta = \dfrac{1}{\frac{\sin\theta}{\cos\theta}} = \dfrac{\cos\theta}{\sin\theta}$

Two more identities may be obtained by dividing $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$ and $\sin^2\theta$ respectively:

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\left(\frac{\sin\theta}{\cos\theta}\right)^2 + 1 = \left(\frac{1}{\cos\theta}\right)^2$$

i.e. $1 + \tan^2\theta = \sec^2\theta$

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$1 + \left(\frac{\cos\theta}{\sin\theta}\right)^2 = \left(\frac{1}{\sin\theta}\right)^2$$

i.e. $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$$\boldsymbol{\cot\theta = \frac{\cos\theta}{\sin\theta} \text{ where } \sin\theta \neq 0}$$

$$\boldsymbol{1 + \tan^2\theta = \sec^2\theta}$$

$$\boldsymbol{1 + \cot^2\theta = \operatorname{cosec}^2\theta}$$

Example 3 Simplify $(\sec\theta - \tan\theta)(\sec\theta + \tan\theta)$ and deduce the value of $\sec\theta + \tan\theta$ if $\sec\theta - \tan\theta = 2$.

Solution:

$$\begin{aligned}(\sec\theta - \tan\theta)(\sec\theta + \tan\theta) &= \sec^2\theta - \tan^2\theta \\ &= (1 + \tan^2\theta) - \tan^2\theta \\ &= \mathbf{1}\end{aligned}$$

$$\therefore \quad \sec\theta + \tan\theta = \frac{1}{\sec\theta - \tan\theta} = \mathbf{\frac{1}{2}}$$

Example 4 Simplify $\dfrac{1 - \operatorname{cosec}^2 x}{(1 - \sin x)(1 + \sin x)}$.

Solution:

$$\begin{aligned}\frac{1 - \operatorname{cosec}^2 x}{(1 - \sin x)(1 + \sin x)} &= \frac{-\cot^2 x}{1 - \sin^2 x} \\ &= \frac{-\left(\dfrac{\cos x}{\sin x}\right)^2}{\cos^2 x} \\ &= -\frac{1}{\sin^2 x} \\ &= \boldsymbol{-\operatorname{cosec}^2 x}\end{aligned}$$

Proving Identities

The basic identities may be used to prove other identities as shown in Example 5.

Example 5 Prove the following identities:

(a) $\sec x - \cos x \equiv \sin x \tan x$

(b) $(1 + \tan\theta)^2 + (1 - \tan\theta)^2 \equiv 2\sec^2\theta$

Solution:

(a)
$$\begin{aligned}\sec x - \cos x &\equiv \frac{1}{\cos x} - \cos x\\ &\equiv \frac{1 - \cos^2 x}{\cos x}\\ &\equiv \frac{\sin^2 x}{\cos x}\\ &\equiv \sin x \frac{\sin x}{\cos x}\\ &\equiv \sin x \tan x\end{aligned}$$

(b)
$$\begin{aligned}(1 + \tan\theta)^2 + (1 - \tan\theta)^2 &\equiv (1 + 2\tan\theta + \tan^2\theta) + (1 - 2\tan\theta + \tan^2\theta)\\ &\equiv 2 + 2\tan^2\theta\\ &\equiv 2(1 + \tan^2\theta)\\ &\equiv 2\sec^2\theta\end{aligned}$$

Note: In proving trigonometric identities, one usually starts with the more complicated side.

Exercise 11.1

1. If $\sin\theta = -\frac{1}{3}$, find the possible values of

   (a) $\cos\theta$, (b) $\tan\theta$.

2. Given that $x = 2\sin\theta - 3$ and $y = 2\cos\theta + 1$, show that $(x + 3)^2 + (y - 1)^2 = 4$.

3. Given $x = 3\cos\theta$ and $y = 2\tan\theta$, find $\sin\theta$ in terms of x and y. Hence or otherwise, show that $4x^2 + x^2y^2 = 36$.

4. Show that $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$.
   Hence find the possible values of $\sin\theta + \cos\theta$ if $\sin\theta\cos\theta = \frac{7}{18}$.

5. Given $x = 3\tan\theta$ and $y = 2\cos^2\theta$, show that $x^2y + 9y = 18$.

6. Simplify the following:

   (a) $\sec x \cos x$ (b) $\sin x \cot x$

   (c) $(\sec x + 1)(\sec x - 1)$ (d) $\dfrac{\sin x}{(1 - \cos x)(1 + \cos x)}$

* 7. Given $\operatorname{cosec} A + \cot A = 3$, evaluate $\operatorname{cosec} A - \cot A$ and $\cos A$.

Prove the identities in questions 8 to 25.

8. $\dfrac{1 + \cos x}{1 + \sec x} \equiv \cos x$

9. $1 - 2\sin^2 x \equiv 2\cos^2 x - 1$

10. $1 - \dfrac{\cos^2 x}{1 + \sin x} \equiv \sin x$

11. $\cot x + \tan x \equiv \operatorname{cosec} x \sec x$

12. $\dfrac{\sin x}{\operatorname{cosec} x - \cot x} \equiv 1 + \cos x$

13. $\sin^4 x - \cos^4 x \equiv \sin^2 x - \cos^2 x$

14. $\sin^4 x - \sin^2 x \equiv \cos^4 x - \cos^2 x$

15. $(1 - \cos x)\left(1 + \dfrac{1}{\cos x}\right) \equiv \sin x \tan x$

16. $\sin^2 x + \tan^2 x \sin^2 x \equiv \tan^2 x$

17. $\tan^2 x - \cot^2 x \equiv \sec^2 x - \operatorname{cosec}^2 x$

18. $\dfrac{1 - \tan^2 x}{1 + \tan^2 x} \equiv \cos^2 x - \sin^2 x$

19. $\dfrac{1}{\sin\theta + 1} - \dfrac{1}{\sin\theta - 1} \equiv 2\sec^2\theta$

20. $(1 + \tan x - \sec x)(1 + \cot x + \operatorname{cosec} x) \equiv 2$

21. $\dfrac{\sin x}{\sec x - 1} + \dfrac{\sin x}{\sec x + 1} \equiv 2\cot x$

22. $\dfrac{\sec\theta}{\sec\theta + \tan\theta} + \dfrac{\tan\theta}{\sec\theta - \tan\theta} \equiv 1 + 2\tan^2\theta$

23. $\dfrac{\cos A + \sec B}{\cos B + \sec A} \equiv \cos A \sec B$

24. $\dfrac{3 - 6\cos^2 x}{\sin x - \cos x} \equiv 3(\sin x + \cos x)$

***25.** $\dfrac{1}{1 - \dfrac{1}{1 - \dfrac{1}{1 - \sec^2 x}}} \equiv \sec^2 x$

***26.** If $x = a\sin\theta - b\cos\theta$ and $y = a\cos\theta + b\sin\theta$, show that $x^2 + y^2 = a^2 + b^2$.

27. Sketch the graphs of $y = \dfrac{2\cos x + \sin x}{\cos x}$ for $0° \leq x \leq 360°$.

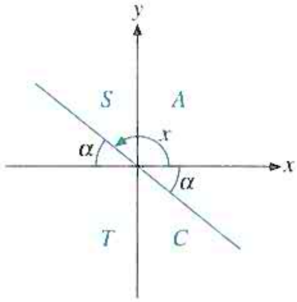
11.2 Trigonometric Equations and More Graphs

In Chapter 10, we solved the basic trigonometric equations $\sin x = k$, $\cos x = k$ and $\tan x = k$. We shall now consider equations which can be reduced to this basic form by simplifying (Example 6) or by factorising (Examples 7 to 9). Variations of this basic form are considered in Example 10.

Example 6 Find all the angles between 0° and 360° which satisfy the equation

$$3\cos x + 2\sin x = 0.$$

Solution:



$$3\cos x + 2\sin x = 0$$

$$2\sin x = -3\cos x$$

$$\frac{\sin x}{\cos x} = -\frac{3}{2} \quad (\because \cos x \neq 0)$$

$$\tan x = -\frac{3}{2}$$

$\tan x < 0 \quad \Rightarrow \quad x$ is in the 2nd or 4th quadrant.

$\tan \alpha = \frac{3}{2} \quad \Rightarrow \quad \alpha = 56.3°$

So, $x = 180° - \alpha,\ 360° - \alpha =$ **123.7°, 303.7°**

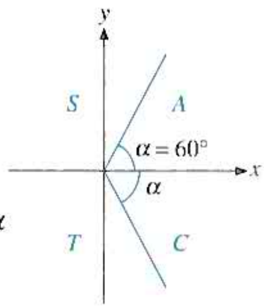
Note: If $\cos x = 0$, the given equation would lead to $\sin x = 0$. However, there is no value of x for which both $\sin x$ and $\cos x$ are zero. Hence $\cos x \neq 0$ and we could divide the equation by $\cos x$.

Example 7 Find all the angles between 0° and 360° inclusive for which

(a) $2\sin x\cos x = \sin x$,

(b) $\cos^2 y - \cos y = 2$.

Solution:



(a) $2\sin x\cos x = \sin x,\ 0° \leqslant x \leqslant 360°$

$$2\sin x\cos x - \sin x = 0$$

$$\sin x(2\cos x - 1) = 0$$

$\Rightarrow \quad \sin x = 0 \quad$ or $\quad \cos x = \frac{1}{2}$

$x = 0°, 180°, 360° \quad x = \alpha, 360° - \alpha$

$= 60°, 300°$

$\therefore \quad x =$ **0, 60°, 180°, 300°, 360°**

(b) $\cos^2 y - \cos y = 2, \quad 0° \leqslant y \leqslant 360°$

$$\cos^2 y - \cos y - 2 = 0$$

$$(\cos y + 1)(\cos y - 2) = 0$$

$\Rightarrow \quad \cos y = -1 \quad$ or $\quad \cos y = 2$ (no solution)

$y =$ **180°**

Note: In (a), we could not cancel $\sin x$ because $\sin x$ might be zero and division by zero is not possible.

Example 8 Find all the angles between 0° and 360° which satisfy the equation

$$\sin y = 4 \tan y.$$

Solution:

$$\begin{aligned}
\sin y &= 4 \tan y, \quad 0° < y < 360° \\
&= 4 \frac{\sin y}{\cos y} \\
\sin y \cos y &= 4 \sin y \\
\sin y(\cos y - 4) &= 0 \\
\Rightarrow \quad \sin y &= 0 \quad \text{or} \quad \cos y = 4 \text{ (no solution)} \\
y &= \mathbf{180°}
\end{aligned}$$

Example 9 Find all the angles between 0° and 360° which satisfy the equation

$$2 \cos^2 y - 1 = \sin y.$$

Solution:

$$2 \cos^2 y - 1 = \sin y, \quad 0° < y < 360°$$

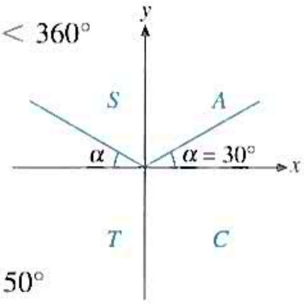
Using $\sin^2 y + \cos^2 y = 1$,

$$\begin{aligned}
2(1 - \sin^2 y) - 1 &= \sin y \\
2 \sin^2 y + \sin y - 1 &= 0 \\
(\sin y + 1)(2 \sin y - 1) &= 0
\end{aligned}$$

$\Rightarrow \quad \sin y = -1 \qquad \text{or} \qquad \sin y = \frac{1}{2}$

$y = 270° \qquad\qquad y = 30°, 150°$

$\therefore \quad y = \mathbf{30°, 150°, 270°}$



Example 10 Find all the angles between 0° and 360° which satisfy the equation

(a) $\cos (x + 30°) = -0.3$,

(b) $\sin 2x = 0.866$,

(c) $\tan (2x - 50°) = -0.7$.

Solution:

(a) $\cos (x + 30°) = -0.3$

$\cos (x + 30°) < 0$

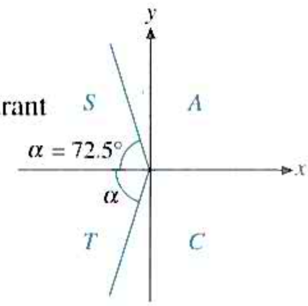
$\Rightarrow \quad x + 30°$ is in the 2nd or 3rd quadrant

and $0° < x < 360°$

$\Rightarrow \quad 30° < x + 30° < 390°$

For angle $x + 30°$ in this interval,

$$\begin{aligned}
x + 30° &= 107.5°, 252.5° \\
x &= \mathbf{77.5°, 222.5°}
\end{aligned}$$



(b) $\sin 2x = 0.866$

$\sin 2x > 0$

$\Rightarrow \quad 2x$ is in the 1st or 2nd quadrant

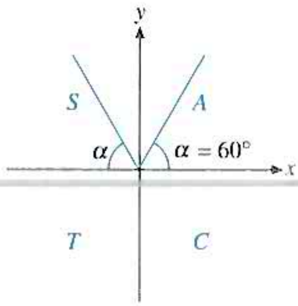
and $0° < x < 360°$

$\Rightarrow \quad 0° < 2x < 720°$

For angle $2x$ in this interval,

$$\begin{aligned}
2x &= 60°, 120°, 60° + 360°, \\
&\quad 120° + 360° \\
&= 60°, 120°, 420°, 480°
\end{aligned}$$

$\therefore \quad x = \mathbf{30°, 60°, 210°, 240°}$



(c) $\tan(2x - 50°) = -0.7$

$\Rightarrow \quad 2x - 50°$ is in the 2nd or 4th quadrant

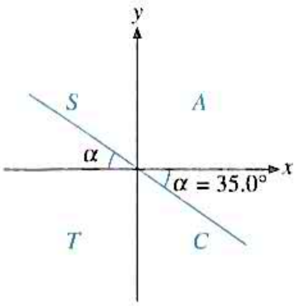
and $0° < x < 360°$

$\Rightarrow \quad -50° < 2x - 50° < 670°$

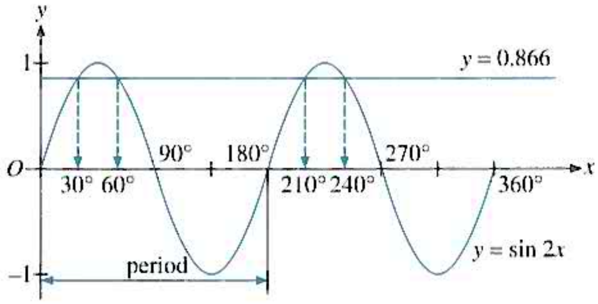
For angle $2x - 50°$ in this interval,

$$2x - 50° = -35.0°,\ 145.0°,\ -35.0° + 360°,\ 145.0° + 360°$$
$$= -35.0°,\ 145.0°,\ 325.0°,\ 505.0°$$

$\therefore \quad x =$ **7.5°, 97.5°, 187.5°, 277.5°**

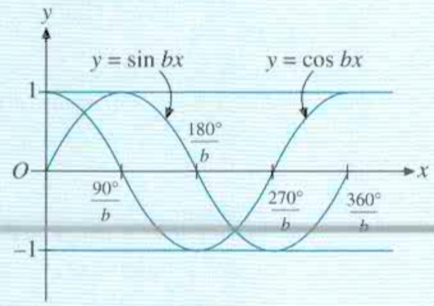


The answers in Example 10(b) are illustrated graphically as follows:



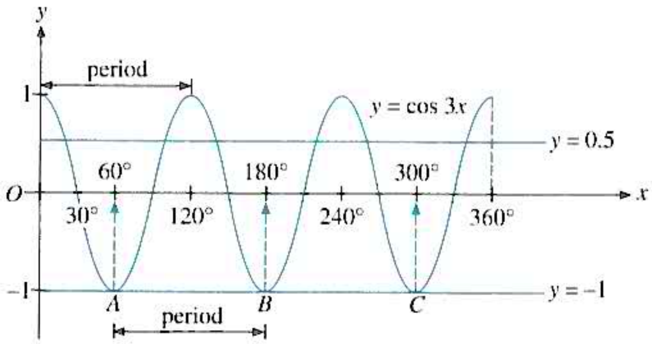
Observe that the graph of $y = \sin 2x$ is of a similar nature to that of $y = \sin x$. However, the curve $y = \sin 2x$ has a period of $\frac{360°}{2}$, i.e. 180° and there are 2 cycles in the interval $0° \leqslant x \leqslant 360°$.

If b is a positive integer, $y = \sin bx$ and $y = \cos bx$ have periods $\frac{360°}{b}$.



Example 11 Sketch the graph of $y = \cos 3x$ for $0° \leqslant x \leqslant 360°$ and hence
(a) find the number of solutions of the equation $\cos 3x = 0.5$,
(b) deduce the solutions of the equation $\cos 3x = -1$.

Solution: $y = \cos 3x$ has a period of $\frac{360°}{3} = 120°$ and there are 3 cycles in the interval $0° \leqslant x \leqslant 360°$. The curve is sketched as follows:

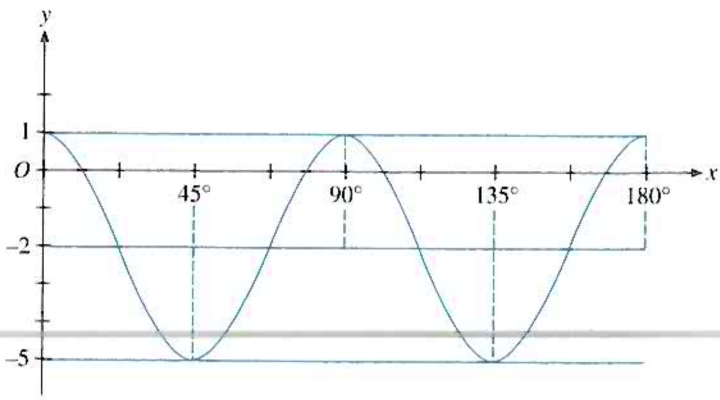


(a) The line $y = 0.5$ cuts the curve $y = \cos 3x$ at 6 points.
$\therefore$ the equation $\cos 3x = 0.5$ has **6** solutions in this interval.

(b) The line $y = -1$ meets the curve $y = \cos 3x$ at 3 points A, B and C.
$\therefore$ $\cos 3x = -1 \Rightarrow x = \mathbf{60°, 180°, 300°}$.

Example 12 Sketch the graph of $y = 3 \cos 4x - 2$ for $0° \leqslant x \leqslant 180°$ and state the corresponding range of y.

Solution: The graph of $y = 3 \cos 4x - 2$ has axis $y = -2$, amplitude 3 and period $= \frac{360°}{4} = 90°$.

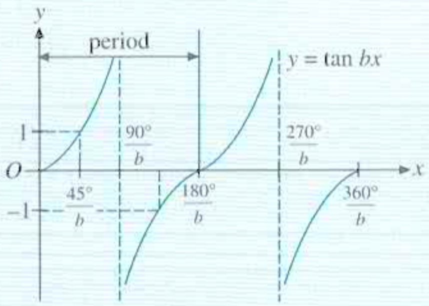


The corresponding range of y is $\{y : -5 \leqslant y \leqslant 1, y \in \mathbb{R}\}$

Recall that $y = \tan x$ has a period $180°$. So:

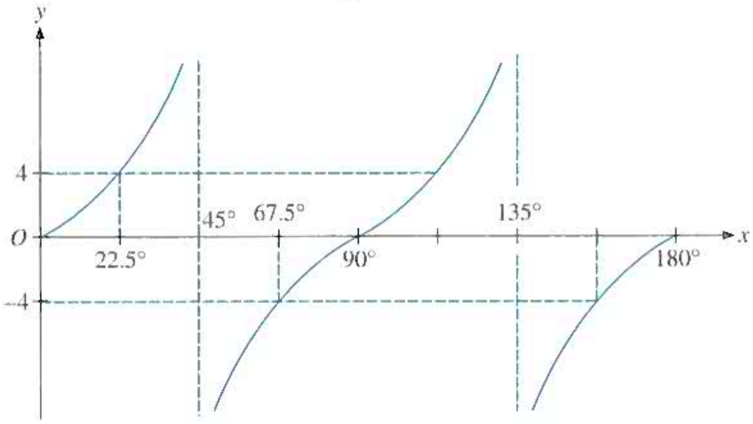
If b is a positive integer,

$y = \tan bx$ has period $\dfrac{180°}{b}$.



Example 13 Sketch the graph of $y = 4 \tan 2x$ for $0° \leqslant x \leqslant 180°$.

Solution: $y = 4 \tan 2x$ has a period of $\dfrac{180°}{2} = 90°$.



Note: When sketching trigonometric curves, take note of the required interval.

Exercise 11.2

1. Find all the angles between 0° and 360° which satisfy the equation
   (a) $5 \cos x + 2 \sin x = 0$, (b) $3(\sin x - \cos x) = \cos x$.
2. Find all the angles between 0° and 360° inclusive which satisfy the following equations.
   (a) $4 \sin x \cos x = \cos x$ (b) $2 \cos^2 x - \cos x = 1$
   (c) $2 \tan x = 4 - \sec^2 x$ (d) $2 \sin x \cos x = \tan x$
3. Find all the angles between 0° and 360° which satisfy the following equations.
   (a) $\cos 2x = 0.5$ (b) $\tan (x - 60°) = \dfrac{1}{\sqrt{3}}$
   (c) $3 \sin 2x + 2 = 0$ (d) $\cos (2x - 40°) = 0.8$
   (e) $\cot (2x + 10°) = -0.5$ (f) $\operatorname{cosec} (2x + 60°) = 4$

4. Find all the angles between 0° and 360° which satisfy the equation

(a) $\sin \frac{1}{2}x = \frac{1}{2}$, **(b)** $3 \cos y = 2 \sec y$, **(c)** $4 \tan z + \cot z = 5$.

5. Find all the angles between 0° and 360° which satisfy the equation

(a) $\tan (2x - 60°) = -1$, **(b)** $2 \sin y = \tan y$, **(c)** $\sec^2 z = 4 \sec z - 3$.

6. Solve the following equations for angles between 0° and 360° inclusive.

(a) $2 \cos 2x + 1 = \sqrt{2}$ **(b)** $3 \cot^2 x = \operatorname{cosec} x \cot x$
(c) $2 \cos^2 x + 3 \sin x = 3$ **(d)** $3 \sin x + 2 \tan x = 0$
(e) $|\sec (x - 50°)| = 3$ **(f)** $\sec^2 x + 2 \tan^2 x = 4$
(g) $2 \sin x \cos x + \cos^2 x = 1$ **(h)** $3 \cos (x + 40°) = 4 \sin (x + 40°)$
(i) $(\cos x - 2)(\cos x + 1) = \sin^2 x$ **(j)** $2 \sin x \tan x = 3$
(k) $\cot \left(\frac{x}{2} - 10°\right) = \frac{1}{2}$ **(l)** $2 \sin^2 x - 5 \sin x \cos x = 3 \cos^2 x$

7. Find the values of x, where $0° < x < 180°$ such that

(a) $2 \cos^2 x + \sin 20° = 1$, **(b)** $\sin (3x + 70°) = 0.2$,
(c) $8 \operatorname{cosec} 2x \cot 2x = 3$, **(d)** $|2 \cos x + 3 \sin x| = \sin x$.

8. Factorise the expression $2 \sin x \cos x - \cos x + 4 \sin x - 2$. Hence solve the equation $2 \sin x \cos x - 2 = \cos x - 4 \sin x$ for $-360° \leqslant x \leqslant 360°$.

***9.** If θ is obtuse and $2 \tan^2 \theta = 5 \sec \theta + 10$, find the value of $\tan \theta$ without using a calculator.

10. Sketch, on separate diagrams, the following graphs for $0° \leqslant x \leqslant 360°$ and state the corresponding range of y.

(a) $y = \sin 2x - 1$ **(b)** $y = 3|\cos 2x|$
(c) $y = 4 \sin 3x + 2$ **(d)** $y = 3 + 4 \cos 2x$

11. Sketch, on separate diagrams, the following graphs for $0° \leqslant x \leqslant 180°$.

(a) $y = 2 \sin 3x$ **(b)** $y = 4 \cos 2x - 1$
(c) $y = 2 \tan 3x - 2$ **(d)** $y = 5 + 3 \sin 2x$

12. Sketch, on the same diagram, the graphs of $y = 4 \sin 2x$ and $y = 2 \cos x - 1$ for $0° \leqslant x \leqslant 360°$. How many solutions are there in this interval such that $4 \sin 2x + 1 = 2 \cos x$?

13. On the same axes, sketch the graphs of $y = |2 \cos 2x|$ and $y = 1 + \sin x$ for $0° \leqslant x \leqslant 360°$. Hence find the number of distinct values of x, in this interval, for which

(a) $2|\cos 2x| = \sin x + 1$, **(b)** $2 \cos 2x = 1 + \sin x$.

14. Use a graphical method to determine the number of solutions of the equation $\cos 3x = \sin x$ in the interval $0° \leqslant x \leqslant 180°$.

***15.** Sketch for $0° \leqslant x \leqslant 180°$ the graphs of $y = 3 \cos 2x - 1$ and $y = \tan x$ on the same axes. Hence find the number of solutions in this interval of the equation

(a) $\sec 2x(1 + \tan x) = 3$, **(b)** $\tan x(3 \cos 2x - 1) = \tan^2 x$.

Important Notes

1. **Basic identities**

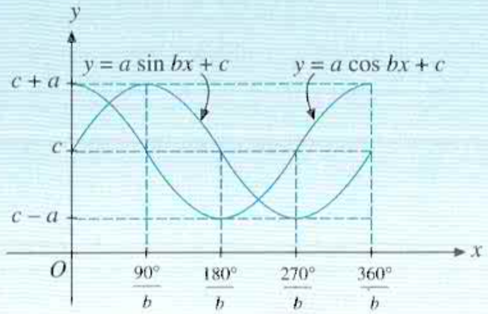
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \sin^2\theta + \cos^2\theta = 1$$

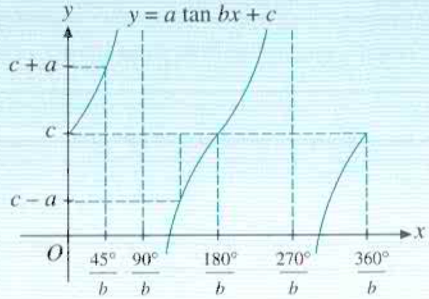
$$\cot\theta = \frac{\cos\theta}{\sin\theta} \qquad 1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

2. **Commonly encountered types of problems**
   (a) Prove other identities using the above basic identities.
   (b) Solve a trigonometric equation which could be simplified to basic equation(s) of the form $\sin x = k$, $\cos x = k$ and $\tan x = k$.
   (c) Solve trigonometric equations of the form $\sin(ax + b) = k$, $\cos(ax + b) = k$ and $\tan(ax + b) = k$.

3. **Graphs of $y = a\sin bx + c$, $y = a\cos bx + c$ and $y = a\tan bx + c$, $a > 0$, $b > 0$**





Miscellaneous Examples

Example 14 Prove the identity $\dfrac{1+\cos x}{\sin x} + \dfrac{\sin x}{1+\cos x} \equiv \dfrac{2}{\sin x}$.
Hence or otherwise, find all the angles between 0° and 360° for which $\dfrac{1+\cos x}{\sin x} + \dfrac{\sin x}{1+\cos x} = 3$.

Solution:

$$\begin{aligned}\frac{1+\cos x}{\sin x} + \frac{\sin x}{1+\cos x} &\equiv \frac{(1+\cos x)^2 + (\sin x)^2}{\sin x(1+\cos x)}\\ &\equiv \frac{\left(1 + 2\cos x + \cos^2 x\right) + \sin^2 x}{\sin x(1+\cos x)}\\ &\equiv \frac{1 + 2\cos x + 1}{\sin x(1+\cos x)}\\ &\equiv \frac{2(1+\cos x)}{\sin x(1+\cos x)}\\ &\equiv \frac{2}{\sin x}\end{aligned}$$

Thus, $\frac{1+\cos x}{\sin x} + \frac{\sin x}{1+\cos x} = 3 \Rightarrow \frac{2}{\sin x} = 3$

$$\sin x = \frac{2}{3}$$

$$x = \mathbf{41.8°,\ 138.2°}$$

Example 15 The points (15°, 2) and (h, 2.5) lie on the curve $y = k \sin 2x + 1$, where $0° \leqslant x \leqslant 90°$. Find h and sketch the curve for this interval.

Solution: (15°, 2) lies on the curve $\Rightarrow 2 = k \sin 30° + 1$

$$= k\left(\frac{1}{2}\right) + 1$$

$$k = 2$$

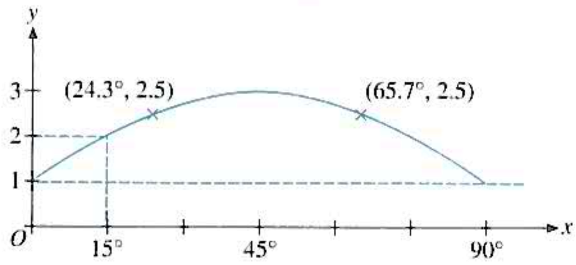
$\therefore\ y = 2 \sin 2x + 1$

At (h, 2.5), $2 \sin 2h + 1 = 2.5$

$$\sin 2h = 0.75$$

$$2h = 48.59°,\ 131.41°$$

$$h = \mathbf{24.3°,\ 65.7°}$$



Miscellaneous Exercise 11

1. Find the relationship between x and y given that
   (a) $x = 2\cos\theta,\ y = 4\sin^2\theta - 1$, **(b)** $x = \tan\theta,\ y = 2\sec\theta$.

2. Given that $\sin\theta = s$, find in terms of s,
   (a) $\cos\theta$, **(b)** $\cot\theta$, **(c)** $\sec(90° - \theta)$.

3. If $x = a\sin\theta$, where $a > 0$ and θ is acute, simplify
   (a) $a^2 - x^2$, **(b)** $\left(1 - \frac{x^2}{a^2}\right)^{\frac{3}{2}}$.

4. Sketch the graph of $y = \frac{2\sin x + \cos x}{\cos x}$ for $0° \leqslant x \leqslant 360°$. Find the values of x in this interval for which $y = 2$.

5. Find all the angles between 0° and 360° inclusive which satisfy the equation.
   (a) $\sec 2x = 2$, **(b)** $4\sin y\cos y = \tan y$, **(c)** $(\tan z + 1)(2\tan z - 1) = 5$.

6. Find all the angles between 0° and 360° which satisfy the following equations.
(a) $2 \tan 2x + \sec 40° = 1$ (b) $2 + \sin y \cos y = 2 \sin^2 y$
(c) $2 \cos^2 z - \sin z + 1 = 0$

7. Find all the angles between 0° and 360° inclusive which satisfy the equation.
(a) $3 \sin x - 2 \operatorname{cosec} x = 1$, (b) $\tan y = 2 \sin y \sin 40°$,
(c) $\cos (2x - 70°) = 0.5$.

8. Find all the angles between 0° and 360° which satisfy the equation
(a) $5 \cos^2 x - 8 \sin x \cos x = 0$,
(b) $5 \tan^2 y + 7 = 11 \sec y$,
(c) $1 + 2 \sin\left(\frac{3z}{2} + 75°\right) = 0$. *(C)*

9. If $8 \cos^2 x + 2 \sin x - 5 = 0$, show that $\sin x = \frac{3}{4}$ and $\sin x = -\frac{1}{2}$. Hence find the possible exact values of $\cot x$.

10. Given that $0° < x < 360°$, $\cos 2x = 0.4$ and $\tan 3x$ is positive, find the values of x.

11. Prove the identity $\frac{1 + \cos A}{1 - \cos A} - \frac{1 - \cos A}{1 + \cos A} \equiv 4 \cot A \operatorname{cosec} A$. *(C)*

12. Given that $\sin x$ and $\cos x$ have the same sign and that $\sin \frac{1}{2}x = 0.75$, calculate the value of x which lies between 0° and 360°.

13. Solve the equation $8x^3 - 2x^2 - 5x - 1 = 0$. Hence find the values of θ, between 0° and 180°, which satisfy the equation $8 \tan^2 \theta - 2 \tan \theta - 5 = \cot \theta$. *(C)*

14. Show that $\frac{2 - \operatorname{cosec}^2 A}{\operatorname{cosec}^2 A + 2 \cot A} \equiv \frac{\sin A - \cos A}{\sin A + \cos A}$. *(C)*

15. Prove the identity $\cos^2 x + \cot^2 x \cos^2 x \equiv \cot^2 x$. Hence solve the equation $\cos^2 x + \cot^2 x \cos^2 x = 4$ for all values of x between 0° and 180°.

16. Prove the identity $\sec \theta \operatorname{cosec} \theta - \cot \theta \equiv \tan \theta$. Hence find all the angles between 0° and 360° for which $\sec 2x \operatorname{cosec} 2x - \cot 2x = 1$.

17. Prove the identity $\frac{1 - \sin x}{\cos x} + \frac{\cos x}{1 - \sin x} \equiv 2 \sec x$. Hence find all the angles between 0° and 360° which satisfy the equation $\frac{1 - \sin x}{\cos x} + \frac{\cos x}{1 - \sin x} = \tan 80°$.

18. The quadratic equation in x, $3x^2 - (4 \cos \theta)x + 2 \sin \theta = 0$, has equal real roots. Find the value of θ, where $0° < \theta < 360°$.

*19. Solve the following equations for $0° \leqslant x \leqslant 360°$.
(a) $\log_2 (\cos x) = -0.1$ (b) $(\sin x)^{\sin 2x - \cos 2x} = 1$
(c) $3 \tan x + \cot x = 3 \sec x$ (d) $2 \sin x \tan x = 3 \sin x + 5 \cos x$

20. If $x = a \sin \alpha \cos \beta$, $y = a \sin \alpha \sin \beta$ and $z = a \cos \alpha$, show that $x^2 + y^2 + z^2 = a^2$.

*21. Given that $\sin x + \sin y = a$ and $\cos x + \cos y = a$, where $a \neq 0$, find $\sin x + \cos x$ in terms of a.

22. If $\sin A - \cos A = \frac{1}{4}$, evaluate
(a) $\sin A \cos A$, (b) $\sec A - \operatorname{cosec} A$.

23. Sketch the graph of $y = 3 \sin 2x - 2$ for $0° \leqslant x \leqslant 360°$ and state the corresponding range of y.

24. Using the same axes, sketch $y = \tan x + 1$ and $y = |2 \cos 2x + 1|$ for $0° \leqslant x \leqslant 180°$. Hence find the number of solutions, in this interval, of the equation
(a) $|2 \cos 2x + 1| = \tan x + 1$, (b) $|\cos x(2 \cos 2x + 1)| = |\sin x + \cos x|$.

*25. Given $f(x) = \dfrac{1}{\cos 2x} - \dfrac{\cos 2x}{1 + \sin 2x}$, show that $f(x) \equiv \tan 2x$. Hence sketch $y = f(x)$ for $0° \leqslant x \leqslant 270°$.

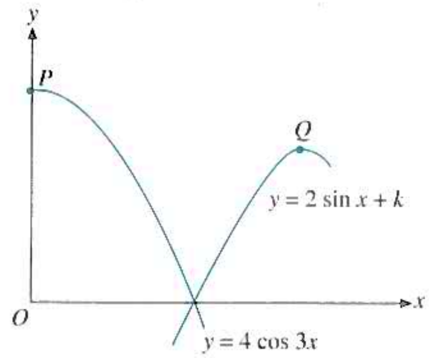
26. Solve for $0° \leqslant x \leqslant 180°$, $0° \leqslant y \leqslant 180°$, the simultaneous equations $x + y = 150°$, $\tan (x - y) = \sqrt{3}$

*27. If $k = \dfrac{1 + \sin x}{\cos x}$, prove that $\dfrac{1}{k} = \dfrac{1 - \sin x}{\cos x}$. Hence find $\sin x$ and $\cos x$ in terms of k.

*28. Given that $\dfrac{\sin^2 A}{1 + 2 \cos^2 A} = \dfrac{3}{19}$, where $90° < A < 180°$, find the value of $\dfrac{\sin A}{1 + 2 \cos A}$ without using a calculator.

*29. Given that $x = \sin \theta - 2 \cos \theta$ and $y = 2 \sin \theta + \cos \theta$, find $\sin \theta$ and $\cos \theta$ in terms of x and y. Hence find a relation between x and y independent of θ.

*30. The diagram shows parts of the graphs of $y = 4 \cos 3x$ and $y = 2 \sin x + k$. Points P and Q are the respective maximum points on these graphs. Given that the two graphs intersect at the x-axis, find the coordinates of P and of Q. Sketch the two graphs on the same diagram for the interval $0° \leqslant x \leqslant 180°$.



31. Sketch, on the same diagram, the curves $y = \tan x$ and $y = 2 \sin x$ for the interval $0° \leqslant x \leqslant 360°$. Find, in this interval,
(a) the **exact** coordinates of the points of intersection of the two curves,
(b) the range of values of x for which $\tan x < 2 \sin x$. *(C)*

32. The following shows the left hand sides (LHS) and right hand sides (RHS) of 3 identities. They have been jumbled up. With the help of a graph plotter, match the right hand side to the left hand side of each identity.

(*Hint*: Plot $y = \dfrac{1}{\tan x + \cot x}$, $y = \cos^2 x - \sin^2 x$, etc and match these graphs.)

LHS	RHS
(a) $\dfrac{1}{\tan x + \cot x} \equiv$	$\cos^2 x - \sin^2 x$
(b) $(\sin x - \cos x)(\tan x + \cot x) \equiv$	$\sin x \cos x$
(c) $\dfrac{\cot x - \tan x}{\cot x + \tan x} \equiv$	$\sec x - \operatorname{cosec} x$

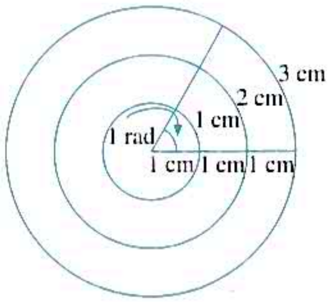
After matching, prove these identities.

12 Circular Measure

12.1 Radian Measure

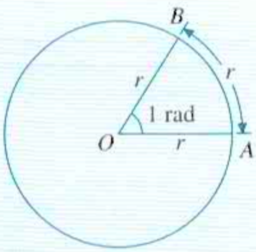
Introducing the Radian

To further our study on trigonometric functions, especially in the calculus (Chapters 18 and 20), a radian measure is introduced as follows:



The diagram on the right shows three sectors of circles with radii 1 cm, 2 cm and 3 cm, having arcs of lengths 1 cm, 2 cm and 3 cm respectively. Notice that the three sectors have the same angle. This angle is defined to be **1 radian**.

The angle subtended at the centre of a circle by an arc equal in length to the radius is 1 radian.



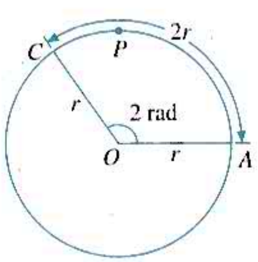


Fig. (a)

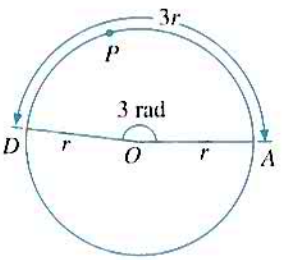


Fig. (b)

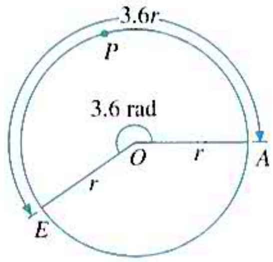
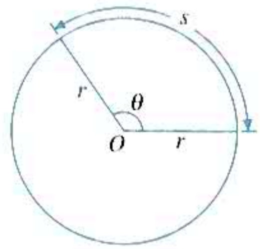


Fig. (c)

In Fig. (a) to (c), we have:

Length of arc $APC = 2r$ and $\angle AOC = 2$ radians
Length of arc $APD = 3r$ and $\angle AOD = 3$ radians
Length of arc $APE = 3.6r$ and $\angle AOE = 3.6$ radians

In general, if the length of the arc is s units and the radius is r units, then $\theta = \frac{s}{r}$. That is, the size of the angle is given by the **ratio of the arc length to the length of the radius**.



For example, if $s = 3$ cm and $r = 2$ cm, then $\theta = \frac{s}{r}$.

$$\begin{aligned}\theta &= \frac{s}{r}\\ &= \frac{3\text{ cm}}{2\text{ cm}}\\ &= 1.5 \text{ radians}\end{aligned}$$

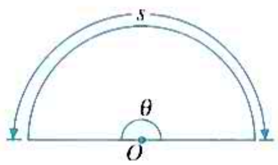
The radian measure of the angle is 1.5. Note that r and s are measured in the same unit, the radian measure of an angle is a number without dimension, although the word 'radian' is often added.

Relation between Radian and Degree Measures

Consider the angle θ in a semicircle of radius r as shown below.
We have:

$$s = \frac{1}{2}(2\pi r) = \pi r$$

and so $$\theta = \frac{s}{r} = \pi \text{ radians}$$



In degrees, $\theta = 180°$.

π radians = 180°

From this we get:

$\frac{\pi}{2}$ **radians = 90°**	$\frac{\pi}{6}$ **radians = 30°**
$\frac{3\pi}{2}$ **radians = 270°**	$\frac{\pi}{4}$ **radians = 45°**
2π **radians = 360°**	$\frac{\pi}{3}$ **radians = 60°**

Furthermore,

$$1 \text{ radian} = \frac{180°}{\pi} \approx 57.3°$$

and $$1° = \frac{\pi}{180} \approx 0.017\,45 \text{ radians.}$$

Trigonometry of Angles in Radians

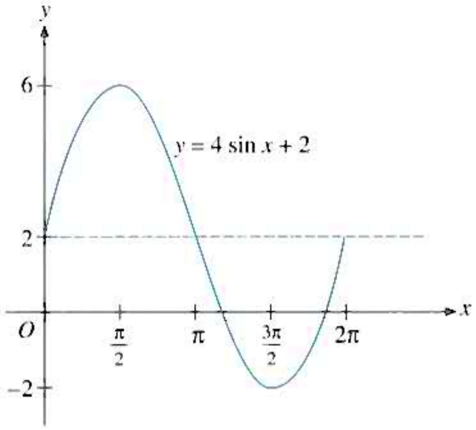
In Chapters 10 and 11, we discussed trigonometry of angles measured in degrees. The rules and identities also apply to angles measured in radians.

Example 1

Sketch the graph of the function $f : x \mapsto 4 \sin x + 2$ for the domain $0 \leqslant x \leqslant 2\pi$ and state the range corresponding to this domain.

Solution:

Observe that x is measured in radians.
The graph of $y = f(x) = 4 \sin x + 2$ has axis $y = 2$, amplitude 4 and period $= 2\pi$.
So, its sketch is as follows:



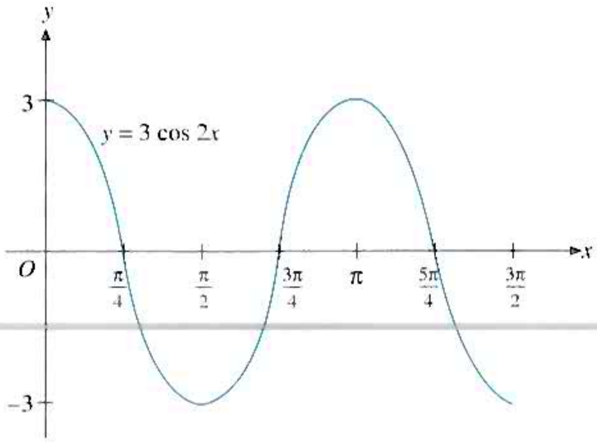
Range of f = $\{y: -2 \leqslant y \leqslant 6, y \in \mathbb{R}\}$.

Example 2

Sketch the graph of $y = 3 \cos 2x$ for $0 \leqslant x \leqslant \dfrac{3\pi}{2}$.

Solution:

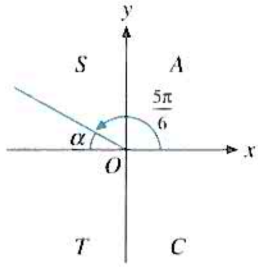
The graph of $y = 3 \cos 2x$ has axis $y = 0$, amplitude 3 and period $= \dfrac{2\pi}{2} = \pi$. So, its sketch is as follows:

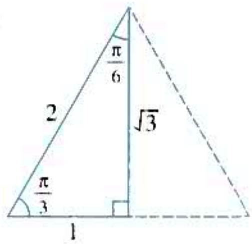


Example 3 Find the exact value of $\cos \frac{5\pi}{6}$.

Solution: basic angle,

$$\alpha = \pi - \frac{5\pi}{6}$$
$$= \frac{\pi}{6}$$
$$\therefore \cos \frac{5\pi}{6} = -\cos \alpha$$
$$= -\cos \frac{\pi}{6}$$
$$= -\frac{\sqrt{3}}{2}$$





Note: In writing $\sin \frac{\pi}{6}$ and $\cos 2$, it is clear that the angles $\frac{\pi}{6}$ and 2 are in radians. Hence $\cos 2 \neq \cos 2°$.

Example 4 Find the values of x, where $0 < x < 2\pi$, which satisfy the equation

$$\sin x + 3 \cos x = 0.$$

Solution:

$\sin x + 3 \cos x = 0$

$\sin x = -3 \cos x$

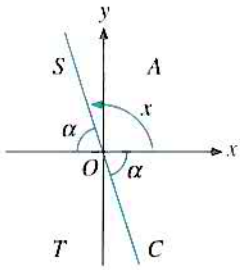
$\tan x = -3$

$\tan x < 0 \;\Rightarrow\; x$ is in the 2nd or 4th quadrant.

$\tan \alpha = 3 \;\Rightarrow\; \alpha = 1.249$

$\therefore\; x = \pi - \alpha,\ 2\pi - \alpha$

$= \mathbf{1.89, 5.03}$



Example 5 Find all the values of x, between 0 and 2, for which $\sin(\pi x) = \frac{1}{2}$, where πx is in radians.

Solution:

$\sin(\pi x) = \frac{1}{2}$,

$\sin(\pi x) > 0 \;\Rightarrow\; \pi x$ is in the 1st or 2nd quadrant,

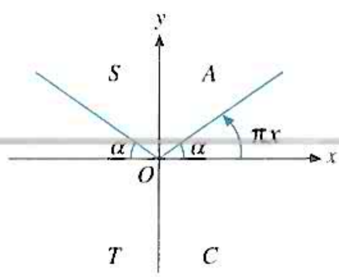
$\sin \alpha = \frac{1}{2} \;\Rightarrow\; \alpha = \frac{\pi}{6}$

Now, $0 < x < 2 \;\Rightarrow\; 0 < \pi x < 2\pi$.

$\therefore\; \pi x = \alpha,\ \pi - \alpha$

$= \frac{\pi}{6}, \frac{5\pi}{6}$

$\Rightarrow\; x = \mathbf{\frac{1}{6}, \frac{5}{6}}$



Exercise 12.1

1. Convert the following angles to degrees.
   (a) $\frac{\pi}{8}$ rad. (b) $\frac{2\pi}{3}$ rad. (c) $\frac{3\pi}{4}$ rad. (d) $\frac{5\pi}{6}$ rad.

2. Convert the following angles to radians. (Leave your answers in terms of π.)
   (a) $210°$ (b) $240°$ (c) $315°$ (d) $330°$

3. Sketch, on separate diagrams, the following graphs for the interval $0 \leqslant x \leqslant 2\pi$ and state the range of y.
   (a) $y = 2\cos x - 1$ (b) $y = |\sin x| + 1$ (c) $y = |2\sin x - 1|$
   (d) $y = \sin 2x + 3$ (e) $y = 5\cos 2x$ (f) $y = \tan x + 1$

4. Sketch, on separate diagrams, the following graphs for the interval $0 \leqslant x \leqslant \pi$.
   (a) $y = 3\cos 2x - 1$ (b) $y = 1 + \sin 2x$
   (c) $y = |\tan 2x| + 1$ (d) $y = 4\sin 3x - 2$

5. By considering the appropriate basic angles, evaluate the following without using a calculator.
   (a) $\cos \frac{3\pi}{4}$ (b) $\sin \frac{2\pi}{3}$ (c) $\tan \frac{7\pi}{4}$ (d) $\sin \frac{7\pi}{6}$

6. Using a calculator, evaluate
   (a) $\sin 1 + 2\cos 0.6$, (b) $\tan 1.2 - \sin 40°$, (c) $2\cos \frac{\pi}{5} + 3\tan \pi°$.

7. Find the values of x, where $0 \leqslant x \leqslant 2\pi$, which satisfy each of the following equations.
   (a) $\cos x = \frac{\sqrt{3}}{2}$ (b) $\tan x = 1$
   (c) $\sin x = -0.5$ (d) $6\sin^2 x - 1 = 0$
   (e) $2\sin x - \cos x = 0$ (f) $3\sin x + 2\cos x = 0$
   (g) $2\sin x\cos x = \cos x\cos 40°$ (h) $10\cos 2x = 7$
   (i) $\operatorname{cosec}(2x - 1.2) = 1.2$ (j) $2\cos^2 x + \sin x = 1$

8. Find all the values of z, between 0 and 10, for which $2\sin\left(\frac{\pi z}{4}\right) = 1$, where $\frac{\pi z}{4}$ is in radians. *(C)*

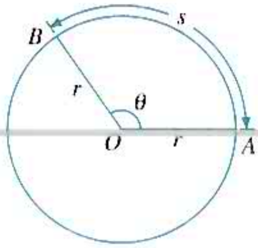
12.2 Arc Length and Area of a Sector

Length of an Arc

From our definition of the radian, we have:

$$\theta = \frac{s}{r}$$

$s = r\theta$, **where θ is in radians**



For example, if $\theta = 2.1$ radians and $r = 3$ cm,

$$\begin{aligned}\text{length of arc } AB,\ s &= r\theta \\ &= 3 \times 2.1 \text{ cm} \\ &= 6.3 \text{ cm}\end{aligned}$$

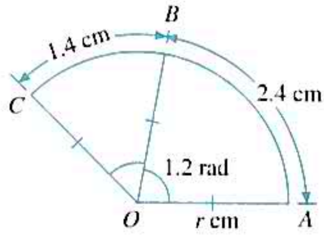
Note: s and r have the same unit.

Example 6

The diagram shows part of a circle, centre O, radius r cm. Calculate

(a) the value of r,

(b) $\angle BOC$ in radians.



Solution:

(a) In the sector AOB, $s = 2.4$ cm and $\theta = 1.2$ radians

$$\begin{aligned}s = r\theta \quad \Rightarrow \quad r &= \frac{s}{\theta} \\ &= \frac{2.4}{1.2} \\ &= \mathbf{2}\end{aligned}$$

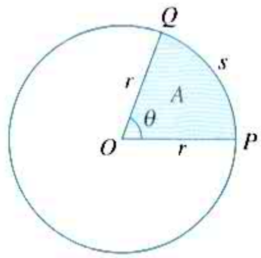
(b) In the sector BOC, $s = 1.4$ cm and $r = 2$ cm.

$$\begin{aligned}\theta &= \frac{s}{r} \\ &= \frac{1.4}{2} \\ &= 0.7 \text{ radians}\end{aligned}$$

$\angle BOC$ is **0.7 radians.**

Area of a Sector

In the diagram, the angle of the sector POQ is θ radians.



By proportion:

$$\frac{\text{area of sector } POQ}{\text{area of circle}} = \frac{\text{angle in sector}}{\text{angle in the circle}}$$

Let the area of sector POQ be A.

Thus,

$$\begin{aligned}\frac{A}{\pi r^2} &= \frac{\theta}{2\pi} \\ A &= \frac{\theta}{2\pi} \times \pi r^2 \\ &= \frac{1}{2} r^2 \theta\end{aligned}$$

Now as $s = r\theta$, we have:

$$A = \frac{1}{2} r(r\theta) = \frac{1}{2} rs$$

The area, A, of a sector is given by:

$$A = \frac{1}{2}r^2\theta, \text{ where } \theta \text{ is in radians or}$$

$$A = \frac{1}{2}rs$$

Example 7

A sector cut from a circle of radius 5 cm has a perimeter of 16 cm. Find the area of this sector.

Solution:

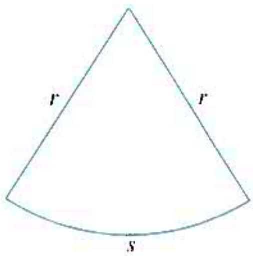
Now, $r = 5$ cm.

Perimeter of sector = 16 cm

$r + r + s = 16$

$5 + 5 + s = 16$

So, $s = 6$



Area of the sector $= \frac{1}{2}rs$

$= \frac{1}{2} \times 5 \times 6$

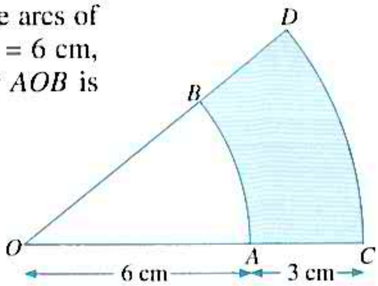
$= \mathbf{15\ cm^2}$

Example 8

In the diagram, arcs AB and CD are arcs of concentric circles, centre O. If $OA = 6$ cm, $AC = 3$ cm and the area of sector AOB is 12 cm², calculate

(a) $\angle AOB$ in radians,

(b) the area and perimeter of the shaded region.



Solution:

(a) Let $\angle AOB = \theta$ radians.

Area of the sector $AOB = 12$ cm²

$$\frac{1}{2}(OA)^2\theta = 12$$

$$\frac{1}{2} \times 6^2 \times \theta = 12$$

$$\theta = \mathbf{\frac{2}{3}}$$

(b) Now $OC = OA + AC = 9$ cm

Area of the sector $COD = \frac{1}{2}(OC)^2\theta$

$= \frac{1}{2} \times 9^2 \times \frac{2}{3}$

$= 27$ cm²

$\therefore$ area of shaded region = area of sector COD – area of sector AOB

$= 27 - 12$

$= \mathbf{15\ cm^2}$

Now, arc $AB = 6 \times \frac{2}{3}$ *(Using $s = r\theta$)*

$= 4$ cm

and arc $CD = 9 \times \frac{2}{3}$

$= 6$ cm

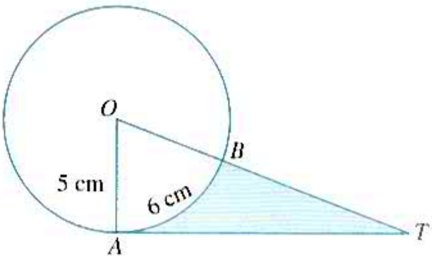
$\therefore$ perimeter of the shaded region = arc $AB + BD +$ arc $CD + AC$

$= 4 + 3 + 6 + 3$

$=$ **16 cm**

Example 9

The diagram shows a circle, centre O, radius 5 cm. The tangent to the circle at A meets OB produced at T. Given that the length of the arc AB is 6 cm, calculate the area of the shaded region, correct to 2 decimal places.



Solution:

Let $\angle AOB = \theta$ radians

$$\theta = \frac{s}{r} = \frac{6}{5} = 1.2$$

For $\triangle OAT$, note that $\angle OAT$ is $90°$.

So, $\frac{AT}{OA} = \tan\theta$

$AT = OA \tan\theta$

$= 5 \tan 1.2$

Area of shaded region = area of the triangle OAT − area of the sector AOB

$= \frac{1}{2} \times OA \times AT - \frac{1}{2}rs$

$= \frac{1}{2} \times 5 \times 5 \tan 1.2 - \frac{1}{2} \times 5 \times 6$

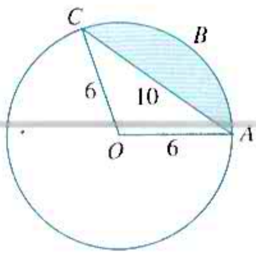
$=$ **17.15 cm²**

Example 10

The diagram shows a circle of radius 6 cm with a chord AC of length 10 cm. Calculate, to 3 significant figures, the area of

(a) the minor sector AOC,

(b) the shaded segment ABC.

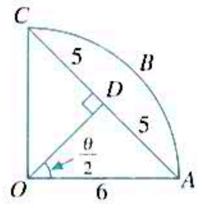


Solution:

(a) Let $\angle AOC = \theta$ radians

Since $\triangle AOC$ is isosceles, the height OD bisects AC at right angles.

So $DA = \frac{1}{2}AC = 5$ cm and $\angle AOD = \frac{1}{2}\theta$.



In $\triangle AOD$, $\sin \frac{\theta}{2} = \frac{5}{6}$

$$\frac{\theta}{2} = 0.985\ 1$$

$$\theta = 1.970$$

$\therefore$ area of the sector $AOC = \frac{1}{2}r^2\theta$

$$= \frac{1}{2} \times 6^2 \times 1.970$$

$$= \mathbf{35.5\ cm^2}$$

(b) Area of the segment ABC = area of the sector AOC – area of the triangle AOC

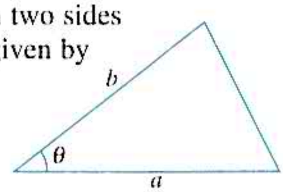
$$= 35.5 - \frac{1}{2} \times 6 \times 6 \sin \theta$$

$$= \mathbf{18.9\ cm^2}$$

Note: **(1)** The segment of a circle is the portion enclosed by a chord and an arc.

(2) The area of any triangle given two sides and the included angle, θ, is given by

$$A = \frac{1}{2}ab \sin \theta.$$

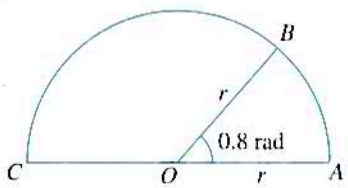


Exercise 12.2

1. The diagram shows a semicircle $OABC$. If the arc AB has length 3.2 cm, calculate

(a) the length of the radius,

(b) the length of the arc BC.



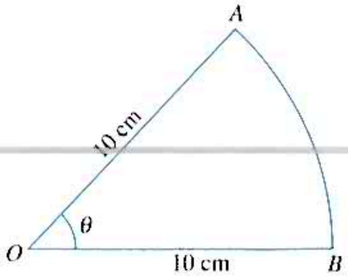
2. The diagram shows a sector AOB whose angle is θ radians. Find

(a) the value of θ if arc AB has length 14 cm,

(b) the length of the arc AB if $\theta = 0.6$,

(c) the area of the sector if the arc AB has length 5 cm,

(d) the length of the arc AB if the area of the sector is 30 cm^2,

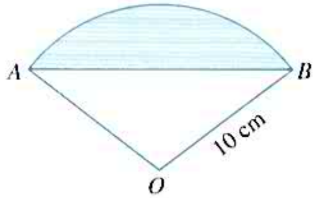


(e) the area of the sector if $\theta = 0.8$,
(f) the value of θ if the area of the sector is 50 cm^2.

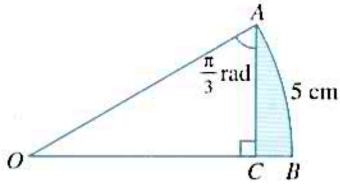
3. A sector cut from a circle of radius 3 cm has a perimeter of 12 cm. Find the area of this sector.

4. A piece of wire 20 cm long is bent to form the shape of a sector. If the arc has length 8 cm, calculate the angle of the sector and the area enclosed by this sector.

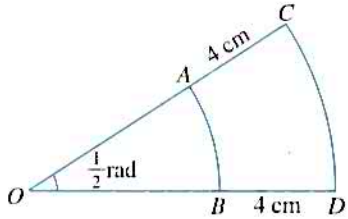
5. The diagram shows part of a circle, centre O, radius 10 cm. Given that the length of the arc AB is 14 cm, calculate, to 3 significant figures,
(a) the angle AOB in radians,
(b) the area of the shaded region.



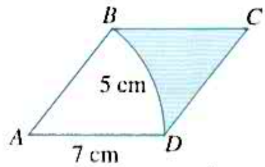
6. OAB is a sector of the circle, centre O, with $\angle OAC = \frac{\pi}{3}$ radians and $\angle OCA$ a right angle. Given that the arc AB has length 5 cm,
(a) show that $OA = 9.55$ cm,
(b) calculate the perimeter of the shaded region,
(c) express the area of the shaded region as a percentage of the area of the sector OAB.



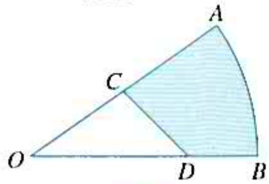
7. The figure shows two sectors in which the arcs AB and CD are arcs of concentric circles, centre O. $BD = AC = 4$ cm and $\angle AOB = \frac{1}{2}$ radians. If the perimeter of $ABDC$ is 16 cm, calculate
(a) OB,
(b) the area of $ABDC$.



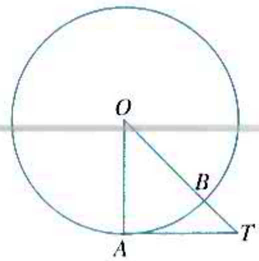
8. The diagram shows a rhombus $ABCD$ with sides 7 cm. An arc BD, centre A, has length 5 cm. Calculate the area of the shaded region.



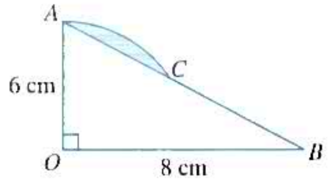
9. OAB is a sector with $\angle AOB = 0.4$ radians. C is the midpoint of OA and D lies on OB.
If $OC = 3$ cm and the area of the shaded region is 4.5 cm^2, calculate the length of DB.



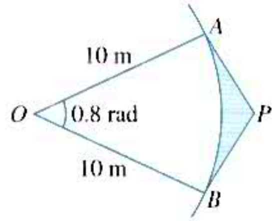
10. The figure shows a circle centre O, radius 6 cm. The tangent to the circle at A meets OB produced at T. If the area of the triangle OAT is 15 cm^2, calculate the area and perimeter of the minor sector OAB.



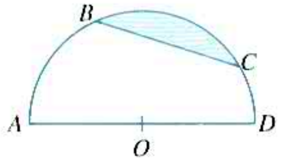
11. OAB is a right-angled triangle with $OA = 6$ cm and $OB = 8$ cm. An arc AC is drawn with centre at O. Calculate
(a) the angle AOC in radians,
(b) the area of the shaded segment.



12. The diagram shows part of a circle, centre O, of radius 10 m. The tangents at the points A and B on the circumference of the circle meet at the point P and the angle AOB is 0.8 radians. Calculate
(a) the length of the perimeter of the shaded region,
(b) the area of the shaded region. *(C)*

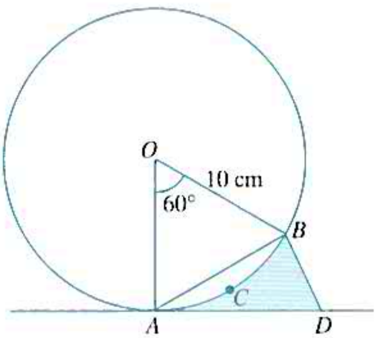


13. The diagram shows a semicircle with centre at O. The lengths of the arcs AB, BC and CD are in the ratio 2 : 3 : 1. If the length of arc BC is 15 cm, calculate the area of the sector AOB and that of the shaded region.



14. A hollow cone has base radius 10 cm and height 24 cm. The cone is unrolled to form a sector of a circle. What are the angle and area of this sector?

15. The diagram shows three points A, B and C on a circle, centre O and radius 10 cm. The line AD is a tangent to the circle. Given that angle $AOB = 60°$, find, to one decimal place,
(a) the length of the arc ACB,
(b) the area of the segment ACB.
Given also that the length of AD equals the length of the arc ACB, find
(c) the area of the shaded region $ACBD$,
(d) the length of BD. *(C)*



Important Notes

1. Radian measure

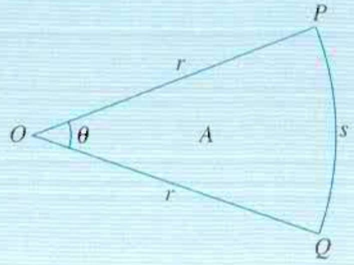
(a) The angle subtended at the centre of a circle by an arc equal in length to the radius is 1 radian.
(b) We have:

$$\frac{\pi}{6} = 30° \qquad \frac{\pi}{2} = 90°$$
$$\frac{\pi}{4} = 45° \qquad \pi = 180°$$
$$\frac{\pi}{3} = 60° \qquad \frac{3\pi}{2} = 270°$$
$$2\pi = 360°$$

(c) Trigonometric functions can be defined for angles in radians. Hence, the methods for
  (i) solving trigonometric equations,
  (ii) sketching trigonometric curves,
are similar to those for angles in degrees.

2. Circular measure

For a sector POQ whose angle θ is in radians,
arc length, $s = r\theta$
area, $A = \frac{1}{2}rs = \frac{1}{2}r^2\theta$



Miscellaneous Examples

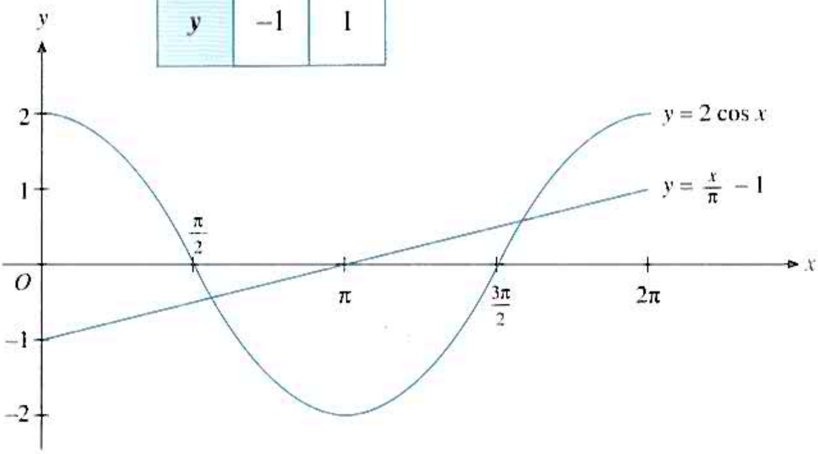
Example 11

Sketch on the same diagram, the graphs of $y = 2\cos x$ and $y = \frac{x}{\pi} - 1$ for the interval $0 \leqslant x \leqslant 2\pi$, labelling each graph clearly. Hence state the number of solutions, in this interval, of the equation $x - 2\pi\cos x = \pi$.

Solution:

Note that $y = \left(\frac{1}{\pi}\right)x - 1$ is of the form $y = mx + c$. To sketch this line, we need 2 points such as those given in the table.

x	0	2π
y	-1	1

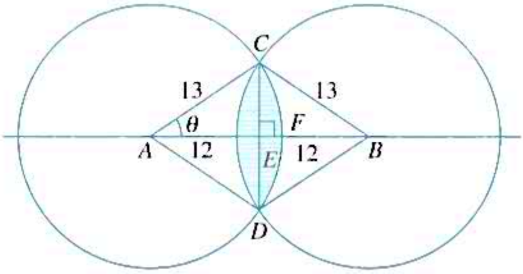


The given equation, $x - 2\pi\cos x = \pi \quad \Rightarrow \quad \frac{x}{\pi} - 2\cos x = 1$

$$\frac{x}{\pi} - 1 = 2\cos x$$

Its solutions are given by the values of x where the two graphs intersect.
Hence there are **2** solutions to this interval.

Example 12 Two circles of equal size, each with radius 13 cm, overlap each other. If their centres are 24 cm apart, calculate the perimeter and area of the common region, correct to 3 significant figures.



Solution:

From $\triangle CAE$, $AE = \frac{1}{2}AB$
$= 12$

$\cos\theta = \frac{12}{13} \Rightarrow \theta = 0.394\ 8$ rad.

Perimeter of common region $= 2 \times \text{arc } CD$
$= 2 \times 13 \times 2\theta$
$=$ **20.5 cm**

Area of common region $= 2 \times$ area of segment CDF
$= 2$(area of sector ACD – area of $\triangle ACD$)
$= 2\left[\frac{1}{2} \times 13^2(2\theta) - \frac{1}{2} \times 13^2 \sin 2\theta\right]$
$=$ **13.4 cm²**

Miscellaneous Exercise 12

1. Sketch, on separate diagrams, the graph of
   (a) $y = 3 \sin x - 2$ for $0 \leqslant x \leqslant 2\pi$, (b) $y = 2 \cos 3x$ for $0 < x < \pi$.

2. Sketch, on the same diagram, the curves $y = 2 \sin x$ and $y = \cos 2x$ for the interval $0 \leqslant x \leqslant 2\pi$, labelling each curve clearly.
   State the number of solutions, in this interval, of the equation
   (a) $\cos 2x = 0.6$, (b) $\cos 2x = 2 \sin x$,
   (c) $\cos 2x - 2 \sin x = 1$, (d) $|\cos 2x \operatorname{cosec} x| = 2$.

3. Sketch the curve $y = \sin 2x$, for values of x from 0 to 2π. Using the same axes, draw the line $y = \frac{x}{\pi}$ and hence state the number of solutions there are, between 0 and 2π, of the equation $x = \pi \sin 2x$.

4. On the same axes, sketch the graphs of $y = 2 \cos x - 1$ and $y = -\frac{2x}{3\pi} + 1$ for $0 \leqslant x \leqslant 2\pi$. Hence state the number of solutions, in this interval, of the equation $-2x + 3\pi = 3\pi(2 \cos x - 1)$.

5. Use a graphical method to determine how many solutions there are of the equation $2x - \pi = \pi \tan 2x$ in the interval $0 < x < \pi$. Deduce how many solutions this equation has in the interval $0 < x < 20\pi$.

6. **Sketch**, on the same diagram, for $0 \leqslant x \leqslant \frac{5}{2}\pi$, the graphs of $y = \cos 2x$ and $y = \frac{2x}{5\pi}$. Hence state the number of roots of the equation $5\pi \cos 2x = 2x$ which are
   (a) between 0 and $\frac{5}{2}\pi$, **(b)** greater than $\frac{5}{2}\pi$.

7. A curve has the equation $y = 5(1 - \cos 2x)$ and is defined for $0 \leqslant x \leqslant \frac{\pi}{2}$ radians. Find
   (a) the value of y when $x = 1$, **(b)** the value of x when $y = 3$. *(C)*

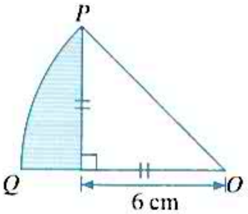
8. Find the value(s) of x, where $0 \leqslant x \leqslant \pi$, for which
   (a) $4 \sin x + 3 \cos x = 0$, **(b)** $2 \sin 2x = 1$,
   (c) $4 \cos x = 1 + 3 \sec x$, **(d)** $2 \cos^2 x = 5 \cos x \sin x$.

9. **(a)** A sector of a circle has an arc length of 20 cm. If the radius of the circle is 12 cm, find the area of the sector.
   (b) Find the value of $\tan 2x$ if $x = 1.6$ radians.

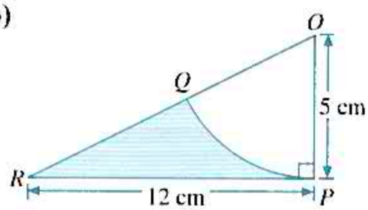
10. Two circles of equal size, each of radius 10 cm, have centres at A and B respectively and the length of AB is 16 cm. The circles intersect at C and D. Find the angle CBA in radians and the area common to both circles.

11. In each of the following figures, PQ is an arc of a circle, centre O. Calculate the perimeter and the area of the shaded region.

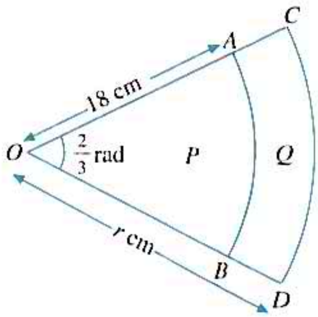
(a)



(b)



12. The diagram shows concentric arcs AB and CD of concentric circles, centre O, with radii 18 cm and r cm respectively. Given that the angle AOB is $\frac{2}{3}$ radians and that the perimeters of the regions P and Q are in the ratio 6 : 5, calculate
    (a) the value of r,
    (b) the ratio of the areas of the regions P and Q.



13. Fig. 1 shows the cross-section of a uniform cylindrical log with centre O and radius 15 cm. Points A, Y, B and X lie on the circumference of the cross-section and the chord AB is 24 cm in length.
    (a) Show that angle AOB is 1.855 radians to 3 decimal places.
    (b) Find the length of the arc AXB.
    (c) Find the area of the shaded region.

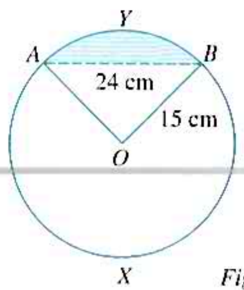


Fig. 1

Fig. 2 shows the cross-section of the same log with the section $ADCBYA$ removed throughout the length of the log. Given that $AD = BC = 13$ cm and that $ABCD$ is a rectangle, find

(d) the area of the new cross-section, $AXBCDA$,

(e) the percentage volume of wood remaining.

(C)

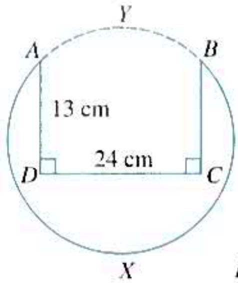


Fig. 2

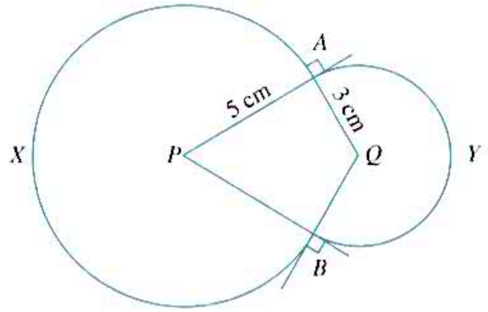
14. The diagram shows the shape $XAYBX$ formed by two intersecting circles. The radii of the circles, centres P and Q, are 5 cm and 3 cm respectively. At each of the points of intersection, A and B, the radius of one circle is perpendicular to the radius of the other.

(a) Show that angle APB is approximately 1.08 radians.

(b) Find the perimeter of the shape $XAYBX$.

(c) Find the area of the shape $XAYBX$.

(C)



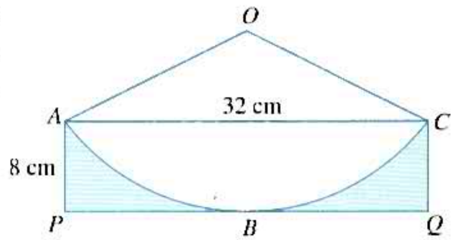
15. A chord of a circle subtends an angle of θ radians at the centre of the circle. The area of the major segment cut off by the chord is 75% of the area of the circle. Prove that $2 \sin \theta = 2\theta - \pi$. Show how the solution of this equation can be obtained graphically.

16. ABC is the segment of a circle, centre O. This segment is enclosed in a rectangle $APQC$. Given that $AC = 32$ cm and $AP = 8$ cm, calculate

(a) the radius of the circle,

(b) the angle AOC in radians,

(c) the area of the shaded region.



17. The diagram shows two circles, C_1, and C_2, touching at A. Circle C_1, has radius 9 cm and centre X; circle C_2 has radius 4 cm and centre Y. A tangent touches the circles C_1 and C_2 at the points P and Q respectively.

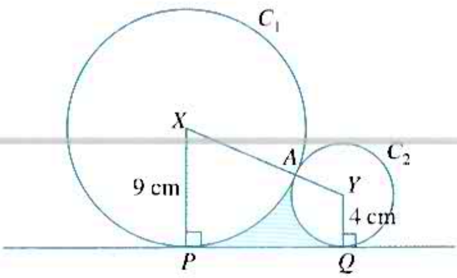
Calculate the length of PQ and show that angle PXY, to 3 decimal places, is 1.176 radians. Find

(a) the length of the minor arc AP of circle C_1,

(b) the length of the minor arc AQ of circle C_2,

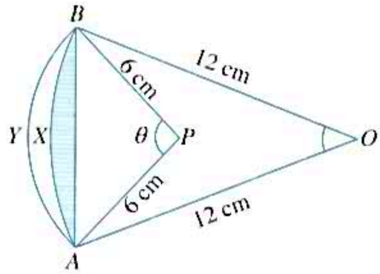
(c) the area of the shaded region.

(C)

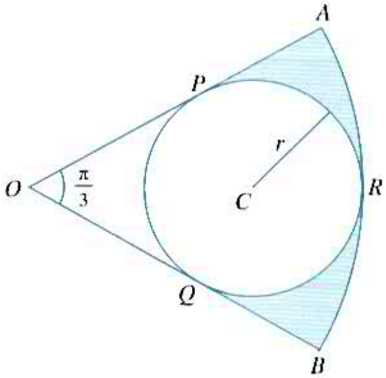


***18.** A semicircle of radius 10 cm has diameter AB. AP and AQ are chords of lengths 6 cm and 8 cm respectively. Calculate the area bounded by these two chords and the arc PQ.

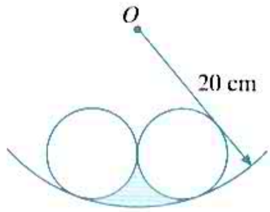
19. In the given figure, AXB is an arc of a circle centre O and radius 12 cm with angle $AOB = 0.6$ radian. AYB is an arc of a circle centre P and radius 6 cm with angle $APB = \theta$. Calculate
(a) the length of the chord AB,
(b) the value of θ in radians,
(c) the difference in length between the arcs AYB and AXB,
(d) the area of the shaded segment.



***20.** In the figure, OAB is a sector of a circle with centre O and radius 24 cm. The circle PQR with centre C and radius r cm is inscribed in the sector. Show that $r = 8$ and that $OQ = 8\sqrt{3}$ cm.
Calculate, in exact form,
(a) the area of the sector OAB,
(b) the area of the quadrilateral $OQCP$.
Hence show that the area of the shaded region is $\frac{32}{3}(5\pi - 6\sqrt{3})$ cm^2.



***21.** Two circles of equal size, each of radius 5 cm, touch each other externally. Both of them touch another circle, centre O and radius 20 cm, internally as shown in the diagram. Calculate the area of the shaded region.



13 Permutations and Combinations

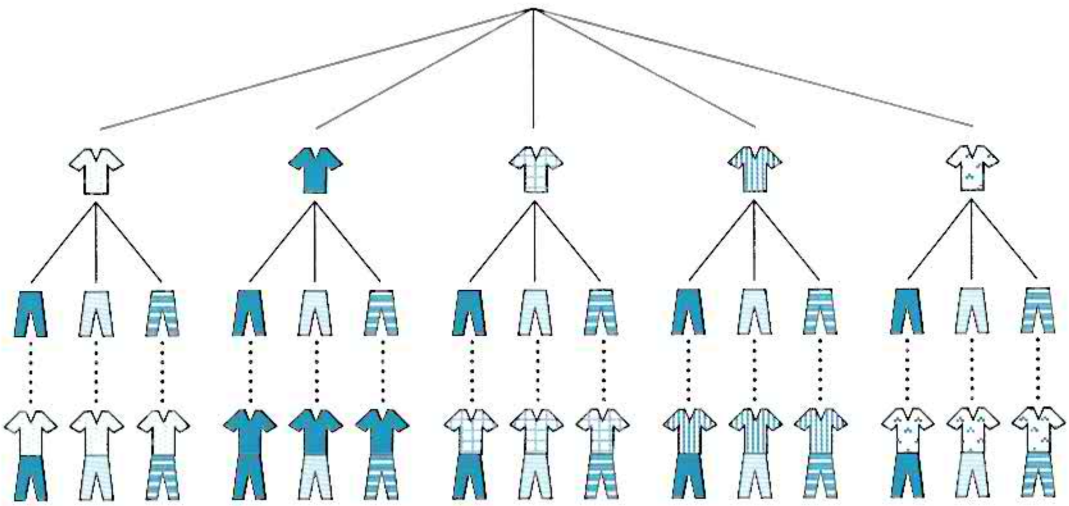
13.1 The Basic Counting Principle

Counting is easy when you have a list of the items to count. It would be more difficult when you do not have a list of the items or when the number of items is large. In this chapter, we shall learn more efficient ways of counting.

Consider the following example.

Kiam Lui has 5 different shirts and 3 different pairs of pants as shown.

In how many different ways can he match his shirt and pants? We can use a **tree diagram** to systematically list the various possibilities as shown below.



With the possibilities clearly listed, we can count the number of ways he can match his shirt and pants, i.e. 15. It is amazing that with just 5 shirts and 3 pairs of pants, Kiam Lui can actually have a different look each day for more than 2 weeks! (If he washes them after use, of course.)

To go out dressed in shirt and pants, Kiam Lui has to decide which shirt and pair of pants to wear. The first task of deciding which shirt can be performed in 5 ways. This is illustrated by the 5 'branches' in the first level of our earlier tree diagram. His next task is to decide which pair of pants to wear. This can be performed in 3 ways and we see that, in the second level, each shirt has 3 further 'branches' because of the 3 pairs of pants. Hence there are $5 \times 3 = 15$ ways of performing the two tasks in succession.

This illustrates the **Basic** (or **Fundamental**) **Counting Principle**:

> **If task A can be performed in m ways, followed by task B which can be performed in n ways, then task A followed by task B can be performed in $(m \times n)$ ways.**

This principle can be extended to more than 2 tasks: To find the number of ways of performing several tasks in succession, multiply the numbers of ways in which each task can be performed.

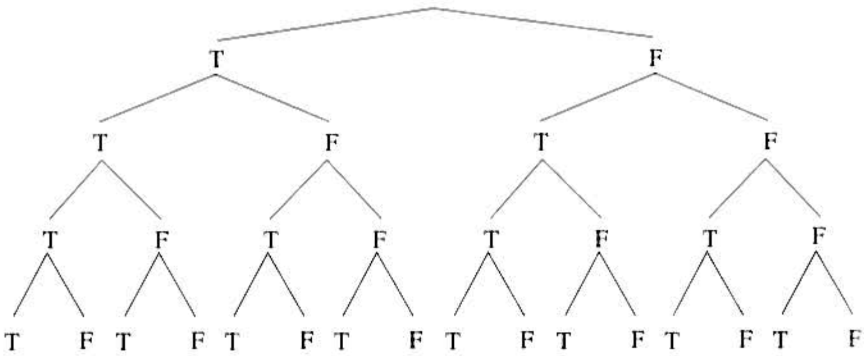
Example 1

Alfred is caught by a surprise test which consists of 4 true-or-false questions. Illustrate on a tree diagram the possible ways in which he can answer all the questions purely by guesswork. Hence deduce the number of possible answers for a 10-question test.

Solution:

He can answer the 1st question in 2 ways, that is, true (T) or false (F).
He can also answer the 2nd question in 2 ways and so forth.
$\therefore$ his possible answers are:

1st question:
2nd question:
3rd question:
4th question:



Using the basic counting principle, we see that there are

$$2 \times 2 \times 2 \times 2 = \mathbf{16} \text{ possible ways.}$$

Hence for 10 questions, there are

$$\underbrace{2 \times 2 \times 2 \times \ldots \times 2}_{10 \text{ factors}} = 2^{10} = \mathbf{1024} \text{ possible answers.}$$

Example 2

In how many ways can Amy, Bernard and Carol arrange themselves in a queue?

Solution:

Let us use 3 boxes to represent the 3 places in the queue.

The 1st box (or place) can be filled in **3** ways, i.e. by any of the **3** persons.

The 2nd box can be filled in 2 ways, i.e. by any of the 2 remaining persons.
The 3rd box will go to the last person.
So, we have

3	2	1

By the basic counting principle,
the number of arrangements is $3 \times 2 \times 1 = 6$.

Can you work out the 6 possible arrangements of these 3 people using a tree diagram?

Exercise 13.1

1. A coin is tossed 3 times. Each toss results in a head or a tail. Find the number of possible outcomes (results) for the 3 tosses. Illustrate these outcomes on a tree diagram.
2. A quiz on the topic of *Gnotosrulb* consists of 5 multiple-choice questions. Each question has four choices, of which one is the correct answer. In how many ways can Sally, who is totally unfamiliar with this topic, answer all the questions?
3. There are 3 feeder bus services plying to and from the nearby town centre. Joe can ride on one of these services or walk to the town centre. However, he decides that on his way back, he will ride on one of these services. In how many ways can he go to the town centre and be back?
4. Find the number of ways to arrange in a row
   (a) 5 people, (b) 6 people.
5. Claire has to do the following during her lunch break: take lunch, post a letter, go to the bank, buy the afternoon papers. In how many ways can she do all these?
6. Six men and five women are available to form a mixed doubles pair for a tennis match. How many pairs are possible?
7. Three friends decide to have dinner together and then go shopping. Five restaurants are proposed for the dinner and four nearby shopping centres are suggested. How many possibilities are there?
8. A big company classifies its employees according to sex, age-group (6 divisions) and employment type (10 categories). How many classifications are there?
9. At a restaurant, a complete dinner meal consists of an appetizer, a main course, a dessert and a beverage. The choices for the appetizer are soup or juice; for the main course are chicken, fish, steak or lamb; for the dessert are Cherries Jubilee, Fresh Peach Cobbler, Chocolate Truffle Cake or Blueberry Rolypoly; for the beverage are coffee, tea or milk. How many complete dinner meals are possible?
10. Eight people have been shortlisted for an interview. In how many ways can the interviewer see them one after another?

13.2 Permutations

Recall Example 2. The 6 possible arrangements of the 3 persons (using their initials) are:

$$ABC \quad ACB \quad BAC \quad BCA \quad CAB \quad CBA$$

These arrangements are also called permutations. A **permutation** is *an arrangement of objects in a definite order*. To **permute** objects is to arrange them in a certain order.

Extending Example 2, the number of permutations of 8 different (or distinguishable) objects would be $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$.

A neater way of writing this long string of numbers is 8! (read as 8 **factorial**).
Hence we have:

The number of permutations of n different objects is
$$n! = n(n-1)(n-2) \times \ldots \times 3 \times 2 \times 1$$

Note: **(1)** n factorial, $n! = n(n-1)(n-2) \times \ldots \times 3 \times 2 \times 1$
$= n(n-1)!$

(2) How should we define 0! so that the above result would still hold?

(3) Factorials can be evaluated on a scientific calculator.

Example 3 Find the total number of different permutations of all the letters of the word SMART.

Solution: Notice that all the letters are different.

$\therefore$ the total number of permutations $= 5!$
$= \mathbf{120}$

Example 4 In how many ways can 9 different books be arranged on a shelf? If another book is added, what is the total number of permutations?

Solution: The number of ways $= 9!$
$= \mathbf{362\ 880}$

With the addition of the 10th book,
the total number of permutations $= 10!$
$= 10 \times 9!$
$= \mathbf{3\ 628\ 800}$

Permutation of r objects from n different objects

In Examples 3 and 4, *all* of the given objects are used in the permutation. We shall now consider permutations which use only *some* of the given objects.

Example 5 For a school concert, 6 items are proposed. Only 4 items will be put up at the concert. How many permutations of 4 concert items are there?

Solution: Let us use 4 boxes to represent the 4 concert items.

For the 1st item, there are 6 choices, i.e. any of the 6 proposed items.
The 2nd item can be any of the 5 remaining items.
The 3rd item can be any of the 4 remaining items.
The 4th item can be any of the 3 remaining items.
So, we have

6	5	4	3

and the number of permutations is $6 \times 5 \times 4 \times 3 = 360$.

We say that the number of permutations of 4 from 6 is ${}^6P_4$ where

$$ {}^6P_4 = \underbrace{6 \times 5 \times 4 \times 3}_{4 \text{ factors}} $$

Using the factorial notation, we have

$$ \begin{aligned} {}^6P_4 &= 6 \times 5 \times 4 \times 3 \\ &= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} \\ &= \frac{6!}{2!} \end{aligned} $$

Similarly,

$$ \begin{aligned} {}^nP_r &= n(n-1)(n-2) \times \ldots \times (n-r+1) \\ &= \frac{n(n-1)(n-2) \times \ldots \times (n-r+1)(n-r)(n-r-1) \times \ldots \times 2 \times 1}{(n-r)(n-r-1) \times \ldots \times 2 \times 1} \\ &= \frac{n!}{(n-r)!} \end{aligned} $$

In general, the number of permutations of r objects from n different objects is

$$ \begin{aligned} {}^nP_r &= \underbrace{n(n-1)(n-2) \times \ldots \times (n-r+1)}_{r \text{ factors}} \\ &= \frac{n!}{(n-r)!} \end{aligned} $$

Note: (1) If your calculator cannot evaluate ${}^nP_r$ directly, use the above result.

(2) When $r = n$, we have ${}^nP_n = \frac{n!}{(n-n)!} = n!$ since $0! = 1$.

(3) When $r = 0$, we have ${}^nP_0 = \frac{n!}{(n-0)!} = 1$.

Example 6 Find the number of 5-letter permutations that can be formed from the letters in the word SINGAPORE.

Solution: Notice that all the 9 letters are different.

$\therefore$ the number of permutations of 5 letters from the given 9 letters is

$$^9P_5 = \mathbf{15\ 120}$$

Example 7 A club has four officials: president, vice-president, secretary and treasurer. If a member cannot hold more than one office, in how many ways can the officials be elected if the club has

(a) 12 members, **(b)** 16 members?

Solution: Since a member cannot hold more than one office, there will be 4 people in the committee.

(a) The number of possible committees is $^{12}P_4 = \mathbf{11\ 880}$

(b) The number of possible committees is $^{16}P_4 = \mathbf{43\ 680}$

Permutations with Restrictions

Often there are restrictions on how the objects are to be arranged. With this type of problems, we settle the restrictions first and apply the basic counting principle.

Example 8 Calculate how many different 5-digit numbers can be formed from the eight digits 1, 3, 4, 5, 6, 7, 8, 9 used without repetition.
How many of these 5-digit numbers are

(a) less than 40 000, **(b)** even?

Solution: There are $^8P_5 = \mathbf{6720}$ different 5-digit numbers.

(a) For the number to be less than 40 000, the 1st digit must be 1 or 3.
So, the 1st box (digit) can be filled in 2 ways:

2				

The other 4 digits can come from the remaining 7 digits in $^7P_4$ ways.
Hence by the basic counting principle,

the number of such numbers $= 2 \times {}^7P_4 = \mathbf{1680}$

(b) For the number to be even, the last digit must be 4, 6 or 8.
So, there are 3 ways to fill the last box:

				3

The other 4 digits can come from the remaining 7 digits in ${}^{7}P_{4}$ ways.

Hence by the basic counting principle,

the number of such numbers $= 3 \times {}^{7}P_{4} =$ **2520**

Example 9

Find the number of permutations of all the letters in the word HISTORY.
Find the number of these permutations in which
(a) the letters O and R are together,
(b) the letters O and R are **not** together.

Solution:

The number of permutations of the 7 different letters $= 7! =$ **5040**.

(a) To ensure that O and R are together, we gather them together to form one block. Within this block, the 2 letters can be permuted in $2!$ ways.
This block with the remaining 5 letters form 6 objects which can be permuted in $6!$ ways.

Hence by the basic counting principle,

the number of such permutations $= 2! \times 6!$
$=$ **1440**

(b) The number of permutations in which O and R are **not** together
$=$ total number of permutations without restrictions
$-$ the number of permutations with O and R together
$= 5040 - 1440$
$=$ **3600**

Example 10

4 boys and 5 girls are to form a line. In how many ways can this be done? Find also, the number of permutations in which
(a) the first two are girls,
(b) the first is a boy and the last is a girl,
(c) the boys are together,
(d) no two girls stand next to each other.

Solution:

Without any restriction, the 9 people can be permuted in $9! =$ **362 880** ways.

(a) If the 1st place must go to a girl, it can be filled by any of the 5 girls.
The 2nd place can be filled by any of the other 4 girls.
Thus far, we have

5	4							

The remaining 7 people can then be permuted in the rest of the places in 7! ways.

Hence by the basic counting principle,
the number of such permutations $= 5 \times 4 \times 7!$
$= \mathbf{100\ 800}$

Alternatively,
the first 2 places can be filled by 2 girls from the 5 girls in ${}^5P_2$ ways and so we have ${}^5P_2 \times 7! = \mathbf{100\ 800}$, as before.

(b) The 1st place can be filled by any of the 4 boys.
The last place goes to any of the 5 girls.
Thus far, we have

4								5

The remaining 7 people can then be permuted in the rest of the places in 7! ways.

Hence by the basic counting principle,
the number of such permutations $= 4 \times 5 \times 7!$
$= \mathbf{100\ 800}$

(c) To ensure the boys are together, we gather them together to form one block. Within this block, these 4 boys can be permuted in 4! ways.
This block and the 5 girls then form 6 objects which can be permuted in 6! ways.
Hence the number of such permutations $= 4! \times 6!$
$= \mathbf{17\ 280}$

(d) If no two girls are together, the positions of the 5 girls relative to the 4 boys must be as follows:

G B G B G B G B G

The girls can be permuted in their positions in 5! ways.
The boys can be permuted in their positions in 4! ways.
Hence the number of such permutations $= 5! \times 4!$
$= \mathbf{2880}$

If no two boys stand next to each other, would our answer in (d) hold good? State your reason.

Exercise 13.2

1. Find the total number of different permutations of all the letters of the word
(a) SIMPLE, **(b)** SECONDARY.

2. Find the total number of different 4-digit numbers using all the digits in the number 4129.

3. The following duties need to be carried out: clean the board, arrange the tables, sweep the floor and clear the waste paper basket. Find the number of ways of assigning these duties to 4 students, given that each student will perform only one duty.

4. In a particular division of a soccer league, there are 9 teams. How many different end-of-the-season rankings are possible? Assume that there are no ties.

5. There are five finalists at an oratorical competition. In how many ways can they be arranged to give their speeches?

6. Four students go to dinner and order a hamburger, a fish burger, a cheeseburger and a beef burger (one burger for each). When the waitress returns with the food, she forgets which student orders which item and simply places a burger before each student. In how many ways can the waitress do this?

7. Without using a calculator, evaluate

   (a) $\frac{4!}{5!}$, (b) $\frac{8!}{6!}$, (c) $\frac{10!}{6!\,4!}$.

8. Simplify

   (a) $\frac{(n-1)!}{n!}$, (b) $\frac{(n-3)!}{n!}$, (c) $\frac{(n!)^2}{(n-1)!(n-2)!}$.

9. Without using a calculator, evaluate

   (a) ${}^7P_3$, (b) ${}^{10}P_2$, (c) ${}^5P_5$.

10. In a Mathematics class with 30 students, the teacher wants 2 different students to present the solutions to problems 3 and 5 on the board. In how many ways can the teacher assign the problems?

11. A shelf will hold only seven books. Given that 11 different books are available, find the number of different arrangements that can be made to fill the shelf.

12. In how many ways can a judge award first, second and third prizes in a contest with 9 participants?

13. In a survey, 10 characteristics of a teacher are listed. You are asked to indicate in order of importance which 4 of these characteristics make a good teacher. How many possible responses are there?

14. How many five-digit numbers can be formed from the digits 2, 3, 5, 7, 8 and 9 if no digit may be repeated?

15. 5 actors and 8 actresses are available for a play which requires 3 male roles and 4 female roles. Find the number of different possible cast lists.

16. In how many ways can 3 boys and 2 girls line up for a group picture? In how many ways can they line up if a boy is to be at each end?

17. A race has a first prize, a second prize and a third prize. 8 runners enter this race and the prizes are awarded for the first, second and third runners in order of merit. Find the number of ways in which these prizes could be won.
    Mei Fong and Evelyn are 2 of the 8 runners. Find the number of different ways in which the prizes could be won if neither of them wins a prize.

18. Amy, Brian, Cheryl, Danny and Eric went to a concert. How many arrangements are possible when they sit in five adjacent seats if
(a) Eric insists on sitting next to Cheryl?
(b) Brian refuses to sit next to Danny?

19. Each of 7 children, in turn, throws a ball once at a target. Calculate the number of ways the children can be arranged in order to take the throws.
Given that 3 of the children are girls and 4 are boys, calculate the number of ways the children can be arranged in order that
(a) successive throws are made by boys and girls alternately,
(b) a girl takes the first throw and a boy takes the last throw. *(C)*

20. Calculate the number of arrangements of the letters of the word INCLUDE if
(a) all the consonants are together,
(b) no two consonants are together,
(c) each arrangement begins with a consonant and ends with a vowel. *(C)*

21. How many numbers between 2000 and 5000 can be made from the digits 1, 2, 4, 5, 7 and 8 if each digit is used only once?

22. At an art exhibition 7 paintings are to be hung in a row along one wall. Find the number of possible arrangements. Given that 3 paintings are by the same artist, find the number of arrangements in which
(a) these 3 paintings are hung side by side,
(b) any one of these paintings is hung at the beginning of the row but neither of the other 2 is hung at the end of the row. *(C)*

23. Calculate the total number of different permutations of all the letters **A**, **B**, **C**, **D**, **E**, **F** when
(a) there are no restrictions,
(b) the letters **A** and **B** are to be adjacent to one another,
(c) the first letter is **A**, **B** or **C** and the last letter is **D**, **E** or **F**. *(C)*

*24. 9 different books are to be arranged on a book-shelf. 4 of these books were written by Shakespeare, 2 by Dickens and 3 by Conrad. How many possible permutations are there if
(a) the books by Conrad must be next to each other?
(b) the books by Dickens are separated from each other?
(c) the books by Conrad are separated from each other?

13.3 Combinations

Having studied permutations where the order of each object is important, let us turn our attention to combinations. A **combination** is *any selection of objects where the order of the objects is immaterial (of no concern).*

For example, the different (ordered) permutations ABC and CAB are considered as the *same combination* when we disregard the order of the letters and realise that both contain the same three letters.

Consider the permutations of 2 people from 4 people, P, Q, R and S. We have learnt that there are ${}^4P_2 = 12$ such permutations.

But if we consider the number of *handshakes* between any 2 of these 4 people, it does not matter whether P shakes Q's hand or Q shakes P's hand, it is counted as only 1 handshake. Observe that the order of P and Q is **not** important here. In such a situation, we are interested in the number of *combinations*, not permutations.

Permutations (order important)		Combinations (order not important)
PQ	QP	PQ
PR	RP	PR
PS	SP	PS
QR	RQ	QR
QS	SQ	QS
RS	SR	RS

Using the letter C for combination, we have:

The number of combinations of r objects from n different objects is ${}^nC_r$.

Can you see from the table above that ${}^4P_2 = {}^4C_2 \times 2!$?

In general, to permute r objects from n different objects, we could first select the r objects in ${}^nC_r$ ways and then arrange these r objects in $r!$ ways.

Hence:

$$ {}^nP_r = {}^nC_r \times r! \quad \Rightarrow \quad {}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{(n-r)!\,r!} $$

Note: Some scientific calculators can evaluate ${}^nC_r$ directly. Otherwise, use the above result.

Example 11 A committee of 5 members is to be selected from 6 seniors and 4 juniors. Find the number of ways in which this can be done if

(a) there are no restrictions,
(b) the committee has exactly 3 seniors,
(c) the committee has at least 1 junior.

Solution: Observe that the order of the members in the committee is *not* important.

(a) If there are no restrictions, the 5 members can be selected from the 10 people in ${}^{10}C_5 = \mathbf{252}$ ways.

(b) If the committee has exactly 3 seniors, these seniors can be selected from the 6 seniors in ${}^6C_3$ ways.
The remaining 2 members must then be juniors which can be selected from the 4 juniors in ${}^4C_2$ ways.
By the basic counting principle,
the number of committees with exactly 3 seniors is
${}^6C_3 \times {}^4C_2 = \mathbf{120}$.

(c) Number of committees with *no* junior (i.e. all 5 members are seniors) $= {}^4C_0 \times {}^6C_5 = 6$

So, the number of committees with at least 1 junior

= total number of committees − number of committees with no junior

$= 252 - 6$

$= 246$

Exercise 13.3

1. Without using a calculator, evaluate ${}^5C_4$ and ${}^9C_3$.
2. Using the result ${}^nC_r = \dfrac{n!}{(n-r)!\,r!}$
   (a) evaluate ${}^nC_0$ and ${}^nC_n$,
   (b) find ${}^nC_1$ and ${}^nC_2$ in terms of n,
   (c) prove that ${}^nC_r = {}^nC_{n-r}$.
3. In a soccer league consisting of twelve sides, each team plays every other team once. How many matches are there?
4. A club has 14 members. It has to send a delegation of 5 members to represent them at a particular event. Find the number of possible delegations.
5. At the library, Peter found 6 books of interest but he can only borrow 4 books. How many possible selections can he make?
6. The second section of a Mathematics paper contains 7 questions and a candidate must answer any 4 questions. In how many ways can the 4 questions be chosen (without regard to order)?
7. On a piece of paper, 8 points are marked such that no 3 points lie on the same straight line. How many lines can be drawn passing through any 2 of these points?
8. Seven points lie on a circle. How many triangles can be drawn using any 3 of these points as vertices?
9. A group of 4 adults and 3 children are to be formed from 8 adults and 5 children. How many possible groups are there?
10. To promote reading, a teacher decides to feature 3 classics, 4 contemporary novels and 2 non-fiction books on a notice board. How many selections can she make from 5 classics, 6 contemporary novels and 4 non-fiction books?
11. Calculate the number of ways in which
    (a) 5 children can be divided into groups of 2 and 3,
    (b) 9 children can be divided into groups of 5 and 4.
    Hence calculate the number of ways in which 9 children can be divided into groups of 2, 3 and 4. *(C)*
12. For a community service project, 10 students volunteer to do some cleaning up at an old folks' home. The teacher-in-charge wants to split them into 3 groups. One group will consist of 5 students to clean the rooms; another group of 3 students to clean the toilets; the last group of 2 students to sweep the corridors. Find the number of ways the teacher-in-charge can form the groups.

13. A club contains the President, the Secretary and 8 other members. The club is asked to send a group of 4 representatives to a conference. Calculate the number of different ways in which the group can be formed if it must contain
   (a) both the President and the Secretary,
   (b) either the President or the Secretary, but not both.

14. A particular firm has 6 vacancies to fill from 15 applicants. Calculate the number of ways in which these vacancies could be filled if there are no restrictions.
   The firm decides that 3 of the 6 vacancies shall be filled by women and 3 by men. The applicants consist of 7 women and 8 men. Calculate the number of ways in which the 6 vacancies could be filled under these conditions.
   One of the 7 women is the wife of one of the 8 men. Calculate the number of ways in which 3 women and 3 men could fill these 6 vacancies, given that both the wife and her husband are among those appointed. *(C)*

15. A delegation of 4 people is to be selected from 5 women and 6 men. Find the number of possible delegations if
   (a) there are no restrictions, (b) there is at least 1 woman,
   (c) there are at least 2 women.
   One of the men cannot get along with one of the women. Find the number of delegations which include this particular man or woman, but not both.

16. A chess club has 10 members, of whom 6 are men and 4 are women. A team of 4 members is selected to play in a match. Find the number of different ways of selecting the team if
   (a) all the players are to be of the same sex,
   (b) there must be an equal number of men and women.
   Given that the 6 men include 2 brothers, find the total number of ways in which the team can be selected if either of the brothers, but not both, must be included. *(C)*

17. Five different letters are chosen from the letters A, B, C, D, E, F, G, H, I and J. Find the number of choices.
   How many of these choices contain
   (a) exactly 3 consonants, (b) no vowel, (c) at least one vowel?

18. An art gallery has a total of 11 paintings by a particular artist; of these, 5 are oil paintings and 6 are watercolour paintings. The art gallery is asked to provide a special exhibition of this artist's work but is restricted to showing only 5 paintings. Calculate the number of ways in which the 5 paintings can be selected for the exhibition if
   (a) there are no restrictions,
   (b) one particular painting must be included,
   (c) there must be exactly 3 watercolour paintings,
   (d) there must be more watercolour paintings than oil paintings. *(C)*

Important Notes

1. **Basic counting principle**
   To find the numbers of ways of performing several tasks *in succession*, multiply the numbers of ways in which each task can be performed.

2. **Factorial**

$$n! = n(n-1)(n-2) \times \ldots \times 3 \times 2 \times 1$$
$$0! = 1$$
$$n! = n(n-1)!$$

3. **Permutations and combinations**

(a) ***Without restrictions***

n different objects

- Permutation (ordered)
  - Permute all n objects ($n!$)
  - Permute only r objects $\left({}^nP_r = \dfrac{n!}{(n-r)!} \right)$
- Combination (unordered)
  - Select only r objects $\left({}^nC_r = \dfrac{n!}{(n-r)!\,r!} \right)$

(b) ***With restrictions***

Settle the restrictions first and apply the basic counting principle.

Miscellaneous Examples

Example 12

2 Mathematics books, 4 Science books and 3 Literature books are to be arranged on a shelf. In how many ways can this be done if

(a) there are no restrictions?

(b) the 2 Mathematics books are not together?

(c) books of the same subject must be placed next to one another?

If 5 of these 9 books are to be transferred and arranged on another shelf, in how many ways can this be done if the first and last books on this shelf are Literature books?

Solution:

(a) If there are no restrictions, number of ways = 9! = **362 880**.

(b) We shall first find the number of ways in which the 2 Mathematics books are together:

Gather these 2 books together to form one block.
Within this block, these 2 books can be permuted in 2! ways.
This block and the other 7 books then form 8 objects which can be permuted in 8! ways.

So, the number of such arrangements = $2! \times 8! = 80\ 640$

$\therefore$ number of ways in which the 2 Mathematics books are **not** together = $362\ 880 - 80\ 640$
$= \mathbf{282\ 240}$

(c) We gather books of the same subject together. This results in 3 blocks which can be arranged in 3! ways.
Within the block of 2 Mathematics books, there are 2! permutations.
Within the block of 4 Science books, there are 4! permutations.
Within the block of 3 Literature books, there are 3! permutations.

Hence by the basic counting principle,
the required number of ways = $3! \times 2! \times 4! \times 3!$
$= \mathbf{1728}$

5 books are to go to another shelf.
The 1st place can be filled by any of the 3 Literature books.
The last place can be filled by any of the remaining 2 Literature books.

Thus far, we have

3				2

The remaining 3 places can be filled by 3 of the remaining 7 books in ${}^7P_3$ ways.

$\therefore$ the required number of ways = $3 \times 2 \times {}^7P_3$
$= \mathbf{1260}$

Example 13

A committee of 5 members are to be formed from 5 married couples. Find the number of committees if

(a) the selection is random (i.e. there are no restrictions),
(b) a particular couple is in the committee,
(c) there are more men than women.

Solution:

(a) Note that there are 10 people (5 men and 5 women) in the 5 married couples. 5 of these people are to be in the committee.
So, number of committees with no restrictions = ${}^{10}C_5 = \mathbf{252}$

(b) If a particular couple (that is, 2 people) is already in the committee, we need to select 3 more people from the remaining 8 people.
$\therefore$ number of committees with a particular couple = ${}^8C_3 = \mathbf{56}$

(c) If there are more men than women, we have the following cases:

No. of men	No. of women	No. of such committees
3	2	${}^5C_3 \times {}^5C_2 = 100$
4	1	${}^5C_4 \times {}^5C_1 = 25$
5	0	${}^5C_5 \times {}^5C_0 = 1$

$\therefore$ the total number of such committees $= 100 + 25 + 1$
$= \mathbf{126}$

Miscellaneous Exercise 13

1. Tammy wishes to contribute three items to a charity gift box. She narrows her choices to seven particular items. Find the number of ways she can select the three items from these seven items.

2. How many 5-letter words ending with a consonant can be formed using letters from the word EDUCATION? (A word here need not appear in a dictionary.)

3. A book store's bestseller list shows its 10 most popular books. Kenneth wishes to buy 4 of them as gifts. How many possibilities are there?

4. Eight friends are to make a journey in 2 taxis, with 4 people in each taxi. In how many ways can this be done?

5. A particular class consists of 25 Chinese, 6 Malays, 4 Indians and 2 Eurasians. Two students from each race are to be selected for a Racial Harmony Day concert item. In how many ways can this be done?

6. There are 6 males and 6 females in the finals of a talent competition. A contest is held to pick the top 3 winners in both the male and female categories in order of merit. How many different entries must be completed to ensure a winning order?

7. A certain contest awards three top prizes of \$50 000, \$20 000 and \$10 000 and five consolation prizes of \$1000 each. Twenty people have each submitted one winning entry and a draw is carried out to pick the eight different winners.
   Firstly, the five winners for the consolation prizes are drawn. How many outcomes are possible?
   Next, the three winners for the top prizes are drawn. How many possible results are there?

8. Find how many numbers between 2500 and 5000 can be formed using digits from 1, 2, 3, 4, 5 and 7, with no digit being repeated.

9. Find the number of ways 3 boys and 4 girls can be arranged in a row if
   (a) there are no restrictions,
   (b) a particular boy and a particular girl must be next to each other,
   (c) the first and last persons are of the opposite sex.

10. Calculate in how many ways each of the following choices can be made:
   (a) 4 books are to be chosen from a list of 9 titles to take away for reading during a holiday.
   (b) 15 people have sent in winning entries for a magazine competition, and three are to be chosen and placed in order of merit so as to receive the 1st, 2nd and 3rd prizes.
   (c) A committee of 3 people comprising the president, secretary and treasurer are to be chosen from 8 possible candidates.

11. Find the total number of arrangements using all of the letters in the word LOGARITHMS.
   How many of these arrangements
   (a) start with H and end with A?
   (b) have three consecutive vowels?
   (c) have the consonants G and H separated?

12. A committee of 6 members is to be chosen from 6 women and 5 men. Calculate the number of ways this can be done if
   (a) the selection is random,
   (b) a minimum of 4 men must be chosen.

13. A tennis team of 4 men and 4 women is to be picked from 6 men and 7 women. Find the number of ways in which this can be done.
   It was decided that 2 of the 7 women must either be selected together or not at all. Find how many possible teams could be selected in these circumstances.
   The selected team is arranged into 4 pairs, each consisting of a man and a woman. Find the number of ways in which this can be done. *(C)*

14. From a group of 6 girls and 7 boys, how many 5-member committees consist of
   (a) 3 girls and 2 boys?
   (b) 3 boys and 2 girls?
   (c) members of the same sex?
   (d) more boys than girls?

15. Find the number of ways in which a team of 6 batsmen, 4 bowlers and a wicket-keeper may be selected from a squad of 8 batsmen, 6 bowlers and 2 wicket-keepers.
   Find the number of ways in which
   (a) this team may be selected if it is to include 4 specified batsmen and 2 specified bowlers,
   (b) the 6 batsmen may be selected from the 8 available, given that 2 particular batsmen cannot be selected together. *(C)*

16. Which of the following statements are incorrect? Give your reasons.
   (a) Group A and Group B are to be formed from 7 people. Group A is to have 4 people and Group B is to have 3 people. This can be done in ${}^7C_3$ ways.
   (b) From a set of 6 distinct elements, ${}^6P_4$ subsets of 4 elements can be formed.
   (c) 9 people are to form three queues of sizes 2, 3 and 4. This can be done in ${}^9C_2 \times {}^7C_3 \times {}^4C_4$ ways.
   (d) ${}^4P_3$ different codes can be formed from the letters in the word FREE. (***Hint:*** Use a tree diagram to list the possibilities.)

17. The number of applicants for a job is 15. Calculate the number of ways in which 6 applicants can be selected for interview.
The 6 selected applicants are interviewed on a particular day. Calculate the number of ways in which the order of the 6 interviews can be arranged.
Of the 6 applicants, 3 have backgrounds in business, 2 have backgrounds in education and 1 has a background in recreation. Calculate the number of ways in which the order of the 6 interviews can be arranged, when applicants having the same background are interviewed in succession. *(C)*

18. Calculate the number of ways of selecting 2 points from 6 distinct points.
Six distinct points are marked on each of two parallel lines. Calculate the number of

(a) distinct quadrilaterals which may be formed using 4 of the 12 points as vertices,
(b) distinct triangles which may be formed using 3 of the 12 points as vertices. *(C)*

19. **(a)** A shelf is to contain 7 different books, of which 4 were written by Dickens and 3 by Hardy. Find the number of arrangements in which
(i) no two books by the same author are adjacent,
(ii) the first two books at the left-hand end are by the same author.

(b) A concert pianist agrees to take part in a charity concert. She offers a choice of 10 works, of which 5 were composed by Chopin, 3 by Liszt and 2 by Schumann. Calculate the number of ways these 4 works can be selected if
(i) there are no restrictions,
(ii) there are 2 works by Chopin, 1 by Liszt and 1 by Schumann,
(iii) there must be at least one work by each composer,
(iv) the selection contains works by exactly two of the three composers. *(C)*

14 Binomial Theorem

14.1 The Binomial Expansion of $(1 + b)^n$

Expressions such as $(x + y)$, $(1 - x)$ and $(a + b)$ which contain two terms each, are called **binomials**. Let us take a binomial $(1 + b)$. If we keep on multiplying $(1 + b)$ by itself, we get $(1 + b)$, $(1 + b)^2$, $(1 + b)^3$, $(1 + b)^4$, $(1 + b)^5$, ...

We know
$$\begin{aligned}(1 + b)^2 &= (1 + b)(1 + b)\\ &= 1 + 2b + b^2\end{aligned}$$
and
$$\begin{aligned}(1 + b)^3 &= (1 + b)(1 + b)^2\\ &= (1 + b)(1 + 2b + b^2)\\ &= 1 + 3b + 3b^2 + b^3\end{aligned}$$

Similarly, we may obtain, by long multiplication, expansions for $(1 + b)^4$, etc. Is there a pattern in such expansions? To answer this, we list the following expansions of $(1 + b)^n$.

		Number of terms
$n = 0$	1	1
$n = 1$	$1 + b$	2
$n = 2$	$1 + 2b + b^2$	3
$n = 3$	$1 + 3b + 3b^2 + b^3$	4
$n = 4$	$1 + 4b + 6b^2 + 4b^3 + b^4$	5
$n = 5$	$1 + 5b + 10b^2 + 10b^3 + 5b^4 + b^5$	6

Based on the expansions above, we draw the following rules concerning the expansions of $(1 + b)^n$.

(a) There are $(n + 1)$ terms.

(b) The power of b starts with 0 and increases to n, i.e. its powers are in ascending order.

For each of the above **binomial expansions**, we list the coefficients, which are known as **binomial coefficients**, as follows:

$n = 0$	**1**
$n = 1$	$\underbrace{1 + 1}$
$n = 2$	$\underbrace{1 + 2} \underbrace{+ 1}$
$n = 3$	$\underbrace{1 + 3} \underbrace{+ 3} \underbrace{+ 1}$
$n = 4$	$\underbrace{1 + 4} \underbrace{+ 6} \underbrace{+ 4} \underbrace{+ 1}$
$n = 5$	**1 5 10 10 5 1**

The array of numbers on page 314 forms the **Pascal's triangle**. What do you observe about it? Firstly, the 'sides' of the Pascal's triangle consist of '1's'. Secondly, the numbers 'inside' the triangle are formed by adding the two numbers just above it as illustrated. Once you have recognised the pattern formed by the numbers, you can deduce subsequent rows of binomial coefficients. For example, the binomial coefficients in the next row $n = 6$ are obtained as follows:

$$\begin{array}{lccccccccccccc} n=5 & & & 1 & + & 5 & + & 10 & + & 10 & + & 5 & + & 1 \\ n=6 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \end{array}$$

$\therefore\ (1 + b)^6 = 1 + 6b + 15b^2 + 20b^3 + 15b^4 + 6b^5 + b^6$

However, there is an inherent weakness in this method. To find any binomial coefficient, you need the two coefficients just above it. A more direct way to get any binomial coefficient is to use the ${}^nC_r$ notation introduced in Chapter 13.

For example, when $n = 4$, there are 5 terms:	T_1,	T_2,	T_3,	T_4,	T_5
Their coefficients are:	1,	4,	6,	4,	1
Using the ${}^nC_r$ notation, they are:	${}^4C_0$,	${}^4C_1$,	${}^4C_2$,	${}^4C_3$,	${}^4C_4$

Using this notation, the Pascal's triangle becomes:

$$\begin{array}{lccccccccccc} n=1 & & & & & {}^1C_0 & & {}^1C_1 & & & & \\ n=2 & & & & {}^2C_0 & & {}^2C_1 & & {}^2C_2 & & & \\ n=3 & & & {}^3C_0 & & {}^3C_1 & & {}^3C_2 & & {}^3C_3 & & \\ n=4 & & {}^4C_0 & & {}^4C_1 & & {}^4C_2 & & {}^4C_3 & & {}^4C_4 & \\ n=5 & {}^5C_0 & & {}^5C_1 & & {}^5C_2 & & {}^5C_3 & & {}^5C_4 & & {}^5C_5 \end{array}$$

Thus, we can easily see that the coefficients of the 3rd and the 5th terms in the expansion of $(1 + b)^5$ are ${}^5C_2$ and ${}^5C_4$ respectively.

In the expansion of $(1 + b)^n$, the coefficient of $T_{r+1} = {}^nC_r$.

Recall that ${}^nC_0 = 1$, ${}^nC_n = 1$ and ${}^nC_r$ may be evaluated using a calculator.

Hence the binomial expansion of $(1 + b)^n$, where $n \in \mathbb{Z}^+$, is given by:

$$(1 + b)^n = 1 + {}^nC_1b^1 + {}^nC_2b^2 + \ldots + {}^nC_rb^r + \ldots + b^n$$

This result is known as the **Binomial Theorem**.

Example 1 Use the Binomial Theorem to expand $(1 + b)^5$. Hence deduce the expansion of
(a) $(1 - b)^5$, **(b)** $(1 + 2x)^5$.

Solution:

$$\begin{aligned}(1 + b)^5 &= 1 + {}^5C_1b + {}^5C_2b^2 + {}^5C_3b^3 + {}^5C_4b^4 + {}^5C_5b^5 \\ &= \mathbf{1 + 5b + 10b^2 + 10b^3 + 5b^4 + b^5}\end{aligned}$$

(a) $(1-b)^5 = [1+(-b)]^5$
$= 1 + 5(-b) + 10(-b)^2 + 10(-b)^3 + 5(-b)^4 + (-b)^5$
$= \mathbf{1 - 5b + 10b^2 - 10b^3 + 5b^4 - b^5}$

(b) Let $b = 2x$.
$(1+2x)^5 = 1 + 5(2x) + 10(2x)^2 + 10(2x)^3 + 5(2x)^4 + (2x)^5$
$= \mathbf{1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5}$

Note: The binomial expansion of $(1-b)^n$ is the same as that of $(1+b)^n$ except that the coefficients have **alternate** + and – signs.

Example 2

Find, in ascending powers of x, the first 4 terms in the binomial expansion of **(a)** $(1+4x)^6$, **(b)** $(1-3x)^7$.

Solution:

(a) $(1+4x)^6 = 1 + {}^6C_1(4x) + {}^6C_2(4x)^2 + {}^6C_3(4x)^3 + \ldots$
$= 1 + 6(4x) + 15(16x^2) + 20(64x^3) + \ldots$
$= \mathbf{1 + 24x + 240x^2 + 1280x^3 + \ldots}$

(b) $(1-3x)^7 = [1+(-3x)]^7$
$= 1 + {}^7C_1(-3x) + {}^7C_2(-3x)^2 + {}^7C_3(-3x)^3 + \ldots$
$= 1 + 7(-3x) + 21(9x^2) + 35(-27x^3) + \ldots$
$= \mathbf{1 - 21x + 189x^2 - 945x^3 + \ldots}$

Example 3

Find the first 4 terms in the expansion of $(1+x^2)^8$ in ascending powers of x. Use your result to estimate the value of $(1.01)^8$.

Solution:

$(1+x^2)^8 = 1 + {}^8C_1(x^2) + {}^8C_2(x^2)^2 + {}^8C_3(x^2)^3 + \ldots$
$= \mathbf{1 + 8x^2 + 28x^4 + 56x^6 + \ldots}$

Comparing $(1.01)^8$ with $(1+x^2)^8$, we note that:
$1 + x^2 = 1.01 \quad \Rightarrow \quad x = \pm 0.1.$
$\therefore \ (1.01)^8 = [1 + (0.1)^2]^8$
$= 1 + 8(0.1)^2 + 28(0.1)^4 + 56(0.1)^6 + \ldots$
$= 1 + 0.08 + 0.002\,8 + 0.000\,056 + \ldots$
$\approx \mathbf{1.082\,856}$

Note: To estimate $(1.01)^8$, one could also use $x = -0.1$.

Example 4

Find, in ascending powers of x, the first 3 terms in the expansion of **(a)** $(1-2x)^5$, **(b)** $(1+3x)^9$.
Hence find the expansion of $(1-2x)^5(1+3x)^9$ up to the terms in x^2.

Solution:

(a) $(1-2x)^5 = [1+(-2x)]^5$
$= 1 + {}^5C_1(-2x) + {}^5C_2(-2x)^2 + \ldots$
$= \mathbf{1 - 10x + 40x^2 + \ldots}$

(b) $(1 + 3x)^9 = 1 + {}^9C_1(3x) + {}^9C_2(3x)^2 + \ldots$

$= \mathbf{1 + 27x + 324x^2 + \ldots}$

$\therefore \quad (1 - 2x)^5(1 + 3x)^9$

$= (1 - 10x + 40x^2 + \ldots)(1 + 27x + 324x^2 + \ldots)$

$= 1(1 + 27x + 324x^2) - 10x(1 + 27x) + 40x^2(1) + \ldots$

$= \mathbf{1 + 17x + 94x^2 + \ldots}$

Instead of expanding from the first term, one can find a **specific** term using the following result:

In the expansion of $(1 + b)^n$, $T_{r+1} = {}^nC_r b^r$.

Example 5 Find the terms in x^2 and x^5 in the expansion of $\left(1 - \frac{x}{2}\right)^{12}$. Hence find the coefficient of x^5 in the expansion of $(3 + 2x^3)\left(1 - \frac{x}{2}\right)^{12}$.

Solution: For $\left(1 - \frac{x}{2}\right)^{12} = \left[1 + \left(-\frac{x}{2}\right)\right]^{12}$, the term in x^r is $T_{r+1} = {}^{12}C_r\left(-\frac{x}{2}\right)^r$.

For x^2, $T_3 = {}^{12}C_2\left(-\frac{x}{2}\right)^2 = 66\left(\frac{x^2}{4}\right) = \mathbf{\frac{33}{2}x^2}$

For x^5, $T_6 = {}^{12}C_5\left(-\frac{x}{2}\right)^5 = 792\left(-\frac{x^5}{32}\right) = \mathbf{-\frac{99}{4}x^5}$

$$\therefore \quad (3 + 2x^3)\left(1 - \frac{x}{2}\right)^{12} = (3 + 2x^3)\left(\ldots + \frac{33}{2}x^2 - \frac{99}{4}x^5 + \ldots\right)$$

$$= \ldots + 3\left(-\frac{99}{4}x^5\right) + 2x^3\left(\frac{33}{2}x^2\right) + \ldots$$

$$= \ldots - \frac{165}{4}x^5 + \ldots$$

$\therefore$ coefficient of x^5 is $\mathbf{-\frac{165}{4}}$.

Exercise 14.1

1. Expand each of the following:
   (a) $(1 - 2x)^4$ (b) $(1 + 3x)^5$ (c) $(1 - ax)^6$

2. Show that $(1 + \sqrt{x})^5 - (1 - \sqrt{x})^5 = 10\sqrt{x} + 20x\sqrt{x} + 2x^2\sqrt{x}$. Hence deduce the value of $(1 + \sqrt{2})^5 - (1 - \sqrt{2})^5$.

3. Find the first 4 terms, in ascending powers of x, in the following expansions:
   (a) $(1 + x)^{10}$ (b) $(1 - x)^{12}$ (c) $(1 - 2x)^8$
   (d) $(1 + 2x)^9$ (e) $(1 - 3x)^8$ (f) $(1 + x^2)^9$
   (g) $(1 - 2x^2)^7$ (h) $\left(1 - \frac{1}{2}x^3\right)^{16}$ (i) $\left(1 + \frac{2x}{y}\right)^8$

4. Find, in ascending powers of x, the first 4 terms in the expansion of $(1 + 3x)^{10}$. Hence deduce the first 4 terms in the expansion of
(a) $(1 - 3x)^{10}$, (b) $(1 + 3x^2)^{10}$.

5. Obtain the first 4 terms in the expansion of $(1 + 2x)^9$ in ascending powers of x. Use this expansion to find an approximate value of $(1.02)^9$.

6. Find, in ascending powers of x, the first 3 terms in the expansion of
(a) $(1 + 4x)^6$, (b) $(1 - 2x)^{14}$.
Hence find the first 3 terms in the expansion of $(1 + 4x)^6(1 - 2x)^{14}$.

7. Expand $(1 - 6x)^4(1 + 2x)^7$ in ascending powers of x up to and including the terms in x^3.

8. Obtain the binomial expansion of $(2 - x)\left(1 + \frac{1}{2}x\right)^8$ in ascending powers of x as far as the term in x^3. Use your result to estimate the value of $1.9 \times (1.05)^8$.

9. Expand $(1 + 2x)^{20} - (1 - 2x)^{20}$ in ascending powers of x up to the term in x^5 and use this result to evaluate $(1.02)^{20} - (0.98)^{20}$ to 3 significant figures.

10. In each of the following expansions, find the indicated term.
(a) $(1 + 4x)^7$, 5th term (b) $(1 - x^2)^{20}$, 8th term
(c) $(1 - 2x)^{10}$, middle term (d) $(1 + 2x^2)^8$, term in x^{12}

11. Find the coefficient of x^3 and of x^6 in the expansion of
(a) $(1 - 3x)^8$, (b) $(1 + 2x)^{12}$, (c) $\left(1 - \frac{x^2}{4}\right)^{10}$.

12. In the expansion of $(1 - 2x)^{11}$ the coefficient of x^3 is k times the coefficient of x^2. Evaluate k. *(C)*

13. (a) Find the ratio of the coefficients of the 4th and 6th terms in the expansion of $(1 + 2x)^{16}$.
(b) The coefficient of x^2 is 69 in the expansion of $(1 - ax)^{24}$, where $a > 0$. Find the coefficient of x^3.

14. Find the expansions of $(1 - x)^6$ and $(1 + 2x)^6$ as far as the terms in x^3. Hence expand $(1 + x - 2x^2)^6$ up to the terms in x^3.

15. When $(1 - x)(1 + ax)^6$ is expanded as far as the term in x^2, the result is $1 + bx^2$. Find the value of a and of b.

16. Find the coefficient of x^{22} in the expansion of $(1 - 3x)(1 + x^3)^{10}$.

17. Expand $(1 - x)^4$. Hence find S if
$$S = (1 - x^3)^4 - 4(1 - x^3)^3 + 6(1 - x^3)^2 - 4(1 - x^3) + 1.$$

*18. If x is sufficiently small such that the terms in x^3 and higher powers of x may be neglected, show that $(1 + x)^5(1 - 4x)^4 \approx 1 - 11x + 26x^2$. Hence or otherwise, expand the following up to the term in x^2.
(a) $(1 - x)^5(1 + 4x)^4$ (b) $(1 + x^2)^5(1 - 2x)^4(1 + 2x)^4$

14.2 The Binomial Expansion of $(a + b)^n$

We shall now derive a more general form of the Binomial Theorem for the expansion of $(a + b)^n$, where n is a positive integer.

$$\begin{aligned}(a+b)^n &= \left[a\left(1+\frac{b}{a}\right)\right]^n \\ &= a^n\left(1+\frac{b}{a}\right)^n \\ &= a^n\left[1 + {}^nC_1\left(\frac{b}{a}\right) + {}^nC_2\left(\frac{b}{a}\right)^2 + {}^nC_3\left(\frac{b}{a}\right)^3 + \ldots + {}^nC_n\left(\frac{b}{a}\right)^n\right] \\ &= a^n\left[1 + {}^nC_1\frac{b}{a} + {}^nC_2\frac{b^2}{a^2} + {}^nC_3\frac{b^3}{a^3} + \ldots + \frac{b^n}{a^n}\right]\end{aligned}$$

$$\boldsymbol{(a+b)^n = a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + {}^nC_3a^{n-3}b^3 + \ldots + b^n}$$

Observe that the rule is similar to the expansion of $(1 + b)^n$ except that now we include the powers of a, descending from n to 0. Observe further that in each term, the powers of a and b always add up to n, the power of the binomial.

Example 6 Expand $(a + b)^3$ and hence find the expansion of $(2 + 3x)^3$.

Solution:

$$\begin{aligned}(a+b)^3 &= a^3 + {}^3C_1a^2b + {}^3C_2ab^2 + b^3 \\ &= \mathbf{a^3 + 3a^2b + 3ab^2 + b^3}\end{aligned}$$

Let $a = 2$ and $b = 3x$.

Then, $(2 + 3x)^3 = 2^3 + 3(2^2)(3x) + 3(2)(3x)^2 + (3x)^3$
$= \mathbf{8 + 36x + 54x^2 + 27x^3}$

Example 7 Find, in descending powers of x, the first 4 terms in the binomial expansion of

(a) $(2x - 3)^5$, **(b)** $\left(x + \frac{1}{x^2}\right)^6$.

Solution:

(a) $(2x - 3)^5$

$$\begin{aligned}&= [2x + (-3)]^5 \\ &= (2x)^5 + {}^5C_1(2x)^4(-3) + {}^5C_2(2x)^3(-3)^2 + {}^5C_3(2x)^2(-3)^3 + \ldots \\ &= 32x^5 + 5(16x^4)(-3) + 10(8x^3)9 + 10(4x^2)(-27) + \ldots \\ &= \mathbf{32x^5 - 240x^4 + 720x^3 - 1080x^2 + \ldots}\end{aligned}$$

(b) $\left(x + \frac{1}{x^2}\right)^6 = x^6 + {}^6C_1x^5\left(\frac{1}{x^2}\right) + {}^6C_2x^4\left(\frac{1}{x^2}\right)^2 + {}^6C_3x^3\left(\frac{1}{x^2}\right)^3 + \ldots$

$= \mathbf{x^6 + 6x^3 + 15 + \frac{20}{x^3} + \ldots}$

Note: In **(b)**, the third term, 15, is independent of x, i.e. the power of x is zero.

Example 8 Given that $\left(p - \frac{1}{2}x\right)^6 = r - 96x + sx^2 + \ldots$, find p, r and s.

Solution:

$$\left(p - \frac{1}{2}x\right)^6 = \left[p + \left(-\frac{1}{2}x\right)\right]^6$$

$$= p^6 + {}^6C_1p^5\left(-\frac{1}{2}x\right) + {}^6C_2p^4\left(-\frac{1}{2}x\right)^2 + \ldots$$

$$= p^6 - 3p^5x + \frac{15}{4}p^4x^2 + \ldots$$

$$\therefore\ p^6 - 3p^5x + \frac{15}{4}p^4x^2 = r - 96x + sx^2$$

Comparing coefficients:

x: $-3p^5 = -96 \Rightarrow p^5 = 32$

$\Rightarrow p = \mathbf{2}$

x^0: $r = p^6 = \mathbf{64}$

x^2: $s = \frac{15}{4}p^4 = \mathbf{60}$

Finding a Specific Term

In the expansion of $(a + b)^n$, $T_{r+1} = {}^nC_ra^{n-r}b^r$.

Example 9 In each of the following expansions, find the indicated term.

(a) $\left(2 + \frac{x^2}{2}\right)^{11}$, 7th term **(b)** $\left(2x - \frac{1}{x^2}\right)^{12}$, term in $\frac{1}{x^{15}}$

Solution:

(a) For $\left(2 + \frac{x^2}{2}\right)^{11}$, $T_{r+1} = {}^{11}C_r2^{11-r}\left(\frac{x^2}{2}\right)^r$

$$\therefore\ T_7 = {}^{11}C_62^5\left(\frac{x^2}{2}\right)^6$$

$$= 462(2^5)\left(\frac{1}{2}\right)^6(x^2)^6$$

$$= \mathbf{231x^{12}}$$

(b) For $\left(2x - \frac{1}{x^2}\right)^{12}$, $T_{r+1} = {}^{12}C_r(2x)^{12-r}(-x^{-2})^r$

Power of x in $T_{r+1} = 12 - r - 2r$
$= 12 - 3r$

For $\frac{1}{x^{15}}$ (i.e. x^{-15}), $12 - 3r = -15$
$r = 9$

$\therefore$ the term in $\frac{1}{x^{15}} = {}^{12}C_9(2x)^3(-x^{-2})^9$
$= -\frac{1760}{x^{15}}$

Exercise 14.2

1. Expand the following:

(a) $(3 - 2x)^4$ (b) $(2 + x^2)^5$ (c) $\left(2 - \frac{1}{2x}\right)^6$

2. Find, in ascending powers of x, the first 4 terms in the expansion of

(a) $(2 + 3x)^6$, (b) $\left(4 - \frac{1}{2}x\right)^5$, (c) $\left(\frac{1}{2x} - 2x^2\right)^8$.

3. Obtain, in ascending powers of x, the first 4 terms in the expansion of $\left(2 - \frac{x}{2}\right)^7$. Hence find the value of $(1.995)^7$ correct to 4 decimal places.

4. Find the non-zero values of a and b for which $(2x - a)^3 = 8x^3 - bx^2 + \frac{3}{2}bx - a^3$.

5. Find the first 3 terms, in ascending powers of x, in the expansion of $(1 - 2x)^9$ and of $(2 + x)^5$. Hence expand $(1 - 2x)^9(2 + x)^5$ up to the terms in x^2.

6. Find the indicated term in each of the following expansions.

(a) $(2 + x)^{10}$, 7th term (b) $(3x - 2)^9$, 4th term

(c) $(y - 2x)^{10}$, 5th term (d) $\left(x + \frac{1}{2x^2}\right)^{12}$, middle term

7. In the expansion of $\left(x^3 - \frac{2}{x^2}\right)^{10}$, find

(a) the term in x^{10}, (b) the coefficient of $\frac{1}{x^5}$, (c) the constant term.

8. Expand $\left(\frac{1}{2} - 2x\right)^5$ up to the term in x^3. If the coefficient of x^2 in the expansion of $(1 + ax + 3x^2)\left(\frac{1}{2} - 2x\right)^5$ is $\frac{13}{2}$, find the coefficient of x^3.

9. In the expansion of $(1 + x)(a - bx)^{12}$, where $ab \neq 0$, the coefficient of x^8 is zero. Find in its simplest form the value of the ratio $\frac{a}{b}$.

Important Notes

1. **Recall from Chapter 13 that the binomial coefficient ^nC_r is given by:**

$$^nC_r = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

2. **The Binomial Theorem for positive integer, n**

   (a) $(a+b)^n = a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + {}^nC_3a^{n-3}b^3 + \dots + b^n$
   There are $n + 1$ terms. The powers of a are in **descending order** while the powers of b are in **ascending order**. The **sum of the powers** of a and b in each term of the expansion is always **equal** to n.

   (b) $T_{r+1} = {}^nC_ra^{n-r}b^r$

3. **In the above result, if $a = 1$,**

   (a) $(1+b)^n = 1 + {}^nC_1b + {}^nC_2b^2 + {}^nC_3b^3 + \dots + b^n$

   (b) $T_{r+1} = {}^nC_rb^r$

Miscellaneous Examples

Example 10 The third term of the binomial expansion of $\left(1 + \frac{1}{4}\right)^n$ is twice the fourth term. Calculate the value of n. Hence evaluate the middle term of this expansion.

Solution: For $\left(1 + \frac{1}{4}\right)^n$, $T_3 = {}^nC_2\left(\frac{1}{4}\right)^2$

$$T_4 = {}^nC_3\left(\frac{1}{4}\right)^3$$

Given $T_3 = 2 \times T_4$

$${}^nC_2\left(\frac{1}{4}\right)^2 = 2 \times {}^nC_3\left(\frac{1}{4}\right)^3$$

$$\frac{n(n-1)}{2!}\left(\frac{1}{4}\right)^2 = 2 \times \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{4}\right)^3$$

Simplifying,

$$\frac{1}{2} = \frac{n-2}{3}\left(\frac{1}{4}\right)$$

$$n - 2 = \frac{3 \times 4}{2}$$

$$\therefore \ n = \mathbf{8}$$

Since $n = 8$, there are $n + 1 = 9$ terms in the expansion.

$\therefore$ the middle term $= T_5$

$$= {}^8C_4\left(\frac{1}{4}\right)^4$$

$$= \mathbf{\frac{35}{128}}$$

Example 11 Write down and simplify the expansion of $(1-p)^5$. Use this result to find the expansion of $(1-x+x^2)^5$ in ascending powers of x as far as the term in x^3. Find the value of x which would enable you to estimate $(1.11)^5$ from this expansion.

Solution: $(1-p)^5 = \mathbf{1 - 5p + 10p^2 - 10p^3 + 5p^4 - p^5}$

Comparing $(1-x+x^2)^5$ with $(1-p)^5$:

$$1 - p = 1 - x + x^2$$
$$p = x - x^2$$

Substitute $p = x - x^2$ into the expansion:

$(1-x+x^2)^5$

$= 1 - 5(x-x^2) + 10(x-x^2)^2 - 10(x-x^2)^3 + \ldots$

$= 1 - 5x + 5x^2 + 10(x^2 - 2x^3 + x^4) - 10(x^3 + \ldots) + \ldots$

(ignoring powers higher than 3)

$= 1 - 5x + 5x^2 + 10x^2 - 20x^3 - 10x^3 + \ldots$

$= 1 - 5x + 15x^2 - 30x^3 + \ldots$

Let $1 - x + x^2 = 1.11$. Therefore $x^2 - x - 0.11 = 0$.

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-0.11)}}{2(1)} \quad \textit{(by quadratic formula)}$$

$= -0.1$ or 1.1

$\therefore$ to estimate $(1.11)^5$, we use $x = -0.1$ so that higher powers of x can be ignored.

Note: We cannot use $x = 1.1$, as higher powers of 1.1 cannot be ignored.

Example 12 Write down, in ascending powers of x, the first three terms in the expansion of $(2+ax)^5$. Given that the first three terms in the expansion of $(b+2x)(2+ax)^5$ are $96 - 176x + cx^2$, find the values of a, b and c.

Solution: $(2+ax)^5 = 2^5 + {}^5C_1 2^4(ax) + {}^5C_2 2^3(ax)^2 + \ldots$

$= \mathbf{32 + 80ax + 80a^2x^2 + \ldots}$

$\therefore$ $(b+2x)(2+ax)^5$

$= (b+2x)(32 + 80ax + 80a^2x^2 + \ldots)$

$= 32b + 80abx + 80a^2bx^2 + 64x + 160ax^2 + \ldots$

$= 32b + (64 + 80ab)x + (80a^2b + 160a)x^2 + \ldots$

Comparing this result with $96 - 176x + cx^2$, we have:

$32b = 96 \quad \Rightarrow \quad b = 3$

$64 + 80ab = -176 \quad \Rightarrow \quad 64 + 80a(3) = -176$

$a = -1$

and $c = 80a^2b + 160a$
$= 80(-1)^2(3) - 160$
$= 80$
$\therefore$ **$a = -1$, $b = 3$ and $c = 80$**

Miscellaneous Exercise 14

1. Find the first four terms in the expansion of $(1 - x)^6$ in ascending powers of x. Using this expansion, or otherwise, find the first four terms in the expansion of $\left(1 - \frac{x^2}{2}\right)^6$.

2. Use a graph plotter to obtain the graphs of
$y = (1 + 2x)^4$
and $y = 1 + 8x + 24x^2 + 64x^3 + 16x^4$.
Deduce from the graphs whether $(1 + 2x)^4 = 1 + 8x + 24x^2 + 64x^3 + 16x^4$. Given that the binomial expansion of $(1 + 2x)^4 - (1 - 2x)^4 = ax + bx^3$, find the values of a and b.

3. Expand $\left(2 + \frac{x}{4}\right)^6$ in ascending powers of x up to the term in x^3. Hence find an approximate value of $(1.997\ 5)^6$.

4. Find the coefficient of x^3 in the expansion of
(a) $\left(1 + \frac{x}{2}\right)^8$, (b) $\left(x - \frac{2}{x^2}\right)^{12}$.

5. Find, in ascending powers of x, the first four terms in the expansion of
(a) $(1 + 3x)^6$, (b) $(1 - 4x)^5$.
Hence find the coefficient of x^2 in the expansion of $(1 + 3x)^6(1 - 4x)^5$.

6. Find the term in x^3 in the expansion of $(1 + 5x)^2(1 - 2x)^6$.

7. Find the first 4 terms in the expansion of $\left(x^2 - \frac{2}{x}\right)^8$ in descending powers of x.
Hence find the coefficient of x^{13} in the expansion of $(x^3 + 1)^2\left(x^2 - \frac{2}{x}\right)^8$.

8. Write down and simplify the expansion of $(1 + p)^4$. Use this result to find the expansion of $(1 + x + x^2)^4$ in ascending powers of x as far as the term in x^3. Hence evaluate $(1.11)^4$ to 3 significant figures.

9. Expand $(2 - p)^6$. Use this result to find the expansion of $\left(2 - \frac{x}{2} + 2x^2\right)^6$ in ascending powers of x as far as the term in x^2.

10. Write down, and simplify, the first three terms of the expansion, in ascending powers of x, of
(a) $\left(1 + \frac{3x}{2}\right)^5$, (b) $(2 - x)^5$.

Hence, or otherwise, obtain the coefficient of x^2 in the expansion of $\left(2 + 2x - \frac{3x^2}{2}\right)^5$. *(C)*

11. Given that the coefficient of x^3 in the expansion of $(a + x)^5 + (1 - 2x)^6$ is -120, calculate the possible values of a. *(C)*

12. In the expansion of $(2 + 3x)^n$, the coefficients of x^3 and x^4 are in the ratio $8 : 15$. Find the value of n.

13. Given that the coefficients of x and x^2 in the expansion of $(1 + ax)^6(2 + bx)^5$ are -112 and 80 respectively, find the integer values of a and b.

14. Write down and simplify the first four terms in the expansion, in ascending powers of x, of $(2 + 3x)^5$. Use this expansion to

(a) estimate the value of $(2.03)^5$,

(b) obtain the expansion of $(2 - x + 3x^2)(2 + 3x)^5$ in ascending powers of x up to the term in x^2.

15. Find the coefficient of x^6 and of x^8 in the binomial expansion of $\left(1 + \frac{1}{2}x^2\right)^{10}$.

Hence find the coefficient of x^8 in the expansion of $(2x^2 + 3)\left(1 + \frac{1}{2}x^2\right)^{10}$.

16. Find the coefficient of $\frac{x^{11}}{y^4}$ in the expansion of $\left(\frac{2y}{x} - \frac{x^2}{y}\right)^{10}$.

17. Find in terms of a, the coefficient of $\frac{1}{x}$ in the expansion of $\left(\frac{a^2}{\sqrt{x}} - \frac{\sqrt{x}}{a}\right)^8$.

18. **(a)** Evaluate the coefficient of x^9 in the expansion of $(1 + 2x)(3 + x)^{11}$.

(b) Evaluate the coefficient of x^5 in the expansion of $\left(x^2 - \frac{2}{x}\right)^7$.

(c) The first three terms in the binomial expansion of $(a + b)^n$, in ascending powers of b, are denoted by p, q and r respectively. Show that $\frac{q^2}{pr} = \frac{2n}{n-1}$. Given that $p = 4$, $q = 32$ and $r = 96$, evaluate n. *(C)*

19. Obtain the binomial expansion of $(2 + \sqrt{3})^5$ in the form $a + b\sqrt{3}$, where a and b are integers. State the corresponding result for the expansion of $(2 - \sqrt{3})^5$ and show that $(2 - \sqrt{3})^5$ is the reciprocal of $(2 + \sqrt{3})^5$.

20. Write down the third and fourth terms in the expansion of $(a + bx)^n$. If these terms are equal, show that $3a = (n - 2)bx$.

21. When $(1 + ax)^8$ is expanded in ascending powers of x, the series expansion is $A + Bx + Cx^2 + 7x^3 + \ldots$ Find the values of a, A, B and C.

22. Given that $(1 + ax)^n = 1 - 12x + 63x^2 + \ldots$, find a and n.

23. Given that the equation $28x^6 - 30x^4 - 1 = 0$ has a root slightly greater than 1. By substituting $x = 1 + h$ and neglecting h^3 and higher powers of h, find a closer approximation to this root.

***24.** In the expansion of $(2 + x)^{14}\left(1 + \frac{2}{x}\right)^{14}$, find the coefficient of x^{12}.

***25.** If $(1 + ax + bx^2)^4 = 1 + 8x + 32x^2 + \dots$, find a and b.

26. Find the term independent of x in the expansion of $\left(1 + x + \frac{1}{x^3}\right)^7$.

27. Read the following argument to explain why the coefficient of b^3 in the binomial expansion of $(1 + b)^5$ is ${}^5C_3$:

$$(1 + b)^5 = (1 + b)(1 + b)(1 + b)(1 + b)(1 + b)$$

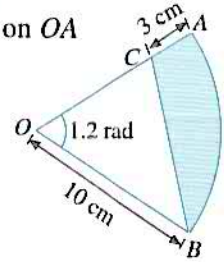
To obtain a b^3 term, we need to get a b from 3 of the brackets and a 1 from the other 2 brackets. One way this can be done is shown below:

$$\begin{array}{ccccccccccc} (1+b) & & (1+b) & & (1+b) & & (1+b) & & (1+b) & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ b & \times & 1 & \times & b & \times & b & \times & 1 & = & b^3 \end{array}$$

The b's can be chosen from the 3 brackets in ${}^5C_3$ ways and the 1's come from the remaining brackets.
It follows that there are ${}^5C_3$ terms of the form b^3, that is the coefficient of b^3 is ${}^5C_3$.

Use a similar argument to explain why:
(a) $(1 + b)^n = 1 + {}^nC_1b + {}^nC_2b^2 + {}^nC_3b^3 + \dots + {}^nC_rb^r + \dots + b^n$
(b) $(a + b)^n = a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + {}^nC_3a^{n-3}b^3 + \dots + {}^nC_ra^{n-r}b^r + \dots + b^n$

Revision Exercise 9



1. OAB is a sector of a circle, centre O and radius 10 cm. C lies on OA such that $AC = 3$ cm. If $A\hat{O}B = 1.2$ radians, calculate
   (a) the length of arc AB,
   (b) the area of sector AOB,
   (c) the area of the shaded region.

2. Find all the angles between 0° and 360° which satisfy the equation
   (a) $\tan(x + 45°) = 1$, **(b)** $\sec 2y = -2$, **(c)** $2 \cos z \cot z = 3$.

3. Sketch, on the same diagram, the graphs of $y = 2 \sin x + 1$ and $y = 3 \cos 2x$ for the interval $0° \leqslant x \leqslant 360°$. Hence state for this interval the number of solutions of the equation
   (a) $2 \sin x + 1 = 3 \cos 2x$,
   (b) $|2 \sin x + 1| = 3 \cos 2x$,
   (c) $|2 \sin x + 1| = 3|\cos 2x|$.

4. **(a)** Without using a calculator, evaluate sin 225° and tan (–240°).
   (b) Given that A is acute and $\cos A = c$, express $\sin A$ and $\cot A$ in terms of c.

5. Sophia is interested in the following CCAs: Choir, Band, Drama Club, Badminton and Tennis. However she feels that she has time only for 3 of them. In how many ways can she decide on her 3 CCAs?
   Having decided on her CCAs, she wants to run for Secretary or Treasurer in each of these 3 CCAs. How many choices are open to her?

6. Find the number of ways 4 boys and 5 girls can be arranged in a row if
   (a) there are no restrictions,
   (b) a particular boy and a particular girl must be next to each other,
   (c) two particular boys must not be next to each other,
   (d) the first and last persons are of the same sex.

7. Write down the first three terms, in ascending powers of x of the expansion $(1 - 2x)^8$. Hence find the coefficient of x^2 in the expansion $(3 + x)^2(1 - 2x)^8$.

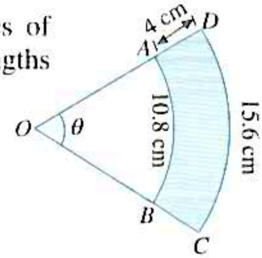
Revision Exercise 10

1. **(a)** Prove the identity $(1 - \sin x)(1 + \operatorname{cosec} x) \equiv \cos x \cot x$.
   (b) Sketch the graph of $y = 2 + 3 \cos x$ for the interval $0° \leqslant x \leqslant 360°$ and state the corresponding range of y.

2. With the help of an equilateral triangle, prove that $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$. Hence find the exact value of $\sin \frac{5\pi}{3}$.

3. Find all the angles between 0° and 360° inclusive which satisfy the equation
   (a) $\frac{3 \sin x}{2 - \sin x} = 2 + \sin x$, **(b)** $\cot(2y - 60°) = 1 - \tan 65°$,
   (c) $9 \sin^2 z - \tan^2 z = 0$, **(d)** $2(\cos x + 5 \sin x) = 3 \sin x$.

4. The diagram shows two sectors in which AB, CD are arcs of concentric circles, centre O. The arcs AB and CD have lengths 10.8 cm and 15.6 cm respectively. If $AD = 4$ cm, calculate



   (a) $A\hat{O}B$ in radians,
   (b) the area of the shaded region $ABCD$.

5. There are 6 men and 4 women.
   (a) Find the number of possible committees which can be formed from 5 of these people.
   (b) Find the number of possible queues which can be formed by 5 of them.
   (c) Find the number of possible groups of 4 if the groups must consist of at most 3 men.

6. (a) Find the term in x^7 in the expansion of $\left(x^2 + \frac{2}{x}\right)^{11}$.

   (b) Given that the first three terms, in ascending powers of x, in the binomial expansion of $(3 + ax)^5$ are $243 - 810x + bx^2$, find the value of a and of b.

7. Write down and simplify the expansion of $(2 - p)^4$. Use this result to find the expansion of $\left(2 + 4x - \frac{x^2}{2}\right)^4$ in ascending powers of x as far as the term in x^2.

Revision Exercise 11

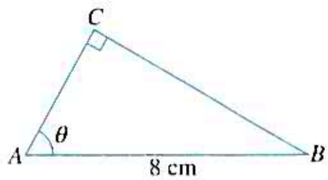
1. Find all the angles between 0° and 360° which satisfy the equation
   (a) $\sqrt{2}\sin 2x + 1 = 0$, (b) $\sec^2 y - \tan y = 3$,
   (c) $|3\sin z - \cos z| = 2\sin z$.

2. Sketch, on the same diagram, the graphs of $y = 3\cos x + 1$ and $y = \frac{1}{2}(x - \pi)^2 - 1$ for the interval $0 \leqslant x \leqslant 2\pi$. Hence state the number of roots in this interval of the equation $6\cos x + 4 = (x - \pi)^2$.

3. (a) Prove the identity $\dfrac{3\sin\theta - 2\cos^2\theta}{\sin^2\theta + 2\sin\theta} \equiv 2 - \operatorname{cosec}\theta$.

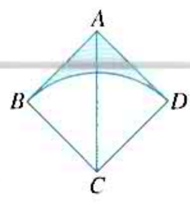
   (b) If $a\cos^2\theta + b\sin^2\theta = c$, show that $\tan^2\theta = \dfrac{a - c}{c - b}$.

4. The perimeter and area of the triangle ABC are P cm and Q cm² respectively.



   (a) Show that $Q = 32\sin\theta\cos\theta$.
   (b) Express P in terms of θ.
   (c) Given that $(P - 8)^2 = 4Q + 80\cos\theta$, find the value of θ, where $0 < \theta < \frac{\pi}{2}$.

5. Two sticks, AC and BD are each 60 cm long and both are used to form the framework of a kite. BD is bent in the shape of an arc of a circle with centre C. If $BC = 40$ cm, find



   (a) $B\hat{C}D$ in radians, (b) the area of the shaded region.

6. (a) Find, in ascending powers of x, the first three terms in the expansion of $(2 - 3x)^7$. Use the expansion to find the value of $(1.997)^7$, correct to the nearest whole number.

   (b) Find the term independent of x in the expansion of $\left(2x^3 - \frac{1}{x}\right)^{12}$.

7. How many 4-figure numbers can be formed from the digits 1, 2, 5, 6, 7 and 9 if each digit is used only once? How many of these numbers are
   (a) even?   (b) less than 6000?

Revision Exercise 12

1. Four letters are to be chosen from the letters in the word PROJECTS. Find the number of choices. How many of these choices contain
   (a) the letter J,   (b) no vowels,   (c) at least one vowel?
   How many 5-letter code words can be formed from the given 8 letters?

2. A cultural event is to showcase 2 song items, 2 plays and 4 dance items. Find the number of possible arrangements.
   Find the number of arrangements
   (a) which begin with a song item,
   (b) in which the 2 plays are not consecutive,
   (c) in which the 4 dance items are separated.

3. (a) Find all the angles between 0° and 360° which satisfy the equation
   (i) $\cot(2x - 40°) = -0.7$,   (ii) $8 \cos y = 1 + \tan^2 y$.
   (b) Find the values of x, where $0 < x < \pi$, for which $\sin x(6 \cos x - 1) = \tan x$.

4. (a) Prove the identity $\cos^2 x + \cot^2 x \cos^2 x \equiv \cot^2 x$.
   Hence find the **exact** value of $\sec x$ if $\cos^2 x + \cot^2 x \cos^2 x = 4$ and $90° < x < 180°$.
   (b) Given that $\tan \theta = t$ and that θ is obtuse, express in terms of t,
   (i) $\sec \theta$,   (ii) $\sin \theta$.

5. (a) Sketch the graph of $y = 4 \cos 3x$ for the domain $0 < x < \pi$ and state the corresponding range.
   (b) Sketch, on the same diagram, the graph of $y = \frac{3x}{2\pi}$ and of $y = 4 \sin x + 1$ for the interval $0 \leqslant x \leqslant 2\pi$. Hence state the number of solutions in this interval of the equation
   (i) $8\pi \sin x + 2\pi = 3x$,   (ii) $2\pi|4 \sin x + 1| = 3x$.

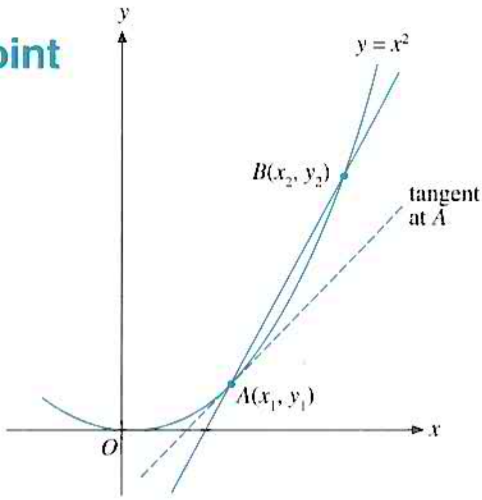
6. Find a if the coefficient of x in the expansion of
$$(1 + 3x)^4\left(1 - \frac{1}{8}x\right)^8 - (1 + ax)^4(1 + x)^3$$
is zero. Find also the coefficient of x^2.

7. A circle of radius r is drawn with its centre on the circumference of another circle of radius r.
   (a) Find, in terms of r, the perimeter of the common region.
   (b) Show that the area of this common region is $\frac{1}{6}r^2(4\pi - 3\sqrt{3})$.

15 Differentiation and Its Technique

15.1 The Gradient Function

Gradient of a Curve at a Point



In the diagram, if $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points on the curve $y = x^2$, the straight line AB has gradient $\frac{y_2 - y_1}{x_2 - x_1}$.

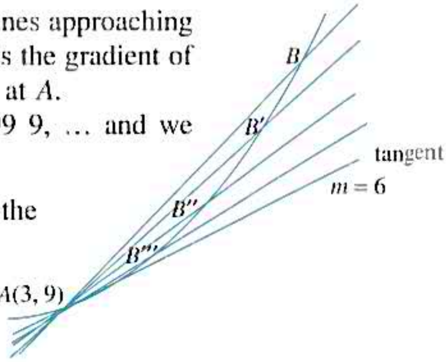
As B approaches A (i.e. x_2 approaches x_1, written as $x_2 \to x_1$) the line AB becomes closer to a line l which is called the **tangent** at A. Therefore the gradient of the line AB, $\frac{y_2 - y_1}{x_2 - x_1}$, approaches a value m which is known as the gradient of the curve at A and is also the gradient of the tangent at A.

Taking $x_1 = 3$, $x_2 = 3.1$, 3.01, 3.001, $3.000\ 1$, as shown in the following table:

x_2	$x_2 - x_1$	y_2	$y_2 - y_1$	$\frac{y_2 - y_1}{x_2 - x_1}$
3.1	0.1	9.61	0.61	6.1
3.01	0.01	9.060 1	0.060 1	6.01
3.001	0.001	9.006 001	0.006 001	6.001
3.000 1	0.000 1	9.000 600 01	0.000 600 01	6.000 1

The gradient, $\frac{y_2 - y_1}{x_2 - x_1}$, approaches the limiting value 6 (= m, the gradient of the tangent at $A(3, 9)$).

The diagram on the right shows graphically a set of lines approaching the tangent at the point $A(3, 9)$. The limiting value 6 is the gradient of the curve $y = x^2$ at A and the gradient of the tangent at A. Similarly, we could take $x_2 = 2.9, 2.99, 2.999, 2.999\,9, \ldots$ and we would have the same limiting value.



We can see that the gradient of a curve at (x_1, y_1) is the limiting value of $\frac{y_2 - y_1}{x_2 - x_1}$ and we write

$$\frac{y_2 - y_1}{x_2 - x_1} \to m \text{ as } x_2 \to x_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

or

$$\lim_{x_2 \to x_1} \left(\frac{y_2 - y_1}{x_2 - x_1} \right) = m \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Alternatively, we let the changes in the x- and y-coordinates from A to B be δx and δy respectively.

Then:

$$\delta x = x_2 - x_1$$
$$\delta y = y_2 - y_1$$
$$\frac{\delta y}{\delta x} = \left(\frac{y_2 - y_1}{x_2 - x_1} \right)$$

As $x_2 \to x_1$, $\delta x = x_2 - x_1 \to 0$.

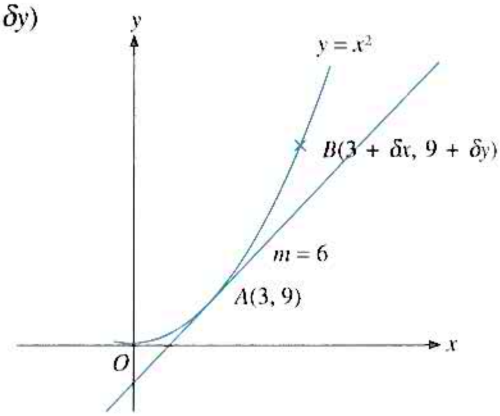
Then (1) and (2) respectively become

$$\frac{\delta y}{\delta x} \to m \text{ as } \delta x \to 0$$

and

$$\lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x} \right) = m.$$

Consider the points $A(3, 9)$ and $B(3 + \delta x, 9 + \delta y)$ on the curve $y = x^2$ shown below.



For the point B, we have

$$\begin{aligned} 9 + \delta y &= (3 + \delta x)^2 \\ &= 9 + 6\delta x + (\delta x)^2 \end{aligned}$$

which gives

$$\begin{aligned} \delta y &= 6\delta x + (\delta x)^2 \\ &= \delta x(6 + \delta x) \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x} \right) &= \lim_{\delta x \to 0} (6 + \delta x) \\ &= 6 \end{aligned}$$

Gradient Function

We are now in a position to obtain a general result for the gradient at any point on the curve of $y = x^2$. For any point $P(x, y)$ and a nearby point $Q(x + \delta x, y + \delta y)$ on the curve,

$$y = x^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

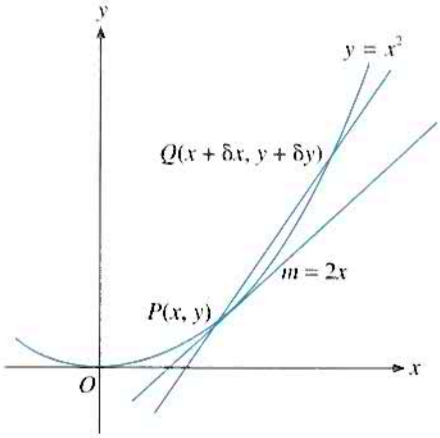
$$\begin{aligned} y + \delta y &= (x + \delta x)^2 \\ &= x^2 + 2x\delta x + (\delta x)^2 \quad \ldots\ldots\ldots\ldots\ldots (2) \end{aligned}$$

Subtracting (1) from (2):

$$\delta y = 2x\delta x + (\delta x)^2$$

$$\frac{\delta y}{\delta x} = 2x + \delta x$$

$$\lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x}\right) = 2x$$



Unlike a straight line, the gradient of the curve $y = x^2$ varies with the x-coordinate of P. So the gradient at $A(1, 1)$ is 2, the gradient at $B(2, 4)$ is 4 and so on. The function $m = 2x$ is called the **gradient function** of $y = x^2$.

By using the above process, we can find the gradient function of any given function as follows:

(a) Consider two points $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ on the curve.

(b) Obtain the ratio $\frac{\delta y}{\delta x}$.

(c) Take the limit of $\frac{\delta y}{\delta x}$ as $\delta x \to 0$.

This process is known as **differentiation from first principles**.

Example 1 Find, from first principles, the gradient function of the curve $y = \frac{1}{x}$ and write down the gradient of the tangent at $\left(2, \frac{1}{2}\right)$.

Solution: For any point $P(x, y)$ on the curve, we have

$$y = \frac{1}{x} \quad \text{.............................} \quad (1)$$

and for a nearby point $Q(x + \delta x, y + \delta y)$,

$$y + \delta y = \frac{1}{x + \delta x} \quad \text{.......................} \quad (2)$$

(2) − (1):

$$\delta y = \frac{1}{x + \delta x} - \frac{1}{x}$$

$$= \frac{x - (x + \delta x)}{(x + \delta x)x}$$

$$= -\frac{\delta x}{(x + \delta x)x}$$

$$\frac{\delta y}{\delta x} = -\frac{1}{(x + \delta x)x}$$

The gradient function is $m = \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x}\right)$

$$= -\frac{1}{x^2}$$

The gradient of the tangent at $A(2, \frac{1}{2})$ is $m = -\frac{1}{4}$

The gradient function of a curve, $\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)$ is denoted by $\frac{dy}{dx}$.

For the curve $y = x^2$, $\frac{dy}{dx} = 2x$.

For the curve $y = \frac{1}{x}$, $\frac{dy}{dx} = -\frac{1}{x^2}$.

These results can be obtained by applying the formula:

> **If $y = x^n$, where n is a constant, then:**
>
> $\frac{dy}{dx} = nx^{n-1}$ **or** $\frac{d}{dx}(x^n) = nx^{n-1}$

So, for $y = x^2$, $\frac{dy}{dx} = 2x^{2-1} = 2x$ or $\frac{d}{dx}(x^2) = 2x$

and for $y = \frac{1}{x} = x^{-1}$, $\frac{dy}{dx} = (-1)x^{-1-1} = -\frac{1}{x^2}$ or $\frac{d}{dx}(x^{-1}) = -x^{-2}$.

Also if $y = ax^n$, where a and n are constants, then:

$$\frac{dy}{dx} = anx^{n-1} \text{ or } \frac{d}{dx}(ax^n) = anx^{n-1}$$

In particular, if $n = 1$, then $y = ax$ (a straight line)

$$\frac{dy}{dx} = a \text{ or } \frac{d}{dx}(ax) = a$$

If $n = 0$, then $y = a$ (a horizontal line)

$$\frac{dy}{dx} = 0 \text{ or } \frac{d}{dx}(a) = 0$$

More generally:

> $\frac{d}{dx}(ax^m + bx^n) = amx^{m-1} + bnx^{n-1}$, **where a, b, m and n are constants.**

The process of obtaining $\frac{dy}{dx}$ of a given function is called **differentiation**. The function $\frac{dy}{dx}$ is also known as the **derived function** or **derivative** or **differential coefficient of y with respect to x**.

Example 2 Find $\frac{dy}{dx}$ if $y = 3x^5 - 2x^3 + 4x$.

Solution:

$$y = 3x^5 - 2x^3 + 4x$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(3x^5 - 2x^3 + 4x)\\ &= \frac{d}{dx}(3x^5) - \frac{d}{dx}(2x^3) + \frac{d}{dx}(4x)\\ &= 3(5x^4) - 2(3x^2) + 4\\ &= \mathbf{15x^4 - 6x^2 + 4}\end{aligned}$$

Example 3 Differentiate $3x^4 - 4\sqrt{x} - 2$ with respect to x.

Solution:

$$\begin{aligned}\frac{d}{dx}(3x^4 - 4\sqrt{x} - 2) &= \frac{d}{dx}(3x^4 - 4x^{\frac{1}{2}} - 2)\\ &= 3(4x^3) - 4\left(\frac{1}{2}x^{-\frac{1}{2}}\right)\\ &= \mathbf{12x^3 - \frac{2}{\sqrt{x}}}\end{aligned}$$

If $y = f(x)$, then the derivative $\frac{dy}{dx}$ may also be denoted by $f'(x)$.

Example 4 Differentiate the following with respect to x.

(a) $f(x) = \frac{x^2 + 5x + 4}{x}$ **(b)** $f(x) = (x - 1)(\sqrt{x} + 3)$

Solution:

(a)

$$\begin{aligned}f(x) &= \frac{x^2 + 5x + 4}{x}\\ &= x + 5 + \frac{4}{x}\\ &= x + 5 + 4x^{-1}\\ f'(x) &= 1 + 4(-x^{-2})\\ &= \mathbf{1 - \frac{4}{x^2}}\end{aligned}$$

(b)

$$\begin{aligned}f(x) &= (x - 1)(\sqrt{x} + 3)\\ &= x\sqrt{x} + 3x - \sqrt{x} - 3\\ &= x^{\frac{3}{2}} + 3x - x^{\frac{1}{2}} - 3\\ f'(x) &= \frac{3}{2}x^{\frac{1}{2}} + 3 - \frac{1}{2}x^{-\frac{1}{2}}\\ &= \mathbf{\frac{3}{2}\sqrt{x} + 3 - \frac{1}{2\sqrt{x}}}\end{aligned}$$

Example 5 Find the gradient of the curve $y = 2x + \frac{1}{x^2}$ at the point $(1, 3)$.

Solution:

$$\begin{aligned}y &= 2x + \frac{1}{x^2}\\ &= 2x + x^{-2}\\ \frac{dy}{dx} &= 2 + (-2x^{-3})\\ &= 2 - \frac{2}{x^3}\end{aligned}$$

At (1, 3), $x = 1$, $\frac{dy}{dx} = 2 - \frac{2}{1^3} = 0$

Therefore, the gradient at (1, 3) is **0**.

Example 6 Find the coordinates of the points on the curve $y = 2x^3 - 4x^2 + x + 1$ at which the gradient is -1.

Solution:

$\frac{dy}{dx} = 6x^2 - 8x + 1$

$$\begin{aligned} \text{Gradient} = -1 \Rightarrow 6x^2 - 8x + 1 &= -1 \\ 3x^2 - 4x + 1 &= 0 \\ (3x - 1)(x - 1) &= 0 \end{aligned}$$

$\Rightarrow x = \frac{1}{3}, y = \frac{26}{27}$ or $x = 1, y = 0$

$\therefore$ the required points are $\left(\frac{1}{3}, \frac{26}{27}\right)$ and **(1, 0)**.

Example 7 Given that the curve $y = ax^2 + \frac{b}{x}$ has gradient 4 at the point (1, 5), calculate the value of a and of b.

Solution:

$$y = ax^2 + \frac{b}{x} \quad \text{.....................} \quad (1)$$

Since the point (1, 5) lies on the curve,

$$5 = a + b \quad \text{.........................} \quad (2)$$

Differentiating (1) w.r.t. x, $\frac{dy}{dx} = 2ax - \frac{b}{x^2}$.

Since gradient at (1, 5) = 4, i.e. $\frac{dy}{dx} = 4$ when $x = 1$,

$$2a - b = 4 \quad \text{.................................} \quad (3)$$

Solving (2) and (3):

$2a - (5 - a) = 4$

$a = \mathbf{3}$ and $b = \mathbf{2}$

Example 8 Differentiate $\frac{x^2 + 1}{x}$ with respect to x and find the coordinates of the points at which the gradient of the curve $y = \frac{x^2 + 1}{x}$ is zero.

Solution:

$$\begin{aligned} \text{Since } \frac{x^2 + 1}{x} &= x + \frac{1}{x} \\ &= x + x^{-1} \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}\left(\frac{x^2 + 1}{x}\right) &= \frac{d}{dx}(x + x^{-1}) \\ &= 1 + (-x^{-2}) \\ &= 1 - \frac{1}{x^2} \end{aligned}$$

So, $\qquad y = \dfrac{x^2 + 1}{x}$ ....(1)

$\Rightarrow \qquad \dfrac{dy}{dx} = 1 - \dfrac{1}{x^2}$

If $\dfrac{dy}{dx} = 0$, then $1 - \dfrac{1}{x^2} = 0$

$$x^2 = 1$$

$$x = \pm 1$$

Substituting $x = 1$ and $x = -1$ into (1), we obtain $y = 2$ and $y = -2$ respectively. Hence the required points are **(1, 2)** and **(–1, –2)**.

Exercise 15.1

1. Differentiate the following with respect to x (where a and b are constants).

(a) $3x^2 + 4x - 1$ **(b)** $x^4 - 7x^2 + 6x$

(c) $2x^3 + 5x^2 - 4x + 9$ **(d)** $4x + \dfrac{2}{x}$

(e) $9x^2 - \dfrac{3}{x^2}$ **(f)** $\dfrac{6}{x^3} - \dfrac{1}{x} + 3$

(g) $3a + bx^2$ **(h)** $5x^2 + \dfrac{4}{x} - 2$

(i) $3x + 2\sqrt{x} - 3$ **(j)** $8x^2 + 3x - \sqrt{x}$

(k) $2x^{\frac{5}{2}} - 4x^{\frac{3}{2}} - 6x + 8$ **(l)** $6x\sqrt{x} - 6\sqrt{x}$

(m) $4x^2\sqrt{x} - \dfrac{6}{\sqrt{x}}$ **(n)** $ax - \dfrac{b}{x}$

2. Differentiate the following with respect to x.

(a) $\dfrac{2x^2 + 4x}{x}$ **(b)** $\dfrac{x^2 - 6x + 4}{x}$ **(c)** $\dfrac{4x^3 - 5x - 3}{2x}$

(d) $\dfrac{x^2 + 4}{2x^2}$ **(e)** $\dfrac{3x^2 + x - 1}{\sqrt{x}}$ **(f)** $\dfrac{6x^2 - \sqrt{x} + 2}{2x}$

3. Find $\dfrac{dy}{dx}$ for the following functions of x.

(a) $(x + 1)(2x - 1)$ **(b)** $x(\sqrt{x} - 2)$ **(c)** $(1 + \sqrt{x})(1 - \sqrt{x})$

(d) $4x^2(3 - \sqrt{x})$ **(e)** $\dfrac{(2x + 1)(x - 2)}{x}$ **(f)** $\dfrac{(1 - x)(4x - 1)}{\sqrt{x}}$

4. Find the value of $f'(x)$ at the given value of x.

(a) $f(x) = 3x^2 - 2x - 4$, $x = 2$ **(b)** $f(x) = 6x - \dfrac{3}{x}$, $x = -1$

(c) $f(x) = 3x - 4\sqrt{x}$, $x = 4$ **(d)** $f(x) = (x - 4)(x + 3)$, $x = 3$

5. Calculate the gradient of the tangent to the curve at the given point.

(a) $y = 4x^2 - 6x + 1$, $(2, 5)$ (b) $y = \frac{6 - 4x}{x}$, $x = -2$

(c) $y = \sqrt{x}(2 - x)$, $x = 9$ (d) $y = \frac{(x + 1)(2x - 3)}{x}$, $x = -1$

6. Calculate the gradient(s) of the curve at the point(s) where y is given.

(a) $y = x^2 - 2x$, $y = -1$ (b) $y = 2x^2 + 3x$, $y = 2$

(c) $y = \frac{x - 9}{x}$, $y = 4$ (d) $y = \frac{x^2 + 4}{x^2}$, $y = 5$

7. Calculate the gradient(s) of the curve at the point(s) where it crosses the given line.

(a) $y = 2x^2 - 5x + 1$, y-axis (b) $y = \frac{x - 4}{x}$, x-axis

(c) $y = 2x^2 - 8$, x-axis (d) $y = \frac{x + 2}{x}$, $y = x$

8. Find the coordinates of the point on the curve $y = x^3 - 3x^2 + 6x + 2$ at which the gradient is 3.

9. The curve $y = ax^2 + \frac{b}{x}$ has gradients 2 and -1 at $x = 1$ and $x = 4$ respectively. Find the value of a and of b.

10. The gradient of the tangent to the curve $y = ax^3 + bx$ at the point $(2, -4)$ is 6. Calculate the values of the constants a and b.

11. Given that the gradient of the curve $y = \frac{a}{x} + bx^2$ at the point $P(3, -15)$ is -13. Find the value of a and of b. Show that the tangent to the curve at the point where $x = 1$ has the same gradient as that at P.

12. The tangent to the curve $y = \frac{a}{x} + bx$ at $(1, 3)$ is parallel to the line $y = 2x + 1$. Calculate the value of a and of b.

13. Given that $f(x) = px^2 + qx$ and that $f(2) = -2$ and $f'(2) = 3$, calculate the value of p and of q.

14. The equation of a curve is $y = 9x + \frac{1}{x}$. Find

(a) the gradient of the curve where $x = 2$,

(b) the coordinates of the points where the tangent is horizontal.

15. Given the curve $y = x^3 - 3x^2 - 9x + 11$, find $\frac{dy}{dx}$. Hence obtain

(a) the x-coordinates of the points where the gradient is 15,

(b) the coordinates of the points where the gradient is zero.

16. The curve $y = x^2 + 2x$ has gradient 3 at the point (a, b). Find the value of a and of b.

17. The curve $y = ax + \frac{3}{x}$ has gradient 1 at $x = 2$. Find the value of a and the x-coordinate of another point at which the gradient is 1.

15.2 Function of a Function (Composite Function)

The function $y = (x^2 + 1)^2$ can simply be expanded as $y = x^4 + 2x^2 + 1$ and so, $\frac{dy}{dx} = 4x^3 + 4x$.

For the function $y = (x^2 + 1)^5$, expansion of the expression $(x^2 + 1)^5$ is tedious. If we introduce a new variable u such that $u = x^2 + 1$, then we have

$$y = u^5$$

and

$$u = x^2 + 1.$$

Note that y is a function of u and u is a function of x.

Let δu be a small change in u corresponding to δx, a small change in x, and δy be a change in y corresponding to δu (or δx).

Then we have

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x} \right)$$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta u} \times \lim_{\delta x \to 0} \frac{\delta u}{\delta x}$$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta u \to 0} \frac{\delta y}{\delta u} \times \lim_{\delta x \to 0} \frac{\delta u}{\delta x} \quad (\delta x \to 0,\ \delta u \to 0)$$

which gives the **chain rule**: $\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}$

Hence for $y = u^5$, $\frac{dy}{du} = 5u^4$ and for $u = x^2 + 1$, $\frac{du}{dx} = 2x$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 5u^4 \times 2x \\ &= 10x(x^2 + 1)^4 \end{aligned}$$

This method enables us to differentiate more complicated functions as shown in the following examples.

Example 9 Differentiate the following w.r.t. x.

(a) $y = (3x^2 + 2x)^7$ **(b)** $y = \frac{1}{(3x - 2)^3}$ **(c)** $y = \sqrt{3x^2 + 5}$

Solution: **(a)** Let $u = 3x^2 + 2x$ and so $y = u^7$.

Then $\frac{du}{dx} = 6x + 2$ and $\frac{dy}{du} = 7u^6$.

By the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 7u^6(6x + 2) \\ &= \mathbf{14(3x + 1)(3x^2 + 2x)^6}\end{aligned}$$

(b) Let $u = 3x - 2$ and so $y = \frac{1}{u^3} = u^{-3}$.

Then $\frac{du}{dx} = 3$ and $\frac{dy}{du} = (-3)u^{-3-1} = -\frac{3}{u^4}$.

By the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= -\frac{3}{u^4}(3) \\ &= -\frac{\mathbf{9}}{\mathbf{(3x - 2)^4}}\end{aligned}$$

(c) Let $u = 3x^2 + 5$ and so $y = \sqrt{u}$.

Then $\frac{du}{dx} = 6x$ and $\frac{dy}{du} = \frac{d}{du}(u^{\frac{1}{2}}) = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$.

By the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2\sqrt{u}}(6x) \\ &= \frac{\mathbf{3x}}{\mathbf{\sqrt{3x^2 + 5}}}\end{aligned}$$

In general, for a function of the form $y = u^n$, where u is a function of x, we have:

$$\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}$$

or

$$\boxed{\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}}$$

We may use this formula to obtain directly the derivatives of the functions in Example 9 as follows:

(a) $y = (3x^2 + 2x)^7$

With $u = 3x^2 + 2x$, $n = 7$

$$\begin{aligned}\frac{dy}{dx} &= 7(3x^2 + 2x)^{7-1}\frac{d}{dx}(3x^2 + 2x) \\ &= 7(3x^2 + 2x)^6(6x + 2) \\ &= \mathbf{14(3x + 1)(3x^2 + 2x)^6}\end{aligned}$$

(b) $y = \dfrac{1}{(3x-2)^3} = (3x-2)^{-3}$

With $u = 3x - 2$, $n = -3$

$$\begin{aligned}\frac{dy}{dx} &= (-3)(3x-2)^{-3-1}\frac{d}{dx}(3x-2)\\ &= -3(3x-2)^{-4}(3)\\ &= -\frac{9}{(3x-2)^4}\end{aligned}$$

(c) $y = \sqrt{3x^2 + 5}$

With $u = 3x^2 + 5$, $n = \frac{1}{2}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}(3x^2+5)^{\frac{1}{2}-1}\frac{d}{dx}(3x^2+5)\\ &= \frac{1}{2\sqrt{3x^2+5}}(6x)\\ &= \frac{6x}{2\sqrt{3x^2+5}}\\ &= \frac{3x}{\sqrt{3x^2+5}}\end{aligned}$$

Since we will encounter the differentiation of expressions such as $\sqrt{3x^2+5}$, $\sqrt{x+2}$ and $\sqrt{3x-x^2}$ which are of the form $\sqrt{f(x)}$, the following formula will be quoted as a standard formula in subsequent discussions.

$$\frac{d}{dx}(\sqrt{u}) = \frac{1}{2\sqrt{u}}\frac{du}{dx}$$

Exercise 15.2

1. Differentiate the following w.r.t. x.

   (a) $(x+2)^5$ **(b)** $(2x-1)^4$ **(c)** $(\frac{1}{4}x+2)^5$

   (d) $(1-4x)^{10}$ **(e)** $(2-3x^2)^4$ **(f)** $(1-x+x^2)^3$

2. Differentiate the following w.r.t. x.

   (a) $\dfrac{3}{(3-4x)^3}$ **(b)** $\dfrac{4}{(2x+7)}$ **(c)** $\dfrac{6}{(2-x)^2}$ **(d)** $\dfrac{2}{(6x^2+5)}$

3. Differentiate the following w.r.t. x.

   (a) $\sqrt{2x-3}$ **(b)** $\sqrt{6-2x}$ **(c)** $\sqrt{x^2-2}$

   (d) $\sqrt{5-3x^2}$ **(e)** $\sqrt{x^2-x+1}$ **(f)** $\sqrt{x^2+2x+2}$

4. Differentiate the following w.r.t. x

(a) $(2-\sqrt{x})^6$ (b) $\dfrac{1}{\left(1-\frac{1}{x}\right)^3}$ (c) $\dfrac{1}{2(3x-2)^2}$

(d) $2\left(\sqrt{x}+2\right)^{\frac{1}{2}}$ (e) $\left(x-\dfrac{1}{x}\right)^3$ (f) $\left(\sqrt{x}+2x\right)^4$

5. Find $\dfrac{dy}{dx}$ and the gradient of the curve at the given value of x.

(a) $y=(3x-1)^4$, $x=1$ (b) $y=\sqrt{5-2x}$, $x=\dfrac{1}{2}$

(c) $y=\dfrac{1}{2x-3}$, $y=1$ (d) $y=(4x-5)^3$, $y=27$

6. Calculate the coordinates of the point on the curve $y=(1-x)^4$ at which the gradient is -4.

7. Calculate the coordinates of the point on the curve $y=\sqrt{x^2-2x+5}$ at which $\dfrac{dy}{dx}=0$.

8. The curve $y=(a-x)^3$ has gradient $-\dfrac{1}{3}$ at $x=2$. Find the possible values of a.

9. Find $\dfrac{dy}{dx}$ and calculate the gradient of the tangent to the curve

(a) $y=(x^2-2x-4)^3$ at the point where $x=-1$,

(b) $y=\dfrac{1}{\sqrt{1+x}}$ at the point where $x=3$.

*10. Given that $f(x)=\sqrt{1+\sqrt{x}}$, where $x \geqslant 0$, show that $f'(x)=\dfrac{1}{4\sqrt{x+x\sqrt{x}}}$.

15.3 Product of Two Functions

Now, we shall proceed to consider the differentiation of functions which are products of two functions such as the following:

$$y=(x^2+1)(x+3)^4$$
$$y=(x^2+1)\sqrt{x+1}$$

For the function $y=(x^2+1)(x+3)^4$,

we let $u=(x^2+1)$

$v=(x+3)^4$

Then $y=uv$ .... (1)

Let δu, δv be changes in u and v corresponding to δx, a small change in x. Let δy be the corresponding change in y. Then

$$y+\delta y=(u+\delta u)(v+\delta v)$$
$$=uv+u\delta v+v\delta u+\delta u\delta v \quad \text{................ (2)}$$

Subtracting (1) from (2):

$$\delta y = u\delta v + v\delta u + \delta u\delta v$$

$$\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x} + \frac{\delta u\delta v}{\delta x}$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = u\lim_{\delta x\to 0}\frac{\delta v}{\delta x} + v\lim_{\delta x\to 0}\frac{\delta u}{\delta x} + \lim_{\delta x\to 0}\frac{\delta u}{\delta x}\delta v$$

Since $\lim_{\delta x\to 0}\frac{\delta u}{\delta x}\delta v = \lim_{\delta x\to 0}\frac{\delta u}{\delta x} \times \lim_{\delta x\to 0}\delta v = \frac{du}{dx} \times \lim_{\delta x\to 0}\delta v$

$= 0$ as $\delta v \to 0$ when $\delta x \to 0$

we have $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\boxed{\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}}$$

This is known as the **product rule** for differentiation.

Example 10 Differentiate $(x^2 + 1)(x + 3)^4$ w.r.t. x.

Solution: Let $y = (x^2 + 1)(x + 3)^4$

$u = x^2 + 1$

$v = (x + 3)^4$

Then $y = uv$

We have $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 4(x + 3)^3$.

So $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$= (x^2 + 1) \times 4(x + 3)^3 + (x + 3)^4 \times 2x$

$= 2(x + 3)^3[2(x^2 + 1) + x(x + 3)]$

$= \mathbf{2(x + 3)^3(3x^2 + 3x + 2)}$

Example 11 Calculate the gradient of the curve $y = x\sqrt{x+3}$ at the point where $x = 1$. Find the coordinates of the point at which the gradient is zero.

Solution:

$$y = x\sqrt{x+3}$$

Using $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = x\frac{d}{dx}(\sqrt{x+3}) + \sqrt{x+3}\,\frac{d}{dx}(x)$$

$$= x\left(\frac{1}{2\sqrt{x+3}}\right) + \sqrt{x+3}$$

$$= \frac{x + 2(x+3)}{2\sqrt{x+3}}$$

$$= \frac{3(x+2)}{2\sqrt{x+3}}$$

At the point where $x = 1$, $\frac{dy}{dx} = \frac{9}{4}$. So the gradient at this point is $2\frac{1}{4}$.

When gradient $\frac{dy}{dx} = 0$

$$\Rightarrow \quad x + 2 = 0$$

$$x = -2 \text{ and } y = -2$$

The required point is $(-2, -2)$.

Exercise 15.3

1. Differentiate the following w.r.t. x.
   (a) $(x - 1)(x + 2)^2$ (b) $x(2x - 1)^3$
   (c) $(1 - 2x)(3x + 2)^4$ (d) $(x^2 + 1)(1 + x)^2$
   (e) $x^2(1 - 4x)^3$ (f) $x(1 - x^2)^2$
   (g) $(x^2 - 2x + 2)(2x + 1)^2$ (h) $(1 - x^2)(1 + 4x)^3$
   (i) $(1 - x^2)^2(1 + 2x^2)$ (j) $(x + 1)^4(1 - 2x)^3$
   (k) $(x^2 - 3x + 4)\left(1 - \frac{1}{2}x\right)^2$ (l) $(x^2 + 6)(1 - 2x - 4x^2)$

2. Differentiate the following w.r.t. x.
   (a) $\sqrt{x}(1 - x)^2$ (b) $x\sqrt{1 + 2x}$ (c) $(2x + 3)\sqrt{1 - 4x}$
   (d) $(x^2 + 1)\sqrt{x + 1}$ (e) $x^2\sqrt{1 - 2x^2}$ (f) $(4x - 1)\sqrt{3x^2 + 1}$

3. By simplifying the following as products of two factors, differentiate
   (a) $x(x + 1)(x + 2)^3$, (b) $x^2(x - 1)\sqrt{5 + 6x}$.

4. Show that if $y = x\sqrt{3 + x^2}$, then $\frac{dy}{dx} = \frac{3 + 2x^2}{\sqrt{3 + x^2}}$.

5. A curve is defined by $y = x\sqrt{x - 1}$, find $\frac{dy}{dx}$ and the gradient at the point $(5, 10)$.

6. Calculate the gradients of the curve $y = (x + 1)^3(x - 1)$ at the points where it crosses the x-axis.

7. Given that $y = (x - 3)\sqrt{x - 1}$, find $\frac{dy}{dx}$ and the value of x for which the gradient is zero.

8. A curve is given by $y = \sqrt{x}(x - 4)^4$. Find $\frac{dy}{dx}$ and the x-coordinates of the points where $\frac{dy}{dx} = 0$.

9. A curve defined by $y = (x - a)\sqrt{x - b}$, for $x \geqslant b$, where a and b are constants, cuts the x-axis at A where $x = b + 1$. Show that the gradient of the curve at A is 1.

10. Find the gradients of the curve $y = x\sqrt{4 - x^2}$ at the points where it crosses the straight line $y = x$.

15.4 Quotient of Two Functions

We shall introduce a rule for differentiating functions which are quotients of two functions such as the following:

$$y = \frac{2x+1}{x^2+1}$$

$$y = \frac{x}{\sqrt{x-3}}$$

If $y = \frac{u}{v}$ where u and v are functions of x, $y = u\left(\frac{1}{v}\right)$.

By the product rule, $\frac{dy}{dx} = u\frac{d}{dx}\left(\frac{1}{v}\right) + \frac{1}{v}\frac{du}{dx}$.

Since $\frac{d}{dx}\left(\frac{1}{v}\right) = \frac{d}{dx}(v^{-1}) = -v^{-2}\frac{dv}{dx} = -\frac{1}{v^2}\frac{dv}{dx}$,

$$\frac{dy}{dx} = u\left(-\frac{1}{v^2}\frac{dv}{dx}\right) + \frac{1}{v}\frac{du}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Hence, we state the **quotient rule**: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Example 12 Given that $y = \frac{2x+1}{x^2+1}$, find $\frac{dy}{dx}$.

Solution: Let $u = 2x + 1$ and $v = x^2 + 1$.

Then $\frac{du}{dx} = 2$ and $\frac{dv}{dx} = 2x$.

$$\begin{aligned}\frac{du}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\\ &= \frac{(x^2+1)(2) - (2x+1)(2x)}{(x^2+1)^2}\\ &= \frac{2(1-x-x^2)}{(x^2+1)^2}\end{aligned}$$

Example 13 (a) Differentiate $\sqrt{4x+1}$ w.r.t. x.

(b) Given that $y = \frac{x}{\sqrt{4x+1}}$, show that $\frac{dy}{dx} = \frac{2x+1}{(4x+1)\sqrt{4x+1}}$.

Solution: (a) $\frac{d}{dx}(\sqrt{4x+1}) = \frac{1}{2\sqrt{4x+1}}\frac{d}{dx}(4x+1)$

$$= \frac{2}{\sqrt{4x+1}}$$

(b) $$\frac{dy}{dx} = \frac{\sqrt{4x+1}\frac{d}{dx}(x) - x\frac{d}{dx}\left(\sqrt{4x+1}\right)}{\left(\sqrt{4x+1}\right)^2}$$

$$= \frac{\sqrt{4x+1} - \frac{2x}{\sqrt{4x+1}}}{(4x+1)}$$

$$= \frac{4x+1-2x}{(4x+1)\sqrt{4x+1}}$$

$$= \frac{2x+1}{(4x+1)\sqrt{4x+1}}$$

Exercise 15.4

1. Differentiate the following w.r.t. x.

(a) $\frac{x}{x+1}$ (b) $\frac{5x}{2x+1}$ (c) $\frac{1-4x}{1+x}$ (d) $\frac{1-x}{1-2x}$

(e) $\frac{x^2}{x+3}$ (f) $\frac{3x^2}{1-4x}$ (g) $\frac{x^2+1}{2x-1}$ (h) $\frac{2x^3}{1-x}$

(i) $\frac{1-x^2}{1+x^2}$ (j) $\frac{2x^2+x+1}{1-2x}$ (k) $\frac{x^2}{(1+x)^2}$ (l) $\frac{1-2x^2}{1-x^3}$

2. Differentiate the following w.r.t. x, simplifying your answers.

(a) $\frac{\sqrt{x}}{1+x}$ (b) $\frac{x}{\sqrt{1-x}}$ (c) $\frac{2x}{\sqrt{2x+1}}$

(d) $\frac{x+1}{\sqrt{1-4x}}$ (e) $\frac{5x}{\sqrt{1-x^2}}$ (f) $\frac{3x^2}{\sqrt{2x^2-3}}$

3. Find $f'(x)$ if

(a) $f(x) = \frac{2x+3}{1-4x}$, (b) $f(x) = \frac{6x}{\sqrt{1+x^2}}$.

4. If $y = \frac{3x^2}{1-4x^2}$, find $\frac{dy}{dx}$ at the point where $x = 1$.

5. Calculate the gradient of the curve $y = \frac{x-2}{1-x}$ at the point where it crosses the x-axis.

6. Calculate the gradient of the tangent to the curve $y = \frac{x+2}{\sqrt{3x+1}}$ at the point where $x = 1$.

7. Calculate the x-coordinates of the points on the curve $y = \sqrt{\frac{1-x}{x^2+3}}$ for which $\frac{dy}{dx} = 0$.

***8.** A curve is defined by $y = \sqrt{\frac{x-a}{b-x}}$ where $a < x < b$ and a and b are constants. Show that the gradient of the curve at $x = \frac{a+b}{2}$ is $\frac{2}{b-a}$.

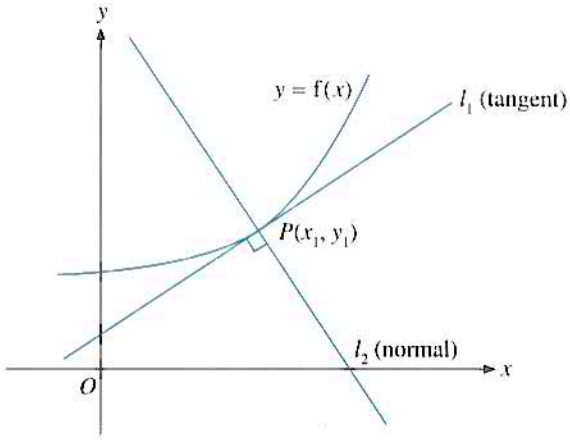
15.5 Equations of Tangent and Normal

In the diagram, the line l_1 is the tangent to the curve $y = f(x)$ at the point $P(x_1, y_1)$. We have learnt that its gradient, m, is given by:

$$m = \text{value of } \frac{dy}{dx} \text{ at } (x_1, y_1)$$

Hence, the equation of the tangent is:

$$y - y_1 = m(x - x_1)$$



The line l_2 is **perpendicular** to the tangent l_1 and is called the **normal** to the curve at $P(x_1, y_1)$. Hence its gradient is $-\frac{1}{m}$ and its equation is:

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

Example 14 Find the equations of the tangent and the normal to the curve $y = 3x^2 - 4x + 5$ at the point where $x = 1$.

Solution:

$$y = 3x^2 - 4x + 5$$

$$\frac{dy}{dx} = 6x - 4$$

When $x = 1,\ y = 3 - 4 + 5$
$= 4$

$$\frac{dy}{dx} = 6 - 4 = 2$$

So, gradient of the tangent at (1, 4) is 2 and its equation is

$$y - 4 = 2(x - 1)$$
$$\mathbf{y = 2x + 2}$$

Gradient of the normal is $-\frac{1}{2}$ and its equation is

$$y - 4 = -\frac{1}{2}(x - 1)$$
$$\mathbf{2y = -x + 9}$$

Example 15 The curve $y = x^2 - 3x + 4$ passes through the points $P(1, 2)$ and $Q(3, 4)$. Find

(a) the equation of the tangent at P,

(b) the equation of the normal at Q,

(c) the coordinates of R, the point of intersection where the tangent and the normal intersect.

Solution:

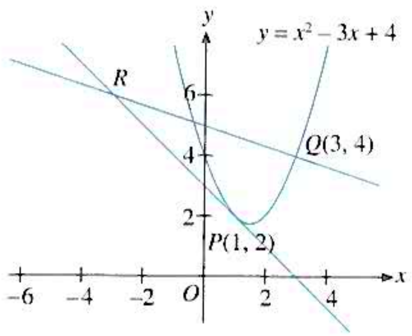
(a)
$$y = x^2 - 3x + 4$$
$$\Rightarrow \quad \frac{dy}{dx} = 2x - 3$$

At $P(1, 2)$, $\frac{dy}{dx} = 2 - 3$

$= -1$

Equation of the tangent at P is

$$y - 2 = -1(x - 1)$$
$$y = -x + 3 \quad \text{.........................(1)}$$



(b) At $Q(3, 4)$, $\frac{dy}{dx} = 2 \times 3 - 3$

$= 3$

Gradient of the normal at Q is $-\frac{1}{3}$ and its equation is

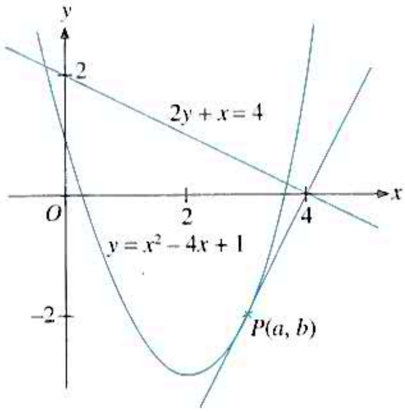
$$y - 4 = -\frac{1}{3}(x - 3)$$
$$y = -\frac{1}{3}x + 5 \quad \text{...................... (2)}$$

(c) For the point R, we solve (1) and (2):

$$-x + 3 = -\frac{1}{3}x + 5$$
$$3(3 - x) = 15 - x$$
$$x = -3 \text{ and } y = 6$$

The coordinates of the point R are $(-3, 6)$.

Example 16 The diagram shows part of the curve $y = x^2 - 4x + 1$ and the tangent line at $P(a, b)$. The tangent line is perpendicular to the line $2y + x = 4$.
Find
(a) the value of a and of b,
(b) the equation of the tangent at P.



Solution:

(a) $$y = x^2 - 4x + 1$$
$$\Rightarrow \quad \frac{dy}{dx} = 2x - 4$$
Gradient of the tangent line at $P(a, b) = 2a - 4$

Gradient of the line $2y + x = 4$ is $-\frac{1}{2}$

Since the lines are perpendicular,
$$(2a - 4)\left(-\frac{1}{2}\right) = -1$$
$\therefore \quad a = \mathbf{3}$ and $b = \mathbf{-2}$

(b) Gradient of the tangent line at $P(3, -2)$ is $6 - 4 = 2$ and its equation is
$$y - (-2) = 2(x - 3)$$
$$\mathbf{y = 2x - 8}$$

Exercise 15.5

1. Find the equations of the tangent and the normal to the curve
(a) $y = 2x^2 - 3x + 1$ at the point $(2, 3)$,
(b) $y = x^3 + 3x^2$ at the point where $x = -1$,
(c) $y = x + \frac{2}{x}$ at the point where $x = 1$.

2. Find the equation of the normal to the curve
(a) $y = 2(x - 1)^3$ at the point where $x = \frac{1}{2}$,
(b) $y = \frac{x^2 + 1}{x - 1}$ at the point where $x = 2$.

3. Find the equations of the tangents to the curve $y = 2x^2 - 3x$ at the point where $y = -1$. Find the coordinates of the point of intersection of the tangents.

4. Find the equations of the normals to the curve $y = 2x^2 - 7$ at the points where $x = 1$ and $x = -1$. Calculate the coordinates of the point of intersection of these normals.

5. Find the equation of the normal to the curve $y = x\sqrt{1 - 2x}$ at the point $(-4, -12)$.

6. Find the equations of the tangents to the curve $y = \frac{x^2 + 5}{x + 1}$ at the points where $x = 1$ and $x = 3$. Find the coordinates of the point where these tangents intersect.

7. Find the equations of the normals to the curve $y = 2x + \frac{8}{x}$ at $x = 1$ and $x = 4$. Find the coordinates of the point where these normals intersect.

8. Find the equations of the tangent and normal to the curve $y = \sqrt{4x - x^2 + 1}$ at the point $(1, 2)$. Show that the tangent is parallel to the line $6y - 3x = 1$.

9. Find the equations of the tangents to the curve $y = x^3 - 11x$ which are parallel to the line $y = x + 2$.

10. Find the equation of the tangent to the curve $y = 3x^2 - 2x + 5$ which is perpendicular to the line $4y + x = 2$.

11. Find the equation of the normal to the curve $y = 3 + 2x - x^2$ which is parallel to the line $2y - x = 3$.

12. Find the equation of the tangent to the curve $y = x^3 - 7x^2 + 14x - 8$ at the point where $x = 1$. Find the x-coordinate of the point at which the tangent is parallel to the tangent at $x = 1$. *(C)*

13. The tangent to the curve $y = 2x^2 + ax + b$ at the point $(-2, 11)$ is perpendicular to the line $2y = x + 7$. Find the value of a and of b. *(C)*

14. The tangent to the curve $y = ax^3 + bx$ at the point $(1, 3)$ crosses the y-axis at $(0, -4)$. Find the value of a and of b.

*15. Find the equation of the tangent to the curve $y = \frac{4}{x^2} + 1$ at the point where $x = a$. This tangent meets the axes at $P(b, 0)$ and $Q(0, b)$. Find the value of a and of b.

16. The normal to the curve $y = x^3 - 2x^2$ at the point $(1, -1)$ passes through the point $(a, 2a)$. Calculate the value of a.

*17. Find the equation of the tangent to the curve $y = x + x^2$ at the point where $x = a$. Find the values of a for which this line passes through the point $P(2, -3)$. Hence find the equations of the tangents from P to the curve.

18. If the equation of the normal to the curve $y = ax + \frac{b}{x}$ at the point $(2, 7)$ is $y + 2x = 11$, find the value of a and of b. Given that this normal meets the curve again at P, find the coordinates of P.

Important Notes

1. For the function $y = ax^n$, where a and n are constants, $\dfrac{dy}{dx} = anx^{n-1}$.

   If $n = 0$, then $y = a$ and $\dfrac{dy}{dx} = 0$.

   If $n = 1$, then $y = ax$ and $\dfrac{dy}{dx} = a$.

2. **Rules for differentiation**

   (a) If y is a function of u and u is a function of x, then

   $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \quad \text{(chain rule)}$$

   (i) For $y = u^n$, we have $\dfrac{d}{dx}(u^n) = nu^{n-1}\dfrac{du}{dx}$.

   (ii) For $y = \sqrt{u}$, we have $\dfrac{d}{dx}(\sqrt{u}) = \dfrac{1}{2\sqrt{u}}\dfrac{du}{dx}$.

   (b) If u and v are functions of x, then

   $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}, \quad \text{(product rule)}$$

   $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \quad \text{(quotient rule)}$$

3. For a curve defined by $y = f(x)$,

   (a) the gradient at $P(x_1, y_1)$, m = gradient of the tangent at P

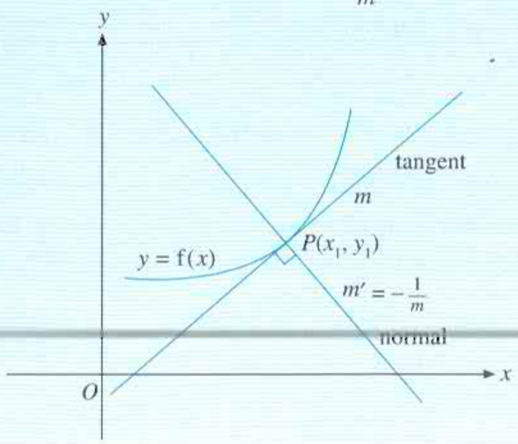
   $$= \text{value of } \frac{dy}{dx} \text{ at } (x_1, y_1),$$

   (b) the equation of the tangent at P is

   $$y - y_1 = m(x - x_1),$$

   (c) the equation of the normal at P is

   $$y - y_1 = -\frac{1}{m}(x - x_1).$$



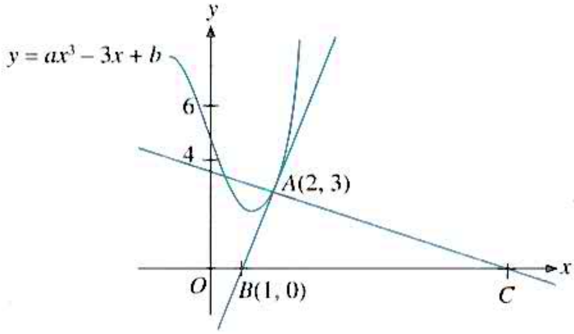
Miscellaneous Examples

Example 17 In the graph, the tangent to the curve $y = ax^3 - 3x + b$ at $A(2, 3)$ meets the x-axis at $B(1, 0)$. The normal to the curve at A meets the x-axis at C. Find

(a) the value of a and of b,

(b) the coordinates of C,

(c) the area of the triangle ABC.



Solution:

(a) $y = ax^3 - 3x + b$

$$\Rightarrow \frac{dy}{dx} = 3ax^2 - 3$$

At $A(2, 3)$, $3 = 8a - 6 + b$

$8a + b = 9$ .... (1)

and $\frac{dy}{dx}$ = gradient of the tangent AB

$$12a - 3 = \frac{3 - 0}{2 - 1} = 3$$

$$a = \mathbf{\frac{1}{2}}$$

From (1), $b = \mathbf{5}$

(b) Gradient of the normal line $= -\frac{1}{3}$ and its equation is

$$y - 3 = -\frac{1}{3}(x - 2)$$

At C, $y = 0$, $x = 11$ and the coordinates of C are **(11, 0)**.

(c) Height of $A(2, 3)$ above the x-axis, $h = 3$

Area of the triangle $ABC = \frac{1}{2} \times BC \times h$

$$= \frac{1}{2}(11 - 1)(3)$$

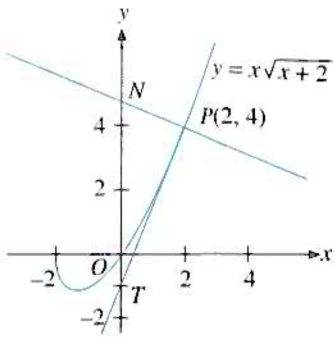
$$= \textbf{15 sq. units}$$

Example 18 The diagram shows part of the curve $y = x\sqrt{x+2}$ for $x \geqslant -2$. The tangent of and the normal to the curve at $P(2, 4)$ meet the y-axis at T and N respectively.

(a) Find the equation of the tangent and the coordinates of T.

(b) Find the equation of the normal and the coordinates of N.

(c) Calculate the area of the triangle PTN.



Solution:

(a) $y = x\sqrt{x+2}$

By the product rule,

$$\frac{dy}{dx} = x\frac{d}{dx}(\sqrt{x+2}) + \sqrt{x+2}\,\frac{d}{dx}(x)$$

$$= x\left(\frac{1}{2\sqrt{x+2}}\right) + \sqrt{x+2}$$

$$= \frac{x + 2(x+2)}{2\sqrt{x+2}} = \frac{3x+4}{2\sqrt{x+2}}$$

At $P(2, 4)$, $\frac{dy}{dx} = \frac{6+4}{2\sqrt{4}} = \frac{5}{2}$

Equation of the tangent is $y - 4 = \frac{5}{2}(x - 2)$,

i.e. $\mathbf{2y = 5x - 2}$.

At the point T, $x = 0$

$$\Rightarrow \quad y - 4 = \frac{5}{2}(-2)$$

$$y = -1$$

Hence, the coordinates of T are $\mathbf{(0, -1)}$.

(b) Equation of the normal is $y - 4 = -\frac{2}{5}(x - 2)$,

i.e. $\mathbf{5y = 24 - 2x}$.

At the point N, $x = 0$

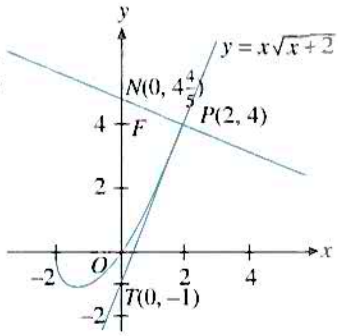
$$\Rightarrow \quad y - 4 = -\frac{2}{5}(-2)$$

$$y = 4\frac{4}{5}$$

Hence, the coordinates of N are $\mathbf{\left(0, 4\frac{4}{5}\right)}$.

(c) Referring to the diagram on the right, F is the foot of the perpendicular from P to the y-axis, and so area of

$$\Delta PTN = \frac{1}{2} \times NT \times PF$$
$$= \frac{1}{2}\left(4\frac{4}{5} - (-1)\right)(2)$$
$$= 5\frac{4}{5} \text{ sq. units}$$



Miscellaneous Exercise 15

1. Differentiate the following with respect to x.

 (a) $\sqrt{1 - 3x^2}$ (b) $x(1 + 2x)^4$ (c) $\dfrac{1 + 2x}{\sqrt{x}}$

2. Differentiate the following with respect to x.

 (a) $\dfrac{4}{2x + 1}$ (b) $\dfrac{x + 2}{1 - 3x}$ (c) $x\sqrt{2 - x}$

3. Calculate the gradient of the curve $y = \left(x - \dfrac{4}{x}\right)^2$ at the point where $x = -1$.

4. (a) Differentiate $(x + 2\sqrt{x})^5$ with respect to x.

 (b) Find the x-coordinates of the points on the curve $y = \dfrac{x^2 + 1}{x}$ at which the tangent is perpendicular to the line $3y + 4x = 2$.

5. The gradient of the curve $y = 8x - 3x^2$ at (a, b) is the same as the gradient of the curve $y = x^2 - 6x + 1$ at $(4, 7)$. Find the value of a and of b.

6. Given that the equation of a curve is $y = \dfrac{10}{x} - x$, find

 (a) the equation of the normal to the curve at $x = 3$,

 (b) the coordinates of the points on the curve at which the tangent has gradient $-\dfrac{7}{2}$.

7. Given that $y = \dfrac{x}{\sqrt{x} - 2}$, prove that $\dfrac{dy}{dx} = \dfrac{\sqrt{x} - 4}{2(\sqrt{x} - 2)^2}$. Hence, obtain the equation of the normal to the curve $y = \dfrac{x}{\sqrt{x} - 2}$ at the point on the curve where $x = 1$.

8. Find the equation of the tangent to the curve $y = 3x^3 - 5x + 4$ at the point $(1, 2)$. Find the coordinates of the other point on the curve at which the gradient is parallel to this tangent.

9. Given that $y = \dfrac{2 + x}{3 - 2x}$, find $\dfrac{dy}{dx}$ and hence obtain the equation of the tangent to the curve at the point where the curve crosses the line $y = 3$.

10. The tangent to the curve $y = x^2 + 6x - 4$ at $(1, 3)$ intersects the normal to the curve $y = x^2 - 6x + 18$ at $(4, 10)$ at the point R. Calculate the coordinates of R.

11. The tangent and normal to the curve $y = 4\sqrt{x + 2}$ at the point $P(7, 12)$ cut the x-axis at M and N respectively. Calculate the area of the triangle PMN.

12. The curve $y = x^2 - 3x - 4$ cuts the x-axis at A and B. The tangents to the curve at A and B meet at T, and the normals to the curve at A and B meet at N. Find the area of the quadrilateral $ATBN$.

13. Find the equation of the tangent to the curve $y = x^3 - 8x^2 + 15x$ at the point $(4, -4)$. Calculate the coordinates of the point where the tangent meets the curve again. *(C)*

14. Given the curve $8y = x^2 - kx + 17$, calculate the value of k such that the tangents at the points with x-coordinates 5 and -3 respectively are perpendicular.

15. Find an expression for the gradient of the curve $y = x^3 + px^2 + 2x + q$, where p and q are constants. Given that the tangents at $A(0, q)$ and $B(2, 5)$ are parallel, find the value of p and of q. *(C)*

16. The tangent to the curve $y = ax^2 + bx + 2$ at $(1, \frac{1}{2})$ is parallel to the normal to the curve $y = x^2 + 6x + 4$ at $(-2, -4)$. Find the value of a and of b.

17. The line $y = 2x - 16$ is a tangent to the curve $y = ax^3 + bx$ at the point where $x = 2$. Find the value of a and of b.

18. Calculate the gradient of the curve $y = \dfrac{2x}{3x + 1}$ at the point $(-1, 1)$. Calculate also the gradient of the normal to the curve at the point where $x = \frac{1}{2}$.

19. The tangent to the curve $y = ax^2 + 2x + b$ at $A(1, 2)$ passes through the point $B(2, 8)$. Calculate the values of the unknown constants a and b.

*20. Find the coordinates of the two points on the curve $y = 4 - x^2$ whose tangents pass through the point $(-1, 7)$.

*21. The tangent at the point $P(a, b)$ on the curve $y = \dfrac{ab}{x}$ meets the x-axis and y-axis at Q and R respectively. Show that $PQ = RP$.

16 Rates of Change

16.1 Constant Rate and Variable Rate of Change

We have so far applied differentiation in obtaining gradient functions of given curves and in simple coordinate geometry. We shall now use differentiation to study the rates of change of certain variables with respect to time and the relationships between them.

Example 1

The radius r (cm) of a circle changes with time t (seconds) and they are related by the equation $r = 0.5t + 1.5$.

(a) Find the initial value of r and the rate of change of r.

(b) Sketch the r–t graph and describe the relationship between r and t.

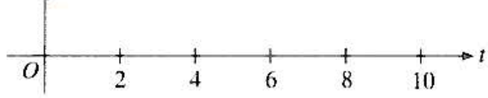
Solution:

(a) $r = 0.5t + 1.5$

When $t = 0$, $r =$ **1.5 cm**, the initial value of r

$\frac{\mathrm{d}r}{\mathrm{d}t} = 0.5 \quad \Rightarrow \quad r$ increases at a **constant rate** of 0.5 cm s^{-1}

(b) The radius r is linearly related to the time t. The rate of change of r with respect to t is the gradient of the line.



In general, if a variable x is linearly related to the time t, then $x = mt + c$, where m and c are constants, $\frac{\mathrm{d}x}{\mathrm{d}t} = m$ and x is said to change at a constant rate m (or at a steady rate).

Example 2

The radius r (cm) of a circle changes with time t (seconds) and they are related by the equation $r = t^2 + 2$.

(a) Find the rate of change of r during the interval $t = 2$ to $t = 2.1$.

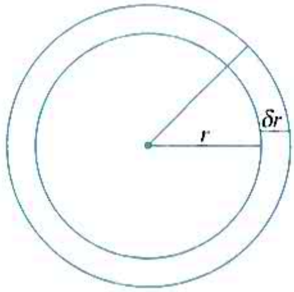
(b) Find the rate of change of r during the interval from t to $t + \delta t$ and hence the rate of change of r at the instant $t = 2$.

Solution:

(a) Change in r, $\delta r = [(2.1)^2 + 2] - [(2)^2 + 2]$
$= 0.41$ cm

Rate of change of r during the interval from $t = 2$ to $t = 2.1$,

$$\frac{\delta r}{\delta t} = \frac{0.41}{0.1}$$
$$= \mathbf{4.1\ cm\ s^{-1}}$$



(b) Change in r, $\delta r = [(t + \delta t)^2 + 2] - [(t)^2 + 2]$
$= 2t\delta t + (\delta t)^2$

Rate of change of r during the interval from t to $t + \delta t$,

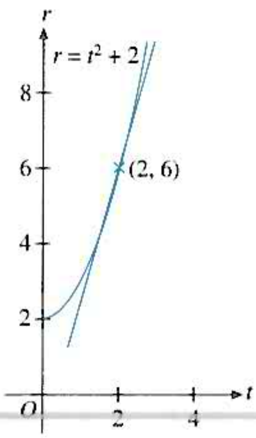
$$\frac{\delta r}{\delta t} = \frac{2t\delta t + (\delta t)^2}{\delta t}$$
$$= 2t + \delta t \text{ cm s}^{-1}$$

As $\delta t \to 0$, $\frac{\delta r}{\delta t} \to 2t$ is the rate of change of r at an instant t.

At $t = 2$, the rate of change of $r = 2 \times 2 = \mathbf{4\ cm\ s^{-1}}$.

As $\delta t \to 0$, $\frac{\delta r}{\delta t} \to 2t$, which is the rate of change at an instant t and is equal to $\frac{dr}{dt} = 2t$.

This shows that the radius r, changes at a rate of $2t$ cm s^{-1} at any instant t. At $t = 2$, the rate of change is 4 cm s^{-1}. We can say that r increases at a rate of 4 cm per second.



In the diagram, the r–t graph shows that r increases as t increases and the rate of change of r at $t = 2$ is the gradient of the r–t curve at $t = 2$.

As we have seen, the gradient function $\frac{dr}{dt}$ is the rate of change of r w.r.t. t.

In general, if a variable x varies with time t, i.e. x is a function of t, then $\frac{dx}{dt}$ is the rate of change of x w.r.t. t at any instant t.

Example 3 The volume, V litres of water in a tank after t seconds is given by

$$V = 5 - \frac{2}{t+1}.$$

(a) What is the initial volume of water?
(b) Find the rate at which the volume of water is increasing when $t = 3$.

Solution:

$$V = 5 - \frac{2}{t+1} \quad \text{.............................. (1)}$$

(a) When $t = 0$, $V = 5 - \frac{2}{0+1} = 3$ and so the initial volume of water is **3 litres.**

(b) Differentiating (1) with respect to t:

$$\frac{dV}{dt} = \frac{2}{(t+1)^2}$$

When $t = 3$, $\frac{dV}{dt} = \frac{2}{4^2} = \frac{1}{8}$ and so the water is increasing at a rate of $\frac{1}{8}$ **litre per second** at this instant.

Exercise 16.1

1. The radius, r cm, of a circle at time t seconds is given by $r = 9t - t^3$. At each of the following instants, find the rate of change of the radius (w.r.t. t) and state whether the radius is increasing or decreasing at these instants.
(a) $t = 1$ **(b)** $t = 2$ **(c)** $t = 2.5$

2. The length, l mm, of an elastic string at time t seconds is given by

$$l = \frac{t^3}{3} - 4t + 10.$$

Find the instant (that is, the value of t) when
(a) the length is increasing at a rate of 5 mm s^{-1},
(b) the length is decreasing at a rate of 4 mm s^{-1}.

3. The radius, r cm, of a spherical balloon at time t seconds is given by

$$r = 3 + \frac{2}{1+t}.$$

What is the initial radius? Find the rate of change of r (w.r.t. t) when $t = 3$.

4. The amount of water, V cm^3, in a leaking tank at time t seconds is given by

$$V = (15 - t)^3 \text{ for } 0 \leqslant t \leqslant 15.$$

Find the rate at which the water leaves the tank when $t = 4$.

5. The volume, V cm^3, of a cube at time t seconds is given by $V = \left(4 + \frac{1}{3}t\right)^3$. Find the rate at which its volume is increasing at the instant when $t = 2$.

6. A rectangle has sides of length x cm and $2x - 4$ cm and the length x cm at time t seconds is given by $x = 2 + 3t$, $(t \geqslant 0)$. Show that the area, A cm^2, of the rectangle, in terms of t is $A = 12t + 18t^2$. Hence find the rate of change of the area at the instant when $t = 2$.

7. The radius, r cm, of a circle at time t seconds is given by $r = 2t^2 + 1$. Express its area, A cm^2, in terms of t and find the rate of change of the area at the instant when $t = 2$. (Leave your answer in terms of π).

8. Water is poured steadily into an empty container. If the volume of water in the container after 5 seconds is 30 cm^3, find
   (a) the rate of change of volume,
   (b) the volume of water after 12 seconds.

9. The volume, V cm^3, of a cone of height h is $\frac{\pi h^3}{12}$. If h increases at a constant rate of 0.2 cm s^{-1} and the initial height is 2 cm, express V in terms of t and find the rate of change of V at time t.

10. The area, A cm^3, of a circle increases at a constant rate of 2 cm^2 s^{-1}. If the initial area of A is 1 cm^2, show that the radius of the circle at time t is given by $r = \sqrt{\frac{2t + 1}{\pi}}$.

16.2 Related Rates of Change

The area, A cm^2, of a circle of radius r cm is given by $A = \pi r^2$.
If r is a function of t where $r = 3t + 1$, we can express A as a function of t:

$$A = \pi(3t + 1)^2$$

The rate of change of A is

$$\frac{dA}{dt} = 2\pi(3t + 1)(3)$$

and the rate of change of r is $\quad \frac{dr}{dt} = 3.$

The relation between the rate of change of A, $\frac{dA}{dt}$, and the rate of change of r, $\frac{dr}{dt}$ can be obtained by using the chain rule as follows:

$$\frac{dA}{dr} = 2\pi r$$

So

$$\begin{aligned}\frac{dA}{dt} &= \frac{dA}{dr} \times \frac{dr}{dt} \\ &= 2\pi r\frac{dr}{dt} \\ &= 2\pi(3t + 1)\frac{dr}{dt}\end{aligned}$$

Example 4 The radius of a circle increases at a rate of 3 cm s^{-1}. Find the rate of increase of the area when
(a) the radius is 5 cm, **(b)** the area is 4π cm^2.

Solution: Let A be the area and r the radius. Then, $A = \pi r^2$.

By the chain rule,
$$\begin{aligned}\frac{dA}{dt} &= \frac{dA}{dr} \times \frac{dr}{dt}\\ &= 2\pi r\frac{dr}{dt}\\ &= 6\pi r \quad (\text{as } \frac{dr}{dt} = 3 \text{ cm s}^{-1})\end{aligned}$$

(a) When $r = 5$, $\frac{dA}{dt} = 30\pi$
Hence the area increases at a rate of **30π cm^2 s^{-1}**.

(b) When $A = 4\pi$,
$$\begin{aligned}\pi r^2 &= 4\pi\\ r^2 &= 4\\ r &= 2 \quad (r \geqslant 0)\\ \therefore \quad \frac{dA}{dt} &= 12\pi\end{aligned}$$
Hence the area increases at a rate of **12π cm^2 s^{-1}**.

In general, if two quantity variables x and y are related by the equation $y = f(x)$, then the rates of change $\frac{dy}{dt}$ and $\frac{dx}{dt}$ are related by:

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} \quad \textbf{(chain rule)}$$

or $$\frac{dy}{dt} = f'(x)\frac{dx}{dt} \text{ where } \frac{dy}{dx} = f'(x).$$

Note: $\frac{dy}{dx}$ is the rate of change of y with respect to x.

Example 5 Show that the rates of change of the volume, V cm^3 and the radius, r cm, of a sphere are related by $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$.

Find, to 2 significant figures, the rate of change of radius r if
(a) the volume V is increasing at a rate of 30 cm^3 s^{-1} when $r = 3$,
(b) the volume V is decreasing at a rate of 20 cm^3 s^{-1} when $r = 4$.

Solution: $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$

By the chain rule, $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

(a) If $\frac{dV}{dt} = 30$ when $r = 3$,

$$30 = 4\pi(3)^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{5}{6\pi} = 0.27$$

Thus, **r changes at a rate of 0.27 cm s^{-1}**.

(b) If $\frac{dV}{dt} = -20$ when $r = 4$,

$$-20 = 4\pi(4)^2 \frac{dr}{dt}$$

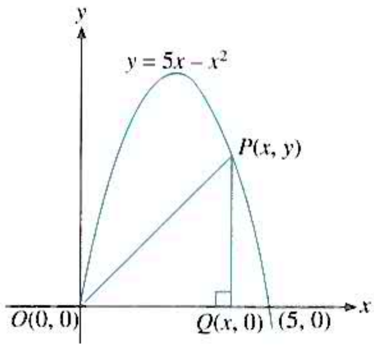
$$\frac{dr}{dt} = -\frac{5}{16\pi} = -0.099$$

Thus, **r changes at a rate of -0.099 cm s^{-1}**.

Example 6 The points $O(0, 0)$ and $P(x, y)$ lie on the curve $y = 5x - x^2$ as shown in the diagram. If Q is the point $(x, 0)$, find the area A, of the triangle OPQ in terms of x.
If x increases at a rate of 2 units per second, find

(a) the rate of increase of A when $x = 3$,

(b) the rate of decrease of A when $x = 4$.



Solution: Since the triangle OPQ is right-angled at Q, the area is

$$A = \frac{1}{2} \times OQ \times QP$$

$$= \frac{1}{2}xy$$

$$= \frac{1}{2}x(5x - x^2)$$

$$= \frac{1}{2}(5x^2 - x^3)$$

Then, $\frac{dA}{dx} = \frac{1}{2}(10x - 3x^2)$.

If x increases at a rate of 2 units per second, then $\frac{dx}{dt} = 2$.

$$\begin{aligned}\frac{dA}{dt} &= \frac{dA}{dx} \times \frac{dx}{dt}\\ &= \frac{1}{2}(10x - 3x^2)(2)\\ &= 10x - 3x^2\\ &= x(10 - 3x)\end{aligned}$$

(a) When $x = 3$, $\frac{dA}{dt} = 3(10 - 9) = 3$

Thus, **A increases at a rate of 3 sq. units s^{-1}.**

(b) When $x = 4$, $\frac{dA}{dt} = 4(10 - 12)$
$= 4(-2)$
$= -8$

Thus, **A decreases at a rate of 8 sq. units s^{-1}.**

Exercise 16.2

1. For each of the following equations connecting x and y, if the rate of change of x is 2 units s^{-1}, find the rate of change of y at the given instant.

(a) $y = 3x^2 - 1$; $x = 2$
(b) $y = 2x^2 + \frac{1}{x}$; $x = 1$
(c) $y = \frac{3}{(2x-3)^3}$; $x = 2$
(d) $y = (3x - 5)^5$; $x = \frac{4}{3}$
(e) $y = x^3 + 2$; $y = 10$
(f) $y = \frac{x}{x+1}$; $y = 2$

2. For each of the following equations connecting x and y, if the rate of change of y is 4 units s^{-1}, find the rate of change of x at the given instant.

(a) $y = x^3 - 2x^2$; $x = 3$
(b) $y = \frac{3x^2}{1+x}$; $x = 2$
(c) $y = \sqrt{2x+7}$; $y = 3$
(d) $y = x(x - 4)$, $x > 0$; $y = 5$

3. The radius of a circle increases at a rate of 2 cm s^{-1}. Find the rate of increase of its area when

(a) the radius is 4 cm,
(b) the area is 9π cm^2.

4. The area of a circle increases at a rate of 2π cm^2 s^{-1}. Calculate the rate of increase of the radius when the radius is 6 cm.

5. The radius of a circular disc increases at a constant rate of 0.02 cm s^{-1}. Find the rate at which the area is increasing when the radius is 10 cm.

6. A circular ripple spreads across a lake. If the area of the ripple increases at a rate of 10π m^2 s^{-1}, find the rate at which the radius is increasing when the radius is 2 m.

7. The radius of a sphere increases at a rate of 2 cm s^{-1}. Find the rate of increase of its volume when the radius is 3 cm.

8. Air is let out of a spherical balloon at a rate of 300 cm^3 s^{-1}. Find the rate at which the radius is decreasing when
   (a) the radius is 2 cm,
   (b) the volume is 36π cm^3.

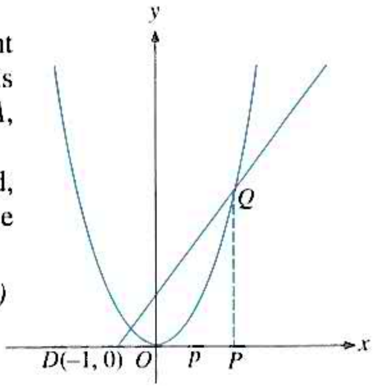
9. The area of a square increases at a rate of 10 cm^2 s^{-1}. Find the rate of change in the length of its side when the area is 4 cm^2.

10. A metal cube is being expanded by heat. At the instant when the length of an edge is 2 cm, the volume of the cube is increasing at the rate of 0.012 cm^3 per second. At what rate is the length of the edge increasing at this instant? *(C)*

11. The surface area of a cube is increasing at 0.2 cm^2 s^{-1}. Find the rate of increase of the volume when the length of a side is 1 cm.

12. (a) Variables x and y are connected by the equation $y = x^2 + \frac{1}{x}$. Given that x is increasing at the rate of $\frac{1}{5}$ unit s^{-1}, find the rate of increase of y when $x = 2$.
    (b) A rectangle has sides of length $2x$ cm and $3x$ cm. Given that the area is increasing at a rate of 36 cm^2 s^{-1}, find the rate of increase of the perimeter when $x = 3$.

13. The figure shows the curve $y = x^2$ and the point $D(-1, 0)$. $P(p, 0)$ is a variable point on the x-axis and PQ is parallel to the y-axis. Express the area, A, of the triangle DPQ in terms of p.
    If p is increasing at the rate of 5 units per second, calculate the rate at which A is increasing at the instant when $p = 4$ units.
    (C)



14. In the triangle ABC, $AB = 2x$, $AC = x + 1$ and $\angle A = 150°$. Show that the area of the triangle is $\frac{1}{2}x(x + 1)$. If x is increasing at the rate of 2 cm s^{-1}, find the rate at which the area is increasing when $x = 4$ cm.

15. The height of a cylinder is twice the radius, x cm, of its base. If x is decreasing at a rate of 0.1 cm s^{-1}, find the rate of change of the volume of the cylinder when $x = 2$.

16. A cylindrical jar, of radius 3 cm, contains water to a depth of 5 cm. The water is then poured at a steady rate into an inverted conical container with its axis vertical. After t seconds, the depth of water in this container is x cm and the volume, V ml, of water that has been transferred is given by $V = \frac{1}{3}\pi x^3$.
    Given that all the water is transferred in 3 seconds, find
    (a) $\frac{dV}{dt}$ in terms of π,
    (b) the rate at which x is increasing at the moment when $x = 2.5$. *(C)*

16.3 Small Changes

We have defined $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$, where $\delta x = x_2 - x_1$ and $\delta y = y_2 - y_1$.

If δx is small, then $\qquad \frac{\delta y}{\delta x} \approx \frac{dy}{dx}$

i.e. $\qquad \delta y \approx \frac{dy}{dx}\delta x$

The smaller the value of δx, the more accurate the approximation.

Example 7 Given $y = x^3$, find

(a) the approximate change in y if x increases from 2 to 2.01,

(b) the approximate change in x if y decreases from 27 to 26.73.

Solution:

(a) We have $\quad \frac{dy}{dx} = 3x^2$

When $x = 2$, $\quad \delta x = 2.01 - 2 = 0.01$

and $\quad \frac{dy}{dx} = 12$

Then, $\quad \delta y \approx \frac{dy}{dx}\delta x = 12(0.01) = 0.12$

Hence the approximate change in y is **0.12**.

(b) When $y = 27$, $\quad x^3 = 27$

$$x = 3$$

$$\delta y = 26.73 - 27 = -0.27$$

$$\frac{dy}{dx} = 27$$

Then, $\quad \delta y \approx \frac{dy}{dx}\delta x$

$$-0.27 \approx 27\delta x$$

$$\delta x \approx \frac{-0.27}{27} \approx -0.01$$

Hence the approximate change in x is **−0.01**.

Note: In **(a)**, as $\delta y > 0$, there is an increase in y.
In **(b)**, as $\delta x < 0$, there is a decrease in x.

Example 8 Given that $y = \frac{1}{2x+1}$, find the value of $\frac{dy}{dx}$ when $x = 2$. Hence find an expression in terms of p, for the approximate increase in y when x decreases from 2 to $2 - p$, where p is small.

Solution:

$$y = \frac{1}{2x+1} = (2x+1)^{-1}$$

$$\frac{dy}{dx} = (-1)(2x+1)^{-2}(2) = -\frac{2}{(2x+1)^2}$$

When $x = 2$, $\frac{dy}{dx} = -\frac{2}{25}$

The approximate change in $y = \delta y$

$$\approx \frac{dy}{dx}\delta x$$

$$\approx -\frac{2}{25} \times (-p)$$

$$\approx \frac{2p}{25}$$

Example 9

Without using a calculator, find the approximate increase in the volume of a cube when each of its sides increases from 10 cm to 10.1 cm. Hence write down the approximate volume of a cube whose sides are 10.1 cm.

Solution:

Let V be the volume and x the length of its edge.

Then, $V = x^3$

$$\frac{dV}{dx} = 3x^2$$

When $x = 10$, $\delta x = 10.1 - 10$

$$= 0.1$$

$$\frac{dV}{dx} = 300$$

Then, $\delta V \approx \frac{dV}{dx}\delta x$

$$\approx 300(0.1)$$

$$\approx 30$$

Hence the approximate increase in volume is **30 cm^3**.
That is, the volume increases from 10^3 (= 1000) to $(10.1)^3$ ($\approx$ 1000 + 30). So the required approximate volume is **1030 cm^3**.

Since $f(x + \delta x) = y + \delta y$

and $\delta y \approx \frac{dy}{dx}\delta x$

$$f(x+\delta x) \approx y + \frac{dy}{dx}\delta x$$

or $f(x+\delta x) \approx f(x) + \frac{dy}{dx}\delta x$

Example 10 Given that $y = \sqrt{x}$, use the calculus to obtain an approximate value for $\sqrt{4.02}$.

Solution:

$$y = \sqrt{x}, \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

When $x = 4$, $y = 2$

$$\delta x = 4.02 - 4 = 0.02$$

$$\frac{dy}{dx} = \frac{1}{4}$$

Using $f(x + \delta x) \approx y + \frac{dy}{dx}\delta x$

$$\sqrt{x + \delta x} \approx y + \frac{dy}{dx}\delta x$$

$$\sqrt{4 + 0.02} \approx 2 + \frac{1}{4}(0.02)$$

$$\sqrt{4.02} \approx \mathbf{2.005}$$

Percentage Changes

If x changes from x to $x + \delta x$, then

percentage change in $x = \frac{\delta x}{x} \times 100\%$

and percentage change in $y = \frac{\delta y}{y} \times 100\%$

Note: x changes by $\frac{\delta x}{x} \times 100\%$ and y changes by $\frac{\delta y}{y} \times 100\%$.

In Example 10, $\frac{\delta x}{x} \times 100 = \frac{0.02}{4} \times 100 = 0.5$

$\therefore$ the percentage change in x is 0.5%.

Also $\frac{\delta y}{y} \times 100 \approx \frac{0.005}{2} \times 100 = 0.25$

$\therefore$ the approximate percentage change in y is 0.25%.

Example 11 Given that $y = 3x^2 - 2x - 3$ and that when $x = 2$, there is a small increase in x of $p\%$. Use the calculus to determine the approximate percentage change in y.

Solution:

$$y = 3x^2 - 2x - 3, \quad \frac{dy}{dx} = 6x - 2$$

When $x = 2$ and $y = 5$, $\frac{dy}{dx} = 10$

$$\delta x = \frac{p}{100}x = \frac{2p}{100} \quad (\delta x = p\% \text{ of } x)$$

$$\text{Then,} \qquad \delta y \approx \frac{dy}{dx}\delta x$$
$$\approx 10\left(\frac{2p}{100}\right)$$
$$\approx 0.2p$$
$$\text{and so} \quad \frac{\delta y}{y} \times 100 \approx \frac{0.2p}{5} \times 100$$
$$\approx 4p$$

Hence the percentage change in y is approximately $4p\,\%$.

Exercise 16.3

1. Given that $y = 2x^3 - 4x^2$, find the approximate change in y as x increases from 1 to 1.05, stating whether this is an increase or a decrease.
2. Given that $y = 8x^2 - x^3$, find the approximate change in y as x increases from 2 to 2.1, stating whether this is an increase or a decrease.
3. Given that $y = 2 + \frac{8}{x}$, find the approximate change in x which will cause y to increase from 6 to 6.01.
4. Given that $y = 2x^{\frac{3}{2}}$, find the approximate change in x when y changes from 16 to 15.9.
5. The function y is defined by $y = x^8 - 2x^4 + 5$. Express in terms of p, the approximate increase or decrease in the value of y when x increases from 2 to $2 + p$, where p is small. *(C)*
6. Use the calculus to determine the approximate change in the area of a circle when the radius increases from 3 cm to 3.02 cm.
7. The area of a circular patch of liquid increases from 4π cm^2 to 4.01π cm^2. Use calculus to find the corresponding increase in the radius.
8. The volume, V cm^3, of a liquid in a container is given by $V = x(4x + 3)$ where x cm is the depth of the liquid. Find $\frac{dV}{dx}$ in terms of x and deduce the approximate change in the volume when the depth decreases from 10 cm to 9.95 cm.
9. The volume, V cm^3, of a spherical balloon of radius r cm is given by $V = \frac{4}{3}\pi r^3$. Write down an expression for $\frac{dV}{dr}$ and use it to find the approximate change in the radius when the volume changes from 36π cm^3 to 36.2π cm^3.
10. The time of swing, T seconds, of a pendulum of length x cm is given by $T = 2\pi\sqrt{\frac{x}{10}}$. Find $\frac{dT}{dx}$ and hence the approximate increase in T when x is increased from 9 to 10.

11. Given that $y = \frac{3}{1 - 2x}$, obtain an expression for $\frac{dy}{dx}$. Hence find, in terms of p, the approximate increase in y when x increases from 2 to $2 + p$, where p is small.

12. Given that $y = \frac{6}{x^3 + 1}$, obtain an expression for $\frac{dy}{dx}$ in terms of x. Hence find, in terms of p, where p is small, the approximate change
(a) in y as x increases from 2 to $2 + p$,
(b) in x when y decreases from 3 to $3 - p$.

13. Use the calculus to find the approximate change in the volume of a spherical soap bubble when its radius increases from 3 cm to 3.01 cm. What is the approximate value of the volume when the radius is 3.01 cm?

14. A cuboid has a square base of side x cm and its height is $(2x + 1)$ cm. Find the approximate increase in its volume when x increases from 10 cm to 10.05 cm. Hence deduce an approximate value for the volume when $x = 10.05$ cm.

15. Given that $y = x^{\frac{1}{3}}$, use the calculus to determine an approximate value for $\sqrt[3]{1003}$.

16. Given that $y = 4x^2 - x - 4$, find the approximate percentage change in y as x increases from 2 to 2.02.

17. Given that $y = \frac{1}{\sqrt{x}}$, calculate the approximate percentage change in y when x increases from 4 by 2%.

18. Given that $y = \frac{1}{2x - 3}$, find the value of $\frac{dy}{dx}$ when $x = 2$. Hence find an expression, in terms of p, for the approximate percentage change in y when x increases from 2 to $2 + p$, where p is small.

19. The function y is defined by $y = 12 - 4x + 5x^2$. Write down its derivative and show that, when $x = 6$, a small increase in x of p% causes an increase in y of approximately $2p$%. *(C)*

20. When a metal cube is heated, its sides expand by 2%. Find the approximate percentage increases in its volume and surface area.

21. The time of oscillation, T, of a pendulum of length x is given by

$$T = k\sqrt{x}, \text{ where } k \text{ is a constant.}$$

A small change δx in x results in a small change δT in T. Use the calculus to obtain, in terms of δx and x, an expression for the approximate value of

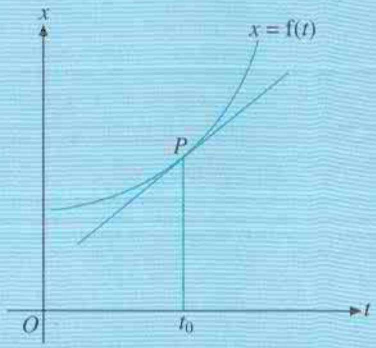
(a) δT, **(b)** $\frac{\delta T}{T}$.

Hence find the approximate percentage change in T corresponding to a small change in x of P%. *(C)*

Important Notes

1. Rates of change

(a) The rate of change of a variable x with respect to time t is $\frac{dx}{dt}$.



(b) In an x–t graph,
rate of change of x at time t_0

$= \frac{dx}{dt}$ when $t = t_0$

$=$ gradient of the curve at $t = t_0$

(c) At the instant $t = t_0$,

$\frac{dx}{dt} > 0 \Leftrightarrow x$ increases as t increases,

$\frac{dx}{dt} < 0 \Leftrightarrow x$ decreases as t increases.

(d) Steady rate:

$\frac{dx}{dt} = m$, constant $\Leftrightarrow x = mt + c$

2. Related rates of change

If x and y are related by the equation $y = f(x)$, then the rates of change $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are related by:

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} \quad \textit{(By the chain rule)}$$

3. Small changes

(a) If $y = f(x)$ and a small change δx in x causes a small change δy in y, then

$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$$

and so $$\delta y \approx \frac{dy}{dx}\delta x.$$

Note that $\delta x > 0 \Leftrightarrow$ an increase in x,
and $\delta x < 0 \Leftrightarrow$ a decrease in x.

(b) The percentage change in x is $\frac{\delta x}{x} \times 100\%$.

(c) $f(x + \delta x) = y + \delta y$

$$\approx y + \frac{dy}{dx}\delta x \quad \text{or} \quad f(x) + \frac{dy}{dx}\delta x$$

Miscellaneous Examples

Example 12 A container holds V cm^3 of a liquid where $V = x^2 + 3x$ and the height of the liquid in the container is x cm. More liquid is added such that the height of the liquid increases at a constant rate of 0.02 cm s^{-1}. If the initial height of the liquid is 9.4 cm, find
(a) the height of the liquid after 30 seconds,
(b) the rate of increase of the volume at this instant.

Solution:

(a) $\frac{dx}{dt} = 0.02 \Rightarrow x = 0.02t + 9.4$
When $t = 30$, $x = 0.02 \times 30 + 9.4 = 10$.
After 30 s, the height is **10 cm**.

(b) Given $V = x^2 + 3x$

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$
$$= (2x + 3)(0.02)$$

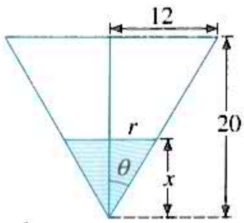
When $x = 10$, $\frac{dV}{dt} = 23(0.02)$
$$= 0.46$$

The volume increases at a rate of **0.46 cm^3 s^{-1}**.

Example 13 A vessel is in the shape of an inverted cone. The radius of the top is 12 cm and the height is 20 cm. If the height of water in the vessel is x cm, show that $V = \frac{3}{25}\pi x^3$.
Water leaks through a hole at the vertex.
(a) Find the approximate decrease in the volume of water when x decreases from 5 cm to 4.98 cm.
(b) Show that a small decrease of $p\%$ in the height will cause a decrease of $3p\%$ in the volume.

Solution:

From the diagram,



$$\tan\theta = \frac{r}{x} = \frac{12}{20}$$
$$r = \frac{3}{5}x$$

The volume of the liquid in the cone is given by:

$$V = \frac{1}{3}\pi r^2 x$$
$$= \frac{1}{3}\pi\left(\frac{3}{5}x\right)^2 x$$
$$= \frac{3}{25}\pi x^3$$

We have $\quad \dfrac{dV}{dx} = \dfrac{9}{25}\pi x^2$

and $\quad \delta V \approx \dfrac{dV}{dx}\delta x = \dfrac{9}{25}\pi x^2 \delta x.$

(a) When $x = 5$, $\delta x = 4.98 - 5 = -0.02$

and so $\quad \delta V \approx \dfrac{9}{25}\pi(5)^2(-0.02) = -0.18\pi$

Hence the volume decreases by approximately **0.18π cm^3**.

(b) If x decreases by $p\%$, $\dfrac{\delta x}{x} \times 100 = -p$

$$\delta x = \frac{-p}{100}(x)$$

and so $\quad \delta V \approx \dfrac{9}{25}\pi x^2\left(-\dfrac{px}{100}\right) = -\dfrac{9}{2500}\pi p x^3$

Then,

$$\frac{\delta V}{V} \times 100 \approx \frac{-\frac{9}{2500}\pi p x^3}{\frac{3}{25}\pi x^3} \times 100 = -3p$$

Hence the volume decreases by approximately **$3p\%$**.

Miscellaneous Exercise 16

1. A viscous liquid is poured onto a flat surface. It forms a circular patch which grows at a steady rate of 4 cm^2 s^{-1}. Find, in terms of π,

(a) the radius of the patch 16 seconds after pouring has commenced,

(b) the rate of increase of the radius at this instant.

2. Under a heating process, the length, x cm, of each side of a metal cube increases from an initial value of 9.9 cm at a constant rate of 0.005 cm s^{-1}. Express the volume, V cm^3, and the surface area, A cm^2, of the cube in terms of x.

Write down expressions for $\dfrac{dV}{dx}$ and $\dfrac{dA}{dx}$.

Hence, find

(a) the rate at which V is increasing when the cube has been heated for 20 s,

(b) the approximate increase in A as x increases from 10 to 10.001 cm. *(C)*

3. The volume, V cm^3, of a sphere of radius r cm, is given by the formula $V = \dfrac{4}{3}\pi r^3$.

(a) Air is leaking from a spherical balloon. Use the calculus to determine the approximate decrease in volume as the radius decreases from 16 cm to 15.8 cm.

(b) A pump puts air into a second spherical balloon at the rate of 1200 cm^3 s^{-1}. Calculate the rate, in cm s^{-1}, at which the radius is increasing at the instant when the radius is 5 cm.
(Answers may be left in terms of π.) *(C)*

4. The volume v of a certain gas varies with the pressure p and is given by $v = \frac{600}{p}$.

(a) Find $\frac{dv}{dp}$ and hence the approximate decrease in v as p decreases from 20 to 19.95.

(b) At the instant when $p = 20$, p increases at the rate of 3 units per second. Find the rate of change of v.

5. Variables x and y are related by the equation $y = \frac{3x + 8}{2x}$.

(a) Obtain an expression for $\frac{dy}{dx}$ and hence find an expression for the approximate decrease in y as x increases from 2 to $2 + p$, where p is small.

(b) Given that x and y are functions of t and that $\frac{dy}{dt} = 1.6$, find the corresponding rate of change of x when $y = 2$.

6. Given that $y = \sqrt{x}$, find $\frac{dy}{dx}$ and use it to calculate an approximate value for

(a) $\sqrt{3.96}$, (b) $\sqrt{4.08}$.

7. A vessel has the shape of an inverted cone. The radius of the top is 8 cm and the height is 20 cm. Water is poured in to a height of x cm. Show that if the volume of the water is V cm^3, then $V = \frac{4}{75}\pi x^3$.

Write down $\frac{dV}{dx}$ and hence find

(a) the approximate increase in V when x increases from 10 to 10.2 cm,

(b) the approximate percentage change in V when x increases by $p\%$.

8. (a) Given that $y = 6 - 4x + 5x^2$, and that the value of x increases from 3 by a small amount $\frac{3p}{100}$. Use the calculus to determine, in terms of p,

(i) the approximate change in y,

(ii) the corresponding percentage change in y.

(b) Liquid is poured into a container at a rate of 12 cm^3 s^{-1}. The volume of liquid in the container is V cm^3, where $V = \frac{1}{2}(h^2 + 4h)$ and h cm is the height of liquid in the container. Find, when $V = 16$,

(i) the value of h,

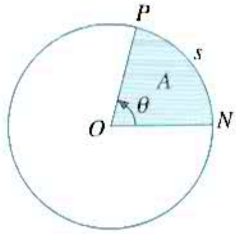
(ii) the rate at which h is increasing.

9. Two variable lengths x cm and y cm are related by the equation $y = \sqrt{\frac{22}{x} - x}$.

(a) Obtain an expression for $\frac{dy}{dx}$ in terms of x.

(b) Given that x and y are functions of t (seconds) and $\frac{dx}{dt} = 2$, find $\frac{dy}{dt}$ when $x = 2$.

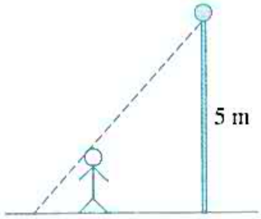
10. N is a fixed point on the circumference of a circle, centre O, radius 8 cm. A variable point P moves round the circumference such that θ (or $\angle PON$) increases at a constant rate of $\frac{\pi}{2}$ rad. per second. Find



(a) the rate of increase of s (arc length from N to P),

(b) the rate of increase of A, the area of the sector NOP.

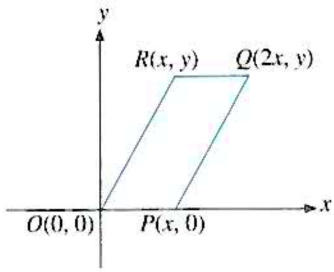
*11. A man 1.5 m tall is walking at a speed of 2 m s^{-1} away from a lamppost which has a lamp 5 m above the ground as shown in the figure. Find



(a) the rate at which the length of his shadow is increasing,

(b) the speed of the top of his shadow.

*12. The figure shows a parallelogram $OPQR$. Given that Q lies on the line $y = 2x + 1$ and x increases at a rate of 1.2 units per second, find



(a) the rate of change of the area of the parallelogram $OPQR$ when $x = 1.5$,

(b) the rate of change of the length of the diagonal OQ at that instant.

*13. A ladder AB of length 5 m has one end A leaning against a vertical wall. The other end B rests on the horizontal ground. When A is at a height of 4 m, it slides down the wall at the rate of 2 m s^{-1}. How fast is the other end, B, sliding along the horizontal ground?

*14. A spherical balloon is released from rest and expands as it rises. After rising for t seconds, its radius is r cm and its surface area is A cm^2, where $A = 4\pi r^2$. The initial radius of the balloon is 16 cm. Given that the rate of increase of the radius is constant and has the value 0.8 cm s^{-1}, find the rate of increase of A when $t = 5$. *(C)*

17 Higher Derivatives and Applications

17.1 Determination of Maximum and Minimum Points

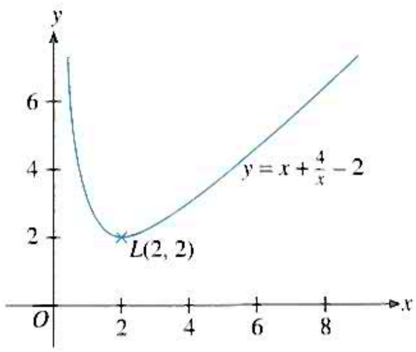
Changes in Gradient

We shall now apply the calculus to find gradient functions of given curves and observe the changes in the gradients to help us find the maximum and minimum points of curves.

Example 1 The diagram shows part of the curve $y = x + \frac{4}{x} - 2$ for $x > 0$, and the point $L(2, 2)$ on the curve.

(a) Show that $\frac{dy}{dx} = 0$ at L.

(b) Observe the change in gradient as x increases through 2.



Solution:

(a)

$$y = x + \frac{4}{x} - 2$$

$$\frac{dy}{dx} = 1 - \frac{4}{x^2}$$

$$= \frac{x^2 - 4}{x^2}$$

$$= \frac{(x + 2)(x - 2)}{x^2}$$

At L, $x = 2$, $\therefore$ $\frac{dy}{dx} = 0$

(b)

$$\frac{dy}{dx} = \frac{x + 2}{x^2}(x - 2)$$

As x increases through 2, $\frac{x + 2}{x^2}$ is always positive and so both $(x - 2)$ and $\frac{dy}{dx}$ change sign from **negative to positive**.

In Example 1, we observe that

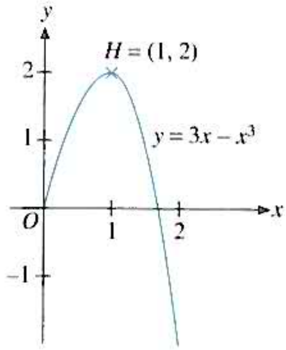
(i) $\frac{dy}{dx} = 0$ when $x = 2$ and so the point $L(2, 2)$ is a **stationary point**.

(ii) the gradient $\frac{dy}{dx}$ changes sign from negative to positive as x increases through 2 and the point L is a **minimum point**.

Example 2 The diagram shows part of the curve defined by $y = 3x - x^3$ for $x \geqslant 0$,

(a) Find $\frac{dy}{dx}$ and show that H is a stationary point.

(b) Show that $\frac{dy}{dx}$ changes sign from positive to negative as x increases through 1.



Solution:

(a) $y = 3x - x^3$

$$\frac{dy}{dx} = 3 - 3x^2 = \mathbf{3(1 - x^2)}$$

At $x = 1$, $\frac{dy}{dx} = 0$ and so **H is a stationary point**.

(b) $\frac{dy}{dx} = 3(1 + x)(1 - x)$

As x increases through 1, $3(1 + x)$ is always positive and so both $(1 - x)$ and $\frac{dy}{dx}$ change sign from **positive to negative**.

In Example 2, we observe that

(i) $\frac{dy}{dx} = 0$ when $x = 1$ and so $H(1, 2)$ is a stationary point.

(ii) the gradient, $\frac{dy}{dx}$, changes sign from positive to negative and the point H is a **maximum point**.

Example 3 **(a)** Show that the curve $y = 3 - \frac{2}{x}$, for $x > 0$, has no stationary point.

(b) Find the stationary point of the curve $y = 1 + x^3$.

Solution:

(a) $y = 3 - \frac{2}{x}$, $x > 0$

$$\frac{dy}{dx} = \frac{2}{x^2}$$

Since $x > 0$, the gradient, $\frac{dy}{dx}$, of the curve is always positive and the curve has no stationary point.

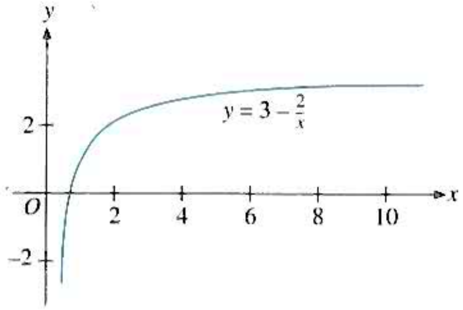
(b) $y = 1 + x^3$

$$\frac{dy}{dx} = 3x^2$$

For stationary point, $\frac{dy}{dx} = 0 \quad \Rightarrow \quad x = 0,\ y = 1.$

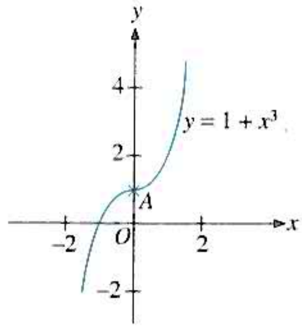
$\therefore$ the stationary point is **$A(0, 1)$**.

In Example 3(a), the curve does not have any stationary point as shown below.



In Example 3(b), $\frac{dy}{dx}$ does not change sign as x increases through 0 and the point A is neither a maximum nor a minimum point as shown in the diagram.

A stationary point is called a **turning point** if it is either a maximum point or a minimum point. Therefore, the points H and L (in Examples 1 and 2) are turning points. But the stationary point A (in Example 3(b)) is not a turning point.



Given a curve $y = f(x)$:

(a) $\frac{dy}{dx} = 0$ **at** $x = a \quad \Rightarrow \quad S(a, f(a))$ **is a stationary point.**

(b) For the stationary point at $x = a$,

(i) if $\frac{dy}{dx}$ changes sign from negative to positive as x increases through a, the point S is a minimum point,

(ii) if $\frac{dy}{dx}$ changes sign from positive to negative as x increases through a, the point S is a maximum point.

(c) A stationary point is called a turning point if it is either a maximum point or a minimum point.

Example 4 A curve is defined by $y = x(x-2)^3$.

(a) Show that $\frac{dy}{dx} = 2(2x-1)(x-2)^2$.

(b) Find the coordinates of the two stationary points on the curve.

(c) Observe the change in sign of $\frac{dy}{dx}$ as x increases through each of the stationary points. Deduce the nature of the point.

Solution:

(a)
$$y = x(x-2)^3$$
$$\Rightarrow \frac{dy}{dx} = x[3(x-2)^2] + (x-2)^3$$
$$= (x-2)^2[3x + (x-2)]$$
$$= 2(2x-1)(x-2)^2$$

(b) When $\frac{dy}{dx} = 0$, $x = \frac{1}{2}$ and $y = -\frac{27}{16}$ or $x = 2$ and $y = 0$ and so the stationary points are $A\left(\frac{1}{2}, -\frac{27}{16}\right)$ and $B(2, 0)$.

(c) As x increases through A, $\frac{dy}{dx}$ changes its sign from negative to positive and hence the stationary A is a minimum point.

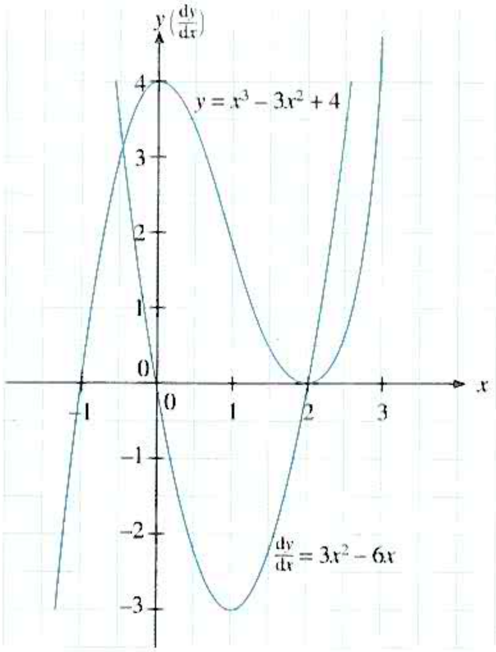
As x increases through B, $\frac{dy}{dx} \geqslant 0$ and hence B is not a turning point. It is in fact a point of inflexion.

Use a graph plotter to draw the curve of $y = x(x-2)^3$ and observe how the stationary points look.

Exercise 17.1

1. Find, by calculus, the stationary point of the curve $y = x^2 - 4x + 6$. Observe the changes in the sign of the gradient of the curve and determine the nature of the stationary point.
2. A curve is defined by $y = x^2(x-2)^2$.
   (a) Show that $\frac{dy}{dx} = 4x(x-1)(x-2)$.
   (b) Find the coordinates of the stationary points on the curve.
   (c) Observe the change in sign of $\frac{dy}{dx}$ as x increases through each of the stationary points and hence deduce the nature of the points.
3. A curve is defined by $y = 2x^3 - 9x^2 + 12$.
   (a) Show that $\frac{dy}{dx} = 6x(x-3)$.
   (b) Find the coordinates of the stationary points on the curve.
   (c) Observe the change in sign of $\frac{dy}{dx}$ as x increases through each of the stationary points and hence deduce the nature of the points.

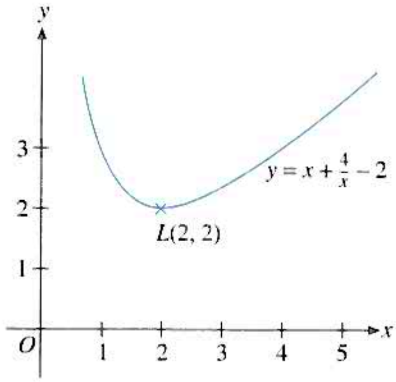
4. The diagram shows an application of a graph plotter, Graphmatica, to draw the graphs of the function $y = x^3 - 3x^2 + 4$ and its derivative function $\frac{dy}{dx} = 3x^2 - 6x$.
Observe the change in sign of $\frac{dy}{dx}$ as x increases through each of the stationary points $(0, 4)$ and $(2, 0)$. Hence deduce the nature of the points.
Repeat the above for the functions given in questions 1 to 3.

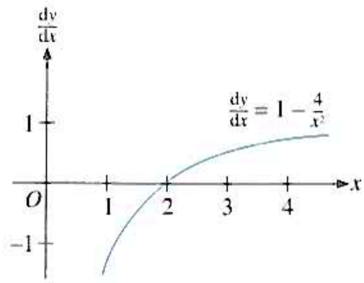


17.2 Maximum and Minimum Points

Higher Derivatives

The diagrams show the curve $y = x + \frac{4}{x} - 2$ in Example 1 and the curve of its gradient function $\frac{dy}{dx} = 1 - \frac{4}{x^2}$.





From the graph of $\frac{dy}{dx}$ against x, we observe that :

$\frac{dy}{dx}$ increases as x increases through 2 $\Leftrightarrow$ rate of change of $\frac{dy}{dx}$ is positive at $x = 2$

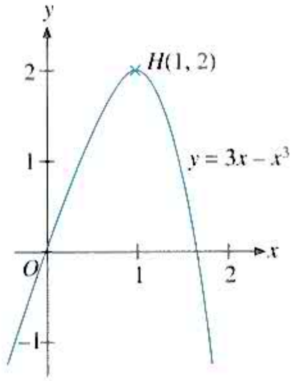
$$\Leftrightarrow \quad \frac{d}{dx}\left(\frac{dy}{dx}\right) > 0 \text{ at } x = 2$$

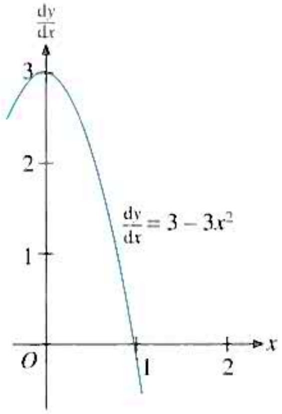
Differentiating $\frac{dy}{dx}$ w.r.t. x, we have $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(1 - \frac{4}{x^2}\right) = \frac{8}{x^3}$.

At $x = 2$, $\frac{d}{dx}\left(\frac{dy}{dx}\right) = 1 > 0$ as observed.

From the above, we can conclude that the stationary point $L(2, 2)$ at which $\frac{dy}{dx} = 0$ and $\frac{d}{dx}\left(\frac{dy}{dx}\right) > 0$ is a minimum point.

Similarly, the following diagrams show the curve $y = 3x - x^3$ in Example 2 and the curve of its gradient function $\frac{dy}{dx} = 3 - 3x^2$.





From the graph of $\frac{dy}{dx}$ against x, we observe that:

$\frac{dy}{dx}$ decreases as x increases through 1, $\Leftrightarrow$ rate of change of $\frac{dy}{dx}$ is negative at $x = 1$

$$\Leftrightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) < 0 \text{ at } x = 1$$

Differentiating $\frac{dy}{dx}$ w.r.t. x, we have $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(3 - 3x^2) = -6x$.

At $x = 1$, $\frac{d}{dx}\left(\frac{dy}{dx}\right) = -6 < 0$ as observed.

From the above, we can conclude that the stationary point $H(1, 2)$ at which $\frac{dy}{dx} = 0$ and $\frac{d}{dx}\left(\frac{dy}{dx}\right) < 0$ is a maximum point.

The derivative of $\frac{dy}{dx}$, that is $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, is usually denoted by $\frac{d^2y}{dx^2}$ and is called the **second derivative** of y with respect to x. Similarly, the derivative of $\frac{d^2y}{dx^2}$, $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ is usually denoted by $\frac{d^3y}{dx^3}$ and is called the **third derivative** of y with respect to x and so on.

For $y = 3x - x^3$, $\frac{dy}{dx} = 3 - 3x^2$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(3 - 3x^2) = -6x$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}(-6x) = -6$$

We may use higher derivatives to determine the nature of a turning point.

Given a curve $y = f(x)$:

(a) $\dfrac{dy}{dx} = 0$ **and** $\dfrac{d^2y}{dx^2} \neq 0$ **at** $x = a \Rightarrow S(a, f(a))$ **is a turning point.**

(b) For a turning point at $x = a$,

(i) if $\dfrac{d^2y}{dx^2} > 0$, then S is a minimum point,

(ii) if $\dfrac{d^2y}{dx^2} < 0$, then S is a maximum point.

Example 5 Find by calculus, the turning point on the quadratic curve $y = 18x - 23 - 3x^2$ for $1 \leqslant x \leqslant 5$. Sketch the curve.

Solution: $y = 18x - 23 - 3x^2$

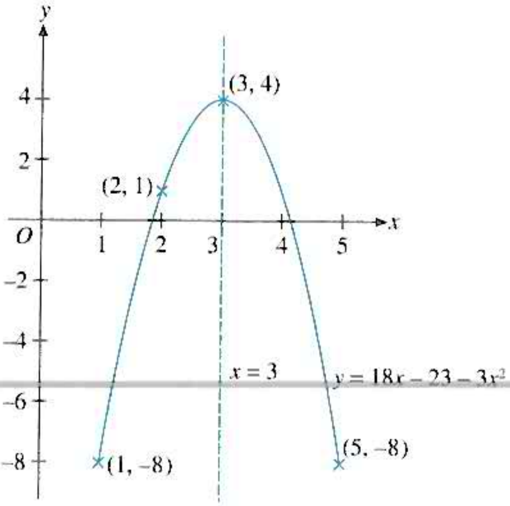
Differentiating w.r.t. x, we have

$$\frac{dy}{dx} = 18 - 6x$$
$$= 6(3 - x)$$
$$\frac{d^2y}{dx^2} = -6$$

When $\dfrac{dy}{dx} = 0$, $x = 3$ and $y = 4$.

Since $\dfrac{d^2y}{dx^2} < 0$, the turning point $(3, 4)$ is a maximum point.

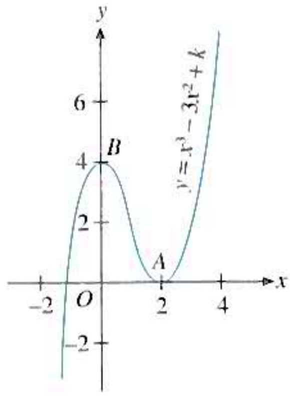
The curve is sketched as shown.



Example 6 The diagram shows part of the curve $y = x^3 - 3x^2 + k$ which touches the x-axis at A and cuts the y-axis at B. Find

(a) the value of k and the value of $\frac{d^2y}{dx^2}$ at A,

(b) the coordinates of B and the value of $\frac{d^2y}{dx^2}$ at B.



Solution:

(a) $y = x^3 - 3x^2 + k$

$\frac{dy}{dx} = 3x^2 - 6x$

The curve touches the x-axis at A,

$\Rightarrow$ A is a stationary point.

$\Rightarrow$ $\frac{dy}{dx} = 0$ at A

$\Rightarrow$ $3x^2 - 6x = 0$

$\Rightarrow$ $x = 0$ or $x = 2$

Since $x > 0$, the coordinates of A are (2, 0).

At A, $2^3 - 3(2)^2 + k = 0$

Hence $k = \mathbf{4}$ and $\frac{d^2y}{dx^2} = 6x - 6 = \mathbf{6}$.

(b) At B, $x = 0$, $y = 4$ and $\frac{d^2y}{dx^2} = \mathbf{-6}$.

The coordinates of B are **(0, 4)**.

Use Graphmatica to draw the graphs of $y = x^3 - 3x^2 + 4$ and $\frac{dy}{dx} = 3x^2 - 6x$ as shown above.

Draw the tangent lines to the curve $\frac{dy}{dx} = 3x^2 - 6x$ at $A(2,0)$ and $B(0, 4)$ and hence find the value of $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ $\left(\text{i.e. } \frac{d^2y}{dx^2}\right)$ at A and at B. Compare your results with the calculated values of $\frac{d^2y}{dx^2}$ in Example 6.

Example 7 Find the value of x, where $x > 0$, for which the curve $y = x^2 + \frac{16}{x}$ has a stationary point and determine whether it is a maximum or a minimum point.

Solution:

$y = x^2 + \frac{16}{x}, x > 0$

$\frac{dy}{dx} = 2x - \frac{16}{x^2}$

At a stationary point, $\frac{dy}{dx} = 0$,

i.e. $$2x - \frac{16}{x^2} = 0$$
$$x^3 = 8$$
$$x = 2$$

Differentiating $\frac{dy}{dx}$ w.r.t. x, we have $\frac{d^2y}{dx^2} = 2 + \frac{32}{x^3}$.

At $x = 2$, $\frac{d^2y}{dx^2} = 6$, which is positive and so the stationary point $S(2, 12)$ is a **minimum point.**

Exercise 17.2

1. Find the coordinates of the stationary points of the following curves and determine the nature of each point.

(a) $y = x^2 - 5x + 1$ **(b)** $y = 5 - 6x + x^2$ **(c)** $y = x^3 - 12x$

(d) $y = x^4 - 8x^2 + 2$ **(e)** $y = x(x - 6)^2$ **(f)** $y = x + \frac{16}{x}$

2. Find the coordinates of the turning points of the following curves. Determine in each case whether the point is a maximum or a minimum point.

(a) $y = x^4 - 4x + 1$ **(b)** $y = x^2(x - 3)$ **(c)** $y = 2x + \frac{18}{x}$

(d) $y = x^2 + \frac{16}{x}$ **(e)** $y = \frac{4x^2 + 9}{x}$ **(f)** $y = \frac{x^2}{x + 1}$

3. Find the coordinates of the turning point of the curve $y = 8x + \frac{1}{2x^2}$ and determine whether this point is a maximum or minimum point.

4. Show that the curve $y = \frac{2x - 1}{1 - x}$ has neither a maximum nor a minimum point.

5. Prove that the curve $y = 3x^4 + kx$ has only one turning point. Find the turning point when $k = 12$ and determine the nature of this point.

6. The curves $y = 2x^2 - 4x + 5$ and $y = x^3 - ax^2 + x + b$ have a common turning point. Find

(a) the coordinates of the turning point, **(b)** the value of a and of b.

7. The graph of $y = 2x^3 + ax^2 + b$ has a stationary point $(-3, 19)$. Find the value of a and b. Determine whether this stationary point is a maximum or a minimum.

8. The curve $y = ax + \frac{b}{2x - 1}$ has a stationary point at $(2, 7)$. Find

(a) the value of a and b, **(b)** the other stationary point.

9. Given that $y = \sqrt{4x - 7} - \frac{2}{3}(x - 4)$, find $\frac{dy}{dx}$ and show that $\frac{d^2y}{dx^2} = \frac{-4}{(4x - 7)\sqrt{4x - 7}}$.
Hence find the maximum point of the curve.

17.3 Maximum and Minimum Values

In Example 5, the curve $y = 18x - 23 - 3x^2$ for $1 \leqslant x \leqslant 5$, has only one stationary point (3, 4) and it is a maximum point. This means that y has the stationary value 4 and it is the **maximum value** when $x = 3$.

Similarly, in Example 7, the curve $y = x^2 + \frac{16}{x}$ for $x > 0$, has only one stationary point (2, 12) and it is a minimum point. This means that y has the stationary value 12 and it is the **minimum value** when $x = 2$.

In general, for a function $y = f(x)$:

(a) If $\frac{dy}{dx} = 0$ at $x = a$, $f(a)$ is a stationary value of y.

(b) If $\frac{dy}{dx} = 0$ only at $x = a$ and $\frac{d^2y}{dx^2} > 0$, then $f(a)$ is the minimum value of y.

(c) If $\frac{dy}{dx} = 0$ only at $x = a$ and $\frac{d^2y}{dx^2} < 0$, then $f(a)$ is the maximum value of y.

Example 8

Two positive numbers x and y vary in such a way that $xy = 18$. Another number z is defined by $z = 2x + y$. Find the values of x and y for which z has a stationary value and show that this value of z is a minimum.

Solution:

$$xy = 18 \Rightarrow y = \frac{18}{x}$$

Then,
$$z = 2x + y = 2x + \frac{18}{x}$$

Differentiating w.r.t. x, we have $\frac{dz}{dx} = 2 - \frac{18}{x^2} = 2\left(1 - \frac{9}{x^2}\right)$

For a stationary value, $\frac{dz}{dx} = 0 \quad \Rightarrow \quad x^2 = 9$.

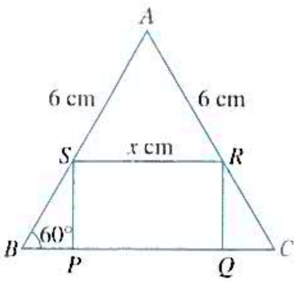
Since $x > 0$, $x = 3$ and thus $y = 6$.

Next, $\frac{d^2z}{dx^2} = \frac{36}{x^3}$.

When $x = 3$, $\frac{d^2z}{dx^2} = \frac{4}{3}$ (> 0), giving a minimum value of z.

Note: The function $z = 2x + \frac{18}{x}$, $x > 0$, has only one stationary value, i.e. when $x = 3$.

Example 9 An equilateral triangle ABC of side 6 cm is to have a rectangle $PQRS$ inscribed in it as shown in the diagram. Show that the area of the rectangle, A cm^2, is given by $A = \frac{\sqrt{3}}{2}x(6 - x)$, where $PQ = x$ cm. Calculate the value of x for which A has a stationary value. Show that this value of x makes A a maximum.



Solution:

$PQ = x$ cm

$BP = \frac{1}{2}(6 - x)$ cm

$SP = BP \tan 60° = \frac{\sqrt{3}}{2}(6 - x)$ cm

Area $A = PQ \times SP = \mathbf{\frac{\sqrt{3}}{2}x(6 - x)}$ **cm^2** (shown)

$\frac{dA}{dx} = \frac{\sqrt{3}}{2}(6 - 2x) = \sqrt{3}(3 - x)$

At $x = \mathbf{3}$, $\frac{dA}{dx} = 0$ and so A has a stationary value.

Since $\frac{d^2A}{dx^2} = -\sqrt{3}$, which is negative, A is a maximum when $x = 3$.

Example 10 An open paper box with a square base of side x cm is made from a vanguard sheet of area 75 cm^2. Show that its volume, V cm^3, is given by $V = \frac{1}{4}(75x - x^3)$. Find the value of x for which V is a maximum and find the maximum volume.

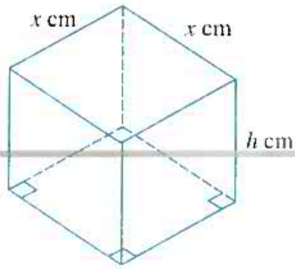
Solution:

Let the height of the box be h cm.

Total surface area of the box = Area of the vanguard sheet

$$x^2 + 4xh = 75$$

$$h = \frac{75 - x^2}{4x}$$



Volume of the box, $V = x^2h$

$$= x^2\left(\frac{75 - x^2}{4x}\right)$$

$$= \frac{1}{4}(75x - x^3)$$

Differentiating w.r.t. x, $\quad \dfrac{dV}{dx} = \dfrac{1}{4}(75 - 3x^2)$

$$= \frac{3}{4}(25 - x^2)$$

and $\quad \dfrac{d^2V}{dx^2} = -\dfrac{3}{2}x$

For a maximum value of V,

$$\frac{dV}{dx} = 0$$

$$\Rightarrow \quad x^2 = 25$$

Since $x \geqslant 0$, $\quad x = 5$

and so, $\quad \dfrac{d^2V}{dx^2} = -\dfrac{15}{2} < 0.$

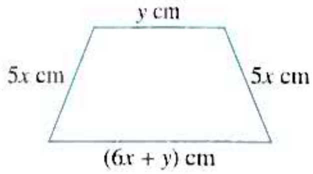
Hence for the volume, V, to be a maximum, $x = 5$ and the maximum volume is $\frac{1}{4}(75 \times 5 - 5^3) = 62\frac{1}{2}$ **cm**^{3}.

Note: $x \geqslant 0$, $h \geqslant 0$ and $V \geqslant 0$. In this solution, $x \geqslant 0$ is used to reject -5 as a value of x.

Exercise 17.3

1. For each of the following expressions, calculate the value of x which gives y a stationary value. Determining whether this value of y is a minimum or a maximum. Give your answers correct to 3 significant figures where necessary.
   (a) $y = 2x^2 - 8x + 3$ **(b)** $y = x^2 + (2x - 1)^2$
   (c) $y = \sqrt{3}x - 6x^2 + 1$ **(d)** $y = 2\pi x - (x - 2)^2$
2. Find the stationary values of the following functions.
   (a) $y = (x - 2)^4 + 3$ **(b)** $y = (x - 3)(x + 2)$
   (c) $y = x\sqrt{1 - 2x}$ **(d)** $y = \dfrac{x}{x^2 + 1}$
3. If $2x + y = 10$ and $A = xy$, find the maximum value of A.
4. Two variables x and y vary in such a way that $x + y = 2$. Another variable z is defined by $z = x^2 + y^2$. Find the values of x and y that make z a minimum.
5. The positive variables x and y are such that $x^4y = 32$. A third variable z is defined by $z = x^2 + y$. Find the values of x and y that give z a stationary value and show that this value of z is a minimum.
6. A rectangle has sides x cm and y cm. If the area of the rectangle is 16 cm^2, show that its perimeter, P cm, is given by $P = 2x + \dfrac{32}{x}$. Hence, calculate the value of x which gives P a stationary value and show that this value of P is a minimum.
7. If the perimeter of a rectangle is to be 80 m, calculate the maximum area.

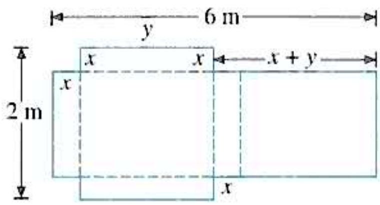
8. A piece of wire of length 104 cm, is bent to form a trapezium as shown in the diagram. Express y in terms of x and show that the area, A cm^2, enclosed by the wire is given by $A = 208x - 20x^2$. Find the value of x and of y for which A is a maximum.



9. A rectangle block has a total surface area of 1.08 m^2. The dimensions of the block are x m, $2x$ m and h m. Show that $h = \dfrac{1.08 - 4x^2}{6x}$ and hence express the volume of the block in terms of x. Find the value of x that makes this volume a maximum. (Proof that it is a maximum is not required.) *(C)*

10. In an upright triangular prism, the triangular base ABC is right-angled at B, $AB = 5x$ cm and $BC = 12x$ cm. The sum of the lengths of all its edges is 180 cm.
    (a) Show that the volume, V cm^3, is given by $V = 1800x^2 - 600x^3$.
    (b) Find the value of x for which V has a maximum value.

11. From a rectangular piece of metal of width 2 m and length 6 m, two squares of side x m and two rectangles of sides x m and $(x + y)$ m are removed as shown in the figure. The metal is then folded about the dotted lines to give a closed box with height x m.
    Show that the volume of the box, V m^3, is given by $V = 2x^3 - 8x^2 + 6x$.
    Find, to 3 significant figures, the value of x and of y for which V has a maximum value and show that this value of V is indeed a maximum.

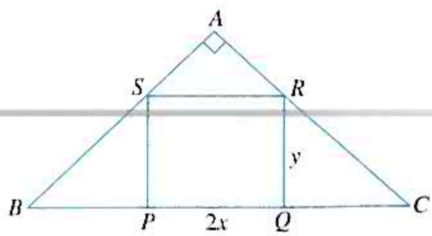


12. A solid circular cylinder has radius r cm and height h cm. It has a fixed volume of 400 cm^3. Show that the total surface area, A cm^2, of the cylinder is given by
$$A = 2\pi r^2 + \frac{800}{r}.$$
    Find, to 3 significant figures, the value of r that gives the cylinder its minimum surface area.

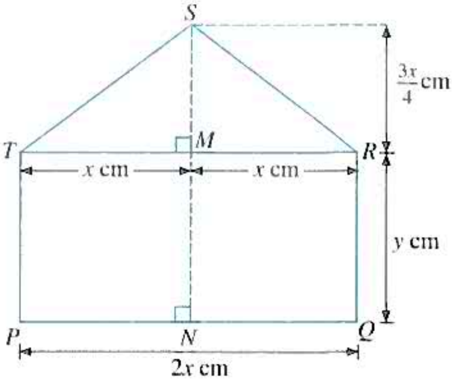
13. A metal sheet of area 100 cm^2 is used to manufacture a closed cylinder can. Find, to two significant figures, the largest possible volume of the can.

14. An open rectangular box of height h cm has a horizontal rectangular base of sides x cm and $2x$ cm. If the volume of the box is 36 cm^3, express h in terms of x and show that the total surface area, A cm^2, of the box is given by $A = 2x^2 + \dfrac{108}{x}$.
    Calculate the value of x and of h which make the total surface area a minimum.

15. The figure shows a triangle ABC where $AB = AC$, $BC = 18$ cm and A is a right angle. $PQRS$ is a rectangle with $PQ = 2x$ cm and $QR = y$ cm. Show that the area, A cm^2, of $PQRS$ is given by $A = 18x - 2x^2$.
    Hence find the maximum area of the rectangle $PQRS$.

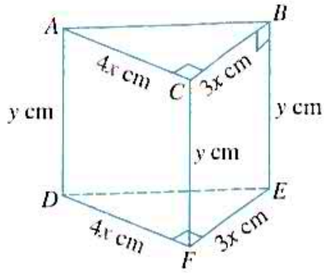


16. The diagram shows a metal plate consisting of a rectangle $PQRT$ and an isosceles triangle RST, where $PQ = 2x$ cm, $QR = y$ cm and $SM = \frac{3x}{4}$ cm. Given that the perimeter of $PQRST$ is 30 cm, show that the area, A cm^2, of $PQRST$ is given by $A = 30x - \frac{15x^2}{4}$.



Calculate
(a) the maximum value of A,
(b) the length of the perpendicular SN from S to PQ when A is a maximum.

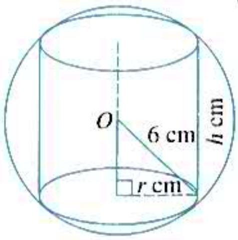
17. The solid prism shown in the diagram has triangular ends perpendicular to the parallel edges AD, BE and CF. The length of each of the edges AD, BE and CF is y cm. The triangles have right angles at C and F. The edges AC and DF are each of length $4x$ cm and the edges BC and EF are each of length $3x$ cm. Given that the volume of the prism is 1500 cm^3,



(a) obtain an expression for y in terms of x,
(b) show that the total surface area, S cm^2, is given by $S = 12x^2 + \frac{3000}{x}$.

Given that the volume remains constant and that x varies, find
(c) an expression for $\frac{dS}{dx}$ in terms of x,
(d) the stationary value of S. *(C)*

18. A cylinder of base radius r cm and height h cm is inscribed in a sphere of radius 6 cm, centre O.
(a) Show that $h^2 + 4r^2 = 144$.
(b) Find the largest possible volume of the cylinder.



Important Notes

1. Maximum and minimum points of a curve $y = f(x)$

(a) If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} \neq 0$ and $x = a$, then $S(a, f(a))$ is a turning point.
(b) For a turning point at $x = a$,
(i) if $\frac{d^2y}{dx^2} > 0$, then S is a minimum point.
(ii) if $\frac{d^2y}{dx^2} < 0$, then S is a maximum point.

2. **Maximum and minimum values of a function $y = f(x)$**

(a) If $\frac{dy}{dx} = 0$ when $x = a$, $f(a)$ is a stationary value of y.

(b) If $f(a)$ is the only stationary value and $\frac{d^2y}{dx^2} > 0$ when $x = a$, then $f(a)$ is the minimum value of y.

(c) If $f(a)$ is the only stationary value and $\frac{d^2y}{dx^2} < 0$ when $x = a$, then $f(a)$ is the maximum value of y.

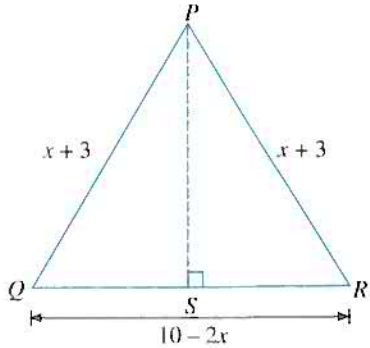
Miscellaneous Examples

Example 11

The lengths of the sides of a triangle in cm are $x + 3$, $x + 3$ and $10 - 2x$. Show that the area, A cm^2, of the triangle is given by $A = 4(5 - x)\sqrt{x - 1}$.
Determine, as x varies within $1 < x < 5$, the maximum area of the triangle.

Solution:

In the diagram,



$$QS = \frac{1}{2}QR = \frac{1}{2}(10 - 2x)$$
$$= 5 - x$$

$$PS^2 = PQ^2 - QS^2$$
$$= (x + 3)^2 - (5 - x)^2$$
$$= 16(x - 1)$$

$$PS = 4\sqrt{x - 1}$$

$$A = \frac{1}{2} \times QR \times PS$$
$$= \frac{1}{2} \times (10 - 2x) \times 4\sqrt{x - 1}$$
$$= \mathbf{4(5 - x)\sqrt{x - 1}}$$

By the product rule,

$$\frac{dA}{dx} = 4\left[(5 - x)\frac{d}{dx}\left(\sqrt{x - 1}\right) + \sqrt{x - 1}\frac{d}{dx}(5 - x)\right]$$
$$= 4\left[(5 - x)\left(\frac{1}{2\sqrt{x - 1}}\right) - \sqrt{x - 1}\right]$$
$$= 4\left[\frac{(5 - x) - 2(x - 1)}{2\sqrt{x - 1}}\right]$$
$$= \frac{2(7 - 3x)}{\sqrt{x - 1}}$$

When $\frac{dA}{dx} = 0$, $x = \frac{7}{3}$ and $A = 4\left(5 - \frac{7}{3}\right)\sqrt{\frac{7}{3} - 1} = \frac{64}{3\sqrt{3}}$

By the quotient rule,

$$\frac{d^2A}{dx^2} = 2\left[\frac{\sqrt{x-1}\frac{d}{dx}(7-3x) - (7-3x)\frac{d}{dx}\left(\sqrt{x-1}\right)}{\left(\sqrt{x-1}\right)^2}\right]$$

$$= 2\left[\frac{-3\sqrt{x-1} - (7-3x)\frac{1}{2\sqrt{x-1}}}{\left(\sqrt{x-1}\right)^2}\right]$$

$$= 2\left[\frac{-6(x-1) - (7-3x)}{2(x-1)\sqrt{x-1}}\right]$$

$$= -\frac{3x+1}{(x-1)\sqrt{x-1}}$$

When $x = \frac{7}{3}$, $\frac{d^2A}{dx^2} = -3\sqrt{3}$, which is negative, and hence the corresponding value of A is a maximum.

Note: **(1)** Alternatively, we can observe the change in the sign of $\frac{dA}{dx}$ to determine the stationary value as a maximum value.

(2) In the $\triangle PQR$, $QP + PR > QR$,
i.e. $x + 3 + x + 3 > 10 - 2x \quad \Leftrightarrow \quad x > 1$
and $QP + QR > PR$.
i.e. $x + 3 + 10 - 2x > x + 3 \quad \Leftrightarrow \quad x < 5$

(3) The perimeter is a constant.

Miscellaneous Exercise 17

1. Calculate the coordinates of the stationary points of the curve $y = x^2 + \frac{16}{x^2}$. Determine whether each of these points is a maximum point or a minimum point.

2. Find the turning point of the curve $y = 4x^2 + \frac{27}{x}$ where $x > 0$. Determine the nature of the point and find the minimum value of y.

3. Find the stationary value of $(x - 2)\sqrt{5 - x}$.

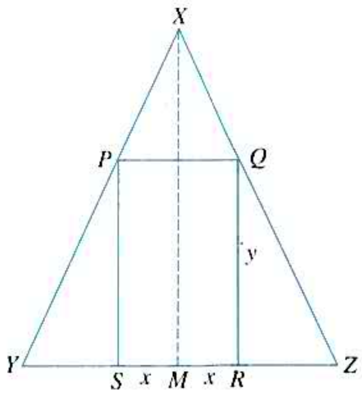
4. The curve $y = ax^3 + 2x^2 + a^2x + b$ has a minimum point at $(-1, 0)$. Find
(a) the value of a and of b,
(b) the coordinates of the other turning point.

5. Two positive quantities p and q vary in such a way that $p^2q = 9$. Another quantity z is defined by $z = 16p + 3q$. Find the values of p and q that make z a minimum.

6. A piece of wire, 360 cm long, is used to make the twelve edges of a rectangular box in which the length is twice the breadth. Denoting the breadth of the box by x cm and the height by h cm, express h in terms of x and show that the volume of the box, V cm^3, is given by $V = 180x^2 - 6x^3$. Determine, as x varies, the maximum volume of the box. Show that this volume is a maximum and not a minimum. *(C)*

7. A piece of wire, 100 cm in length, is divided into two parts. One part is bent to form a square of side x cm and the other is bent to form a square of side y cm. Express y in terms of x. Show that A cm^2, the total area enclosed by the two squares is given by $A = x^2 + (25 - x)^2$. Calculate the value of x for which A has a stationary value. Determine whether this value of x makes A a maximum or a minimum.

8. The triangle XYZ shown in the figure has $XY = XZ = 13$ cm and $YZ = 10$ cm. The rectangle $PQRS$ is inscribed in the triangle so that PQ is parallel to YZ and PS and QR are parallel to XM where M is the midpoint of YZ. Given that $SM = MR = x$ cm and $QR = y$ cm, express y in terms of x, and show that if the area of the rectangle is A cm^2, then $A = \frac{24x}{5}(5 - x)$. Hence find the value of x for which the area is a maximum and the value of the maximum area. *(C)*



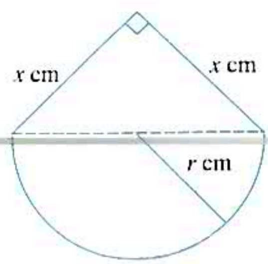
9. A rectangular box is such that the sides of its base are x cm and $2x$ cm. The volume of the box is 100 cm^3. Show that the total surface area, S cm^2, of the box is $S = 4x^2 + \frac{300}{x}$. Calculate, to 3 significant figures, the stationary value of S. Determine whether this value is a maximum or a minimum.

10. A piece of wire, 80 cm long, is bent to form the shape shown in the figure. This shape consists of a semicircular arc, radius, r cm, and two sides, each of length x cm, of a right-angled triangle.

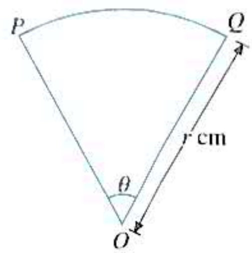
Show that the area enclosed, A cm^2, is given by $A = \frac{1}{2}\pi r^2 + \frac{1}{8}(80 - \pi r)^2$.

Hence determine, to 3 significant figures, the value of r for which A is either a maximum or a minimum.

Determine whether this value of r makes A a maximum or a minimum.



11. A piece of wire, of fixed length L cm, is bent to form the boundary $OPQO$ of a sector of a circle. The circle has centre O and radius r cm. The angle of the sector is θ radians.



Show that the area, A cm^2, of the sector is given by $A = \frac{1}{2}rL - r^2$.

(a) Find the relationship between r and L for which A has a stationary value and find the corresponding value of θ. Determine the nature of this stationary value.

(b) Show that, for this value of θ, the area of the triangle OPQ is approximately 45.5% of the area of the sector OPQ. *(C)*

12. A closed cylindrical can, of base radius r cm and height h cm, is to be constructed to hold 250π cm^3. Show that the total area, A cm^2, of material required to make the can is given by $A = 2\pi r^2 + \dfrac{500\pi}{r}$.

The material for the curved side of the can costs 0.05¢ per square centimetre and the material for the top and base costs 0.03¢ per square centimetre. Find the dimensions of the can if

(a) the area of the material used is to be the least,

(b) the cost of the material used is to be the least (where necessary, give your answers correct to 3 significant figures).

13. The lengths of the sides of a triangle are $6 + x$, $6 + x$ and $16 - 2x$ units.

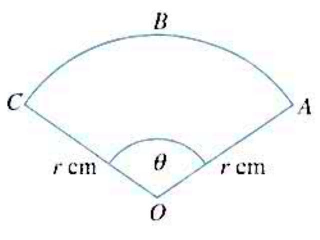
(a) Show that the area of the triangle is $(8 - x)\sqrt{28(x - 1)}$.

(b) Find the value of x which makes this area a maximum.

14. A piece of wire of length l cm is bent to form a rectangle. Show that the area of the rectangle is maximum when it is a square.

15. If α is a positive root of the equation $2x^2 + px + 18 = 0$, calculate the value of α for which p has a stationary value and determine whether this value of p is a maximum or a minimum.

16. **(a)** The diagram shows a piece of wire bent to form the perimeter $OABCO$ of a sector of a circle, centre O, radius r cm, where angle AOC is θ radians. The wire is of length 100 cm and r and θ may vary. Find

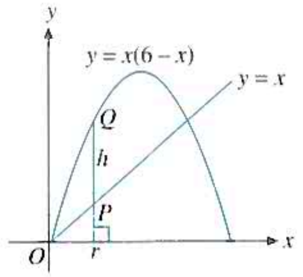


(i) the value of r for which the area enclosed by the wire is a maximum,

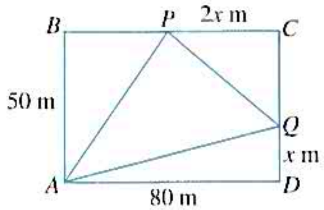
(ii) the corresponding value of the angle AOC in **degrees**.

(b) Given that $y = -\dfrac{2}{3x^3} + \dfrac{5}{2x^2} - \dfrac{2}{x}$, find the two values of x for which y is stationary. Show that the larger of these values of x corresponds to a minimum value of y. *(C)*

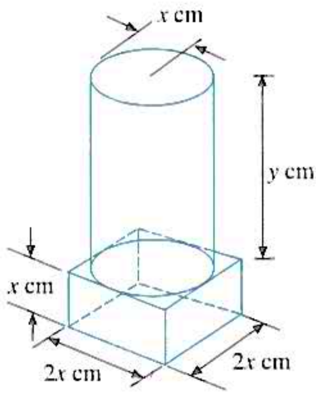
17. **(a)** The diagram shows part of the line $y = x$ and part of the curve $y = x(6 - x)$. The point P lies on the line and the point Q lies on the curve. The x-coordinate of both P and Q is r, where $0 \leqslant r \leqslant 5$, and the distance PQ is denoted by h. Express h in terms of r and hence find the greatest value of h as r varies.



(b) The diagram shows a rectangular field $ABCD$ with $AB = 50$ m and $AD = 80$ m. The field is partitioned by three straight fences AP, AQ and PQ. The distance of P from C is twice the distance of Q from D. Given that $DQ = x$ m, show that the area, A m^2, of triangle APQ is given by $A = x^2 - 40x + 2000$.
Given that x varies, find the stationary value of A and determine whether this is a maximum or a minimum value. *(C)*

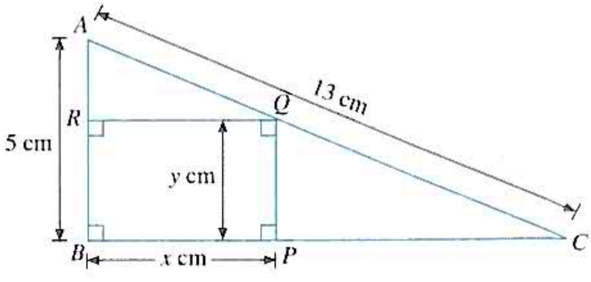


18. The diagram shows a solid body which consists of a right circular cylinder fixed, with no overlap, to a rectangular block. The block has a square base of side $2x$ cm and a height of x cm. The cylinder has a radius of x cm and a height of y cm. Given that the total volume of the solid is 27 cm^3, express y in terms of x.
Hence show that the total surface area, A cm^2, of the solid is given by $A = \dfrac{54}{x} + 8x^2$.



Find
(a) the value of x for which A has a stationary value,
(b) the value of A and of y corresponding to this value of x. Determine whether the stationary value of A is a maximum or a minimum. *(C)*

19. **(a)** Find the coordinates of the stationary point on the curve $y = \dfrac{16x^3 + 4x^2 + 1}{2x^2}$ and determine the nature of this point.



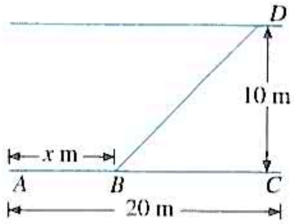
(b) In the triangle ABC, angle $ABC = 90°$, $AB = 5$ cm and $AC = 13$ cm. The rectangle $BPQR$ is such that its vertices P, Q and R lie on BC, CA and AB respectively.
Given that $BP = x$ cm and $PQ = y$ cm, prove that $y = \dfrac{60 - 5x}{12}$.
Express the area of the rectangle in terms of x and hence calculate the maximum value of this area as x varies. *(C)*

20. Given that $y = \frac{4}{2-x} + \frac{9}{x-3}$, find

(a) $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$,

(b) the stationary values of y and determine the nature of these values.

***21.** In the diagram, a man walks a distance of x m at a speed of 1.25 m s^{-1} from A to B. Then he swims across a river directly from B to D at a speed of 1 m s^{-1}. Given that D is directly opposite to C, $CD = 10$ m and $AC = 20$ m, show that the time, t seconds, taken to reach D is given by $t = \sqrt{100 + (20-x)^2} + \frac{4x}{5}$.



Find the value of x for which t is a minimum. (Proof that it is a minimum is not required.)

***22.** Which point on the curve $y = x^2$ is nearest to the point (3, 0)?

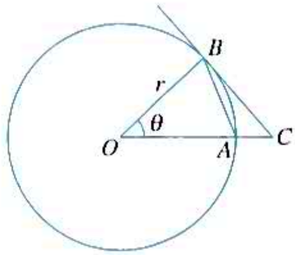
18 Derivatives of Trigonometric Functions

In this chapter, we will first find the derivatives, from first principles, of the sine function $y = \sin x$ and the cosine function $y = \cos x$ and then apply the chain rule, the product rule and the quotient rule for differentiation of related functions.

18.1 Differentiation of Trigonometric Functions

Differentiation of sin x, cos x and tan x

The diagram below shows the tangent at B to a circle of radius r and the angle subtended by the arc AB is θ radians where $0 < \theta < \frac{\pi}{2}$. Then,



area of $\triangle OAB <$ area of sector $OAB <$ area of $\triangle OBC$,

$$\frac{1}{2}r^2 \sin\theta < \frac{1}{2}r^2\theta < \frac{1}{2}r^2 \tan\theta,$$

$$\sin\theta < \theta < \tan\theta.$$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta} \quad \because \sin\theta > 0$$

As $\theta \to 0$, $\cos\theta \to 1$ and so $\dfrac{1}{\cos\theta} \to 1$.

The value of $\dfrac{\theta}{\sin\theta}$ is 'squeezed' between 1 and $\dfrac{1}{\cos\theta}$ and so $\dfrac{\theta}{\sin\theta} \to 1$ as $\theta \to 0$,

i.e. $\displaystyle\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

The Derivative of sin x

Let $y = \sin x$, where x is in radians.
Let a small change in x be δx and the consequent change in y be δy.

Then, $\quad y + \delta y = \sin(x + \delta x)$
and so $\quad \delta y = \sin(x + \delta x) - \sin x$.

Using $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$, we have:

$$\delta y = 2\cos\left(\frac{x+\delta x+x}{2}\right)\sin\left(\frac{x+\delta x+x}{2}\right)$$

$$= 2\cos\left(x+\frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)$$

$$\frac{\delta y}{\delta x} = \cos\left(x+\frac{\delta x}{2}\right)\frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}$$

As $\delta x \to 0$, $\cos\left(x+\frac{\delta x}{2}\right) \to \cos x$, $\dfrac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)} \to 1$, and so $\frac{\delta y}{\delta x} \to \cos x$.

Hence $\frac{dy}{dx} = \cos x$.

The Derivative of cos x

Let $y = \cos x$, where x is in radians.
Let a small change in x be δx and the consequent change in y be δy.
Then, $\quad y + \delta y = \cos(x + \delta x)$
and so $\quad \delta y = \cos(x + \delta x) - \cos x$.

Using $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$$\delta y = -2\sin\left(\frac{x+\delta x+x}{2}\right)\sin\left(\frac{x+\delta x-x}{2}\right)$$

$$= -2\sin\left(x+\frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)$$

$$\frac{\delta y}{\delta x} = -\sin\left(x+\frac{\delta x}{2}\right)\frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}$$

As $\delta x \to 0$, $\sin\left(x+\frac{\delta x}{2}\right) \to \sin x$, $\dfrac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)} \to 1$, and so $\frac{\delta y}{\delta x} \to -\sin x$.

Hence $\frac{dy}{dx} = -\sin x$.

The Derivative of tan x

Let $y = \tan x$, where x is in radians, i.e. $y = \frac{\sin x}{\cos x}$.

By the quotient rule, $\dfrac{dy}{dx} = \dfrac{\cos x \dfrac{d}{dx}(\sin x) - \sin x \dfrac{d}{dx}(\cos x)}{\cos^2 x}$

$$= \frac{\cos^2 x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

$\dfrac{d}{dx}(\sin x) = \cos x$**, where** x **is in radians.**

$\dfrac{d}{dx}(\cos x) = -\sin x$**, where** x **is in radians.**

$\dfrac{d}{dx}(\tan x) = \sec^2 x$**, where** x **is in radians.**

Example 1

Differentiate with respect to x.

(a) $x + 2\cos x$ **(b)** $2 + 3\sin x$

Solution:

(a) $\dfrac{d}{dx}(x + 2\cos x) = 1 + 2\dfrac{d}{dx}(\cos x)$

$= \mathbf{1 - 2\sin x}$

(b) $\dfrac{d}{dx}(2 + 3\sin x) = 3\dfrac{d}{dx}(\sin x)$

$= \mathbf{3\cos x}$

Example 2

Differentiate with respect to x.

(a) $x^2 \sin x$ **(b)** $(x + 1)\tan x$ **(c)** $\dfrac{\cos x}{x}$

Solution:

(a) By the product rule,

$\dfrac{d}{dx}(x^2 \sin x) = x^2\dfrac{d}{dx}(\sin x) + \sin x\dfrac{d}{dx}(x^2)$

$= \mathbf{x^2 \cos x + 2x \sin x}$

(b) $\dfrac{d}{dx}[(x + 1)\tan x] = (x + 1)\dfrac{d}{dx}(\tan x) + \tan x\dfrac{d}{dx}(x + 1)$

$= \mathbf{(x + 1)\sec^2 x + \tan x}$

(c) By the quotient rule,

$$\frac{d}{dx}\left(\frac{\cos x}{x}\right) = \frac{x\dfrac{d}{dx}(\cos x) - \cos x\dfrac{d}{dx}(x)}{x^2}$$

$$= \frac{-x\sin x - \cos x}{x^2}$$

$$= -\frac{x\sin x + \cos x}{x^2}$$

Example 3 By using $\sec x = \frac{1}{\cos x}$, show that $\frac{d}{dx}(\sec x) = \sec x \tan x$.

Solution: By the quotient rule,

$$\frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(1) - 1 \times \frac{d}{dx}(\cos x)}{\cos^2 x}$$
$$= \frac{0 - (-\sin x)}{\cos^2 x}$$
$$= \frac{1}{\cos x} \times \frac{\sin x}{\cos x}$$
$$= \sec x \tan x$$

Hence, $\frac{d}{dx}(\sec x) = \boldsymbol{\sec x \tan x}$

Example 4 Differentiate with respect to x.

(a) $(3 + \sin x)^3$ **(b)** $\sqrt{2 + \cos x}$

Solution: **(a)** Using $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$, we have:

$$\frac{d}{dx}[(3 + \sin x)^3] = 3(3 + \sin x)^2 \frac{d}{dx}(3 + \sin x)$$
$$= 3(3 + \sin x)^2(\cos x)$$
$$= \boldsymbol{3 \cos x(3 + \sin x)^2}$$

(b) Using $\frac{d}{dx}\left(\sqrt{u}\right) = \frac{1}{2\sqrt{u}}\frac{du}{dx}$, we have:

$$\frac{d}{dx}\left(\sqrt{2 + \cos x}\right) = \frac{1}{2\sqrt{2 + \cos x}}\frac{d}{dx}(2 + \cos x)$$

$$= \boldsymbol{\frac{-\sin x}{2\sqrt{2 + \cos x}}}$$

Example 5 The tangent and normal to the curve $y = \sin x$ at the point P where $x = \frac{\pi}{3}$ cut the x-axis at A and B respectively.

(a) Show that $AB = \frac{5\sqrt{3}}{4}$.

(b) Find the area of the triangle PAB.

Solution: **(a)** $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$

At P, $x = \frac{\pi}{3}$, $y = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{dy}{dx} = \cos\frac{\pi}{3} = \frac{1}{2}$.

The equation of the tangent at P is

$$y - \frac{\sqrt{3}}{2} = \frac{1}{2}\left(x - \frac{\pi}{3}\right)$$

and the coordinates of A are

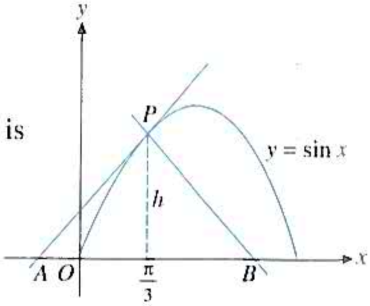
$$\left(\frac{\pi}{3} - \sqrt{3}, 0\right).$$

The equation of the normal at P is

$$y - \frac{\sqrt{3}}{2} = -2\left(x - \frac{\pi}{3}\right)$$

and the coordinates of B are

$$\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}, 0\right).$$



Hence $AB = \left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right) - \left(\frac{\pi}{3} - \sqrt{3}\right) = \frac{5\sqrt{3}}{4}$ **units.**

(b) Area of $\triangle PAB = \frac{1}{2} \times AB \times h = \frac{1}{2} \times \frac{5\sqrt{3}}{4} \times \frac{\sqrt{3}}{2}$

$= \frac{15}{16}$ **sq. units.**

Example 6

Find the approximate change in $\sin x$ if x increases from $\frac{\pi}{3}$ to $\frac{11\pi}{30}$ radians. Hence write down the value of $\sin\left(\frac{11\pi}{30}\right)$, giving your answer correct to 3 significant figures.

Solution:

Let $y = \sin x$. Then $\frac{dy}{dx} = \cos x$.

When $x = \frac{\pi}{3}$, $\frac{dy}{dx} = \cos\frac{\pi}{3} = \frac{1}{2}$.

Now, $\delta x = \frac{11\pi}{30} - \frac{\pi}{3} = \frac{\pi}{30}$

The approximate change in $\sin x$ is $\delta y \approx \frac{dy}{dx}\delta x = \frac{1}{2} \times \frac{\pi}{30} = \frac{\pi}{60}$

Next, using $f(x + \delta x) \approx f(x) + \frac{dy}{dx}\delta x$, we have:

$\sin\left(\frac{11\pi}{30}\right) \approx \sin\frac{\pi}{3} + \frac{\pi}{60} = \frac{\sqrt{3}}{2} + \frac{\pi}{60} = $ **0.918** (to 3 sig. fig.)

Example 7

Determine the value of x, where $0 \leqslant x \leqslant \pi$, for which the curve

$$y = 2\sin x + 5\cos x$$

has a stationary point and determine whether this is a maximum point or a minimum point.

Solution:

$$y = 2\sin x + 5\cos x$$

$$\frac{dy}{dx} = 2\cos x - 5\sin x$$

At the stationary point, $\frac{dy}{dx} = 0$.

$$\Rightarrow \quad 2\cos x - 5\sin x = 0$$
$$\tan x = \frac{2}{5}$$
$$x = \mathbf{0.381} \quad \text{(to 3 sig. fig.)}$$

When $x = 0.381$, $\frac{d^2y}{dx^2} = -2\sin x - 5\cos x$

$$= -5.39$$

Since $\frac{d^2y}{dx^2} < 0$, the point is a **maximum point**.

Note: Without evaluating $\frac{d^2y}{dx^2}$, we can easily observe that $\frac{d^2y}{dx^2} < 0$ as x is in the 1st quadrant.

Differentiation of sin ($ax + b$), cos ($ax + b$) and tan ($ax + b$), where a and b are constants

Let $y = \sin(ax + b)$

$u = ax + b$

Then $y = \sin u$

By the chain rule,
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \cos u \times a$$
$$= a\cos(ax + b)$$

Next, let $y = \cos(ax + b)$

$u = ax + b$

Then $y = \cos u$

By the chain rule,
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= -\sin u \times a$$
$$= -a\sin(ax + b)$$

Hence:

$$\frac{d}{dx}[\sin(ax + b)] = a\cos(ax + b)$$

$$\frac{d}{dx}[\cos(ax + b)] = -a\sin(ax + b)$$

$$\frac{d}{dx}[\tan(ax + b)] = a\sec^2(ax + b)$$

Example 8 Differentiate with respect to x.

(a) $\sin(2x+1)$ **(b)** $\cos\left(3x-\frac{\pi}{2}\right)$ **(c)** $\tan(\pi-x)$

Solution:

(a)
$$y = \sin(2x+1)$$
$$\frac{dy}{dx} = \mathbf{2\cos(2x+1)}$$

(b)
$$y = \cos\left(3x-\frac{\pi}{2}\right)$$
$$\frac{dy}{dx} = \mathbf{-3\sin\left(3x-\frac{\pi}{2}\right)}$$

(c)
$$y = \tan(\pi-x)$$
$$\frac{dy}{dx} = \mathbf{-\sec^2(\pi-x)}$$

Example 9 Differentiate with respect to x.

(a) $x\sin 2x$ **(b)** $\sin 3x\cos 2x$

Solution: By the product rule,

(a)
$$\begin{aligned}\frac{d}{dx}(x\sin 2x) &= x\frac{d}{dx}(\sin 2x) + \sin 2x\frac{d}{dx}(x)\\ &= x(2\cos 2x) + \sin 2x\\ &= 2x\cos 2x + \sin 2x\end{aligned}$$

(b)
$$\begin{aligned}\frac{d}{dx}(\sin 3x\cos 2x) &= \sin 3x\,\frac{d}{dx}(\cos 2x) + \cos 2x\,\frac{d}{dx}(\sin 3x)\\ &= \sin 3x\,(-2\sin 2x) + \cos 2x\,(3\cos 3x)\\ &= \mathbf{-2\sin 2x\sin 3x + 3\cos 2x\cos 3x}\end{aligned}$$

Differentiation of $\sin^n x$ and $\cos^n x$ where n is a constant

Let
$$y = \sin^n x$$

Using
$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}, \text{ we have:}$$
$$\begin{aligned}\frac{dy}{dx} &= n\sin^{n-1}x\frac{d}{dx}(\sin x)\\ &= n\sin^{n-1}x\cos x\end{aligned}$$

Similarly, for $\cos^n x$ where n is a constant,
$$\begin{aligned}\frac{dy}{dx} &= n\cos^{n-1}x\frac{d}{dx}(\cos x)\\ &= -n\cos^{n-1}x\sin x\end{aligned}$$

$$\frac{d}{dx}(\sin^n x) = n\sin^{n-1}x\cos x$$
$$\frac{d}{dx}(\cos^n x) = -n\cos^{n-1}x\sin x$$

Example 10 Differentiate with respect to x.

(a) $\sin^3(2x+1)$ (b) $\tan^3 x$

Solution: Using $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$, we have:

(a) $$\frac{d}{dx}[\sin^3(2x+1)] = 3\sin^2(2x+1)\frac{d}{dx}[\sin(2x+1)]$$
$$= 3\sin^2(2x+1) \times 2\cos(2x+1)$$
$$= \mathbf{6\cos(2x+1)\sin^2(2x+1)}$$

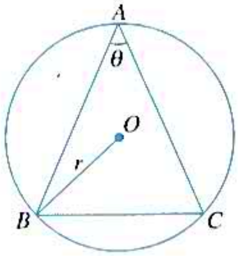
(b) $$\frac{d}{dx}(\tan^3 x) = 3\tan^2 x\frac{d}{dx}(\tan x)$$
$$= \mathbf{3\tan^2 x\sec^2 x}$$

Example 11 Given that $y = \sin 3x + \cos^3 x$, find the value of $\frac{dy}{dx}$ when $x = \frac{\pi}{4}$.

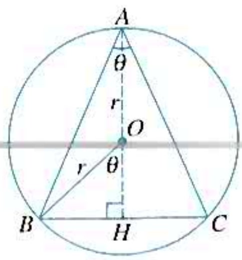
Solution:
$$y = \sin 3x + \cos^3 x$$
$$\frac{dy}{dx} = 3\cos 3x + 3\cos^2 x(-\sin x)$$
$$= 3(\cos 3x - \sin x\cos^2 x)$$

When $x = \frac{\pi}{4}$,
$$\frac{dy}{dx} = 3\left(\cos\frac{3\pi}{4} - \sin\frac{\pi}{4}\cos^2\frac{\pi}{4}\right)$$
$$= 3\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)^2\right)$$
$$= -\frac{9}{2\sqrt{2}}$$

Example 12 In the figure, a circle with centre O and radius r circumscribes an isosceles triangle ABC with $AB = AC$. If $\angle BAC = \theta$, where $0 < \theta < \frac{\pi}{2}$ show that the area of $\triangle ABC$ is given by $A = r^2\sin\theta(1+\cos\theta)$.
Show that A has its maximum value when $\theta = \frac{\pi}{3}$.



Solution: Now, $\angle BOH = \angle BAC = \theta$.



$$A = \frac{1}{2} \times BC \times AH$$
$$= \frac{1}{2}(2r\sin\theta)(r + r\cos\theta)$$
$$\therefore \quad A = r^2\sin\theta(1+\cos\theta)$$

$$\text{Next, } \frac{dA}{d\theta} = r^2[\sin\theta(-\sin\theta) + \cos\theta(1 + \cos\theta)]$$
$$= r^2[-(1 - \cos^2\theta) + \cos\theta + \cos^2\theta]$$
$$= r^2[2\cos^2\theta + \cos\theta - 1] \text{ .... (1)}$$
$$\frac{dA}{d\theta} = r^2(2\cos\theta - 1)(\cos\theta + 1)$$

$$\frac{dA}{d\theta} = 0 \quad \Rightarrow \quad \cos\theta = \frac{1}{2} \text{ or } \cos\theta = -1$$
$$\Rightarrow \quad \theta = \frac{\pi}{3} \text{ or } \theta = \pi \quad \text{(rejected, } \because \ \theta \text{ is acute)}$$

Differentiating (1) with respect to θ,

$$\frac{d^2A}{d\theta^2} = r^2[4\cos\theta(-\sin\theta) - \sin\theta]$$
$$= -r^2\sin\theta(1 + 4\cos\theta)$$

When $\theta = \frac{\pi}{3}$, $\frac{d^2A}{d\theta^2} < 0$.

$\therefore$ A has its maximum value when $\theta = \frac{\pi}{3}$.

Exercise 18.1

1. Differentiate the following with respect to x.

(a) $4\sin x - 3$ **(b)** $x^2 - 5\cos x$
(c) $2\sin x + 3\cos x$ **(d)** $4x^2 + 3\tan x$

2. Differentiate the following with respect to x.

(a) $x^2\cos x$ **(b)** $x\tan x$ **(c)** $(x + 1)^2\sin x$ **(d)** $\frac{1 - 2\sin x}{\cos x}$

3. Using $\frac{d}{dx}(\sin x) = \cos x$, show that $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$.

4. Differentiate the following with respect to x.

(a) $(1 - \cos x)^3$ **(b)** $(3\sin x + 2)^2$
(c) $\sqrt{2 - \tan x}$ **(d)** $\sqrt{\sin x + 2\cos x}$

5. Differentiate the following with respect to x.

(a) $\sin 3x + \cos 4x$ **(b)** $4\sin\frac{1}{2}x$
(c) $\sin(2x - 5)$ **(d)** $\cos\left(2x + \frac{\pi}{3}\right)$
(e) $3\tan^2 2x$ **(f)** $6\tan\frac{1}{2}x$
(g) $2\cos\left(\frac{\pi}{4} - x\right)$ **(h)** $8\sin\left(\frac{3x - 5}{4}\right)$

6. Differentiate the following with respect to x.

(a) $\sin x\cos 3x$ **(b)** $(1 + x^2)\tan 5x$ **(c)** $\frac{x}{\cos 2x}$ **(d)** $\frac{\cos x}{\sin 3x}$

7. Differentiate the following with respect to x.

(a) $2\sin^3 x$ (b) $\sin 2x - 3\cos^4 x$

(c) $\cos^2 3x$ (d) $x + 3\sin^5 2x$

(e) $4\tan^2 5x + 3$ (f) $x\cos^7 2x$

(g) $(1-x)^2 \sin^4 x$ (h) $\dfrac{\cos^2 x}{x}$

8. Given that $y = x + \cos x$, find $\dfrac{dy}{dx}$ and the approximate change in y when x increases from $\frac{\pi}{6}$ to $\frac{\pi}{5}$.

9. Find the value of $\dfrac{dy}{dx}$ for $y = 1 - 3\cos 2x$ at the point where $x = \frac{\pi}{12}$. Obtain the approximate change in y when x increases from $\frac{\pi}{12}$ to $\frac{\pi}{11}$.

10. Given that $r = \sqrt{\cos\theta}$, find the approximate error in r while θ is measured as $\frac{\pi}{3}$ and is increased by 2%. If θ increases at a constant rate of 0.5 radians per second, find the rate of change of r when $\theta = \frac{\pi}{3}$.

11. Find an approximate percentage change in $\sqrt{1+\sin x}$ when x increases from $\frac{\pi}{6}$ by 5%. Given that x changes at a constant rate of $\frac{\pi}{30}$ radians per second, find the rate of change of $\sqrt{1+\sin x}$ when $x = \frac{\pi}{3}$.

12. Find the equations of the tangent and the normal to the curve $y = 1 + \cos x$ at the point where $x = \frac{\pi}{6}$. Show that the area enclosed by the tangent, the normal and the x-axis is $\frac{5}{4}\left(1 + \frac{\sqrt{3}}{2}\right)^2$.

13. Find the value of x, where $0 \leqslant x \leqslant \pi$ for which $y = x - 2\sin x$ has a stationary value.

14. Determine the value of x, where $0 \leqslant x \leqslant \pi$, for which the curve

$$y = 2\cos x + 3\sin x$$

has a stationary point and determine the nature of this point.

15. Find the coordinates of the stationary points of the curve $y = x - \tan x$, where $0 \leqslant x < \pi$.

16. The equation of a curve is $y = 3\cos x + 4\sin x$, where $0 \leqslant x \leqslant 2\pi$. Calculate the values of x for which the tangents to the curve are parallel to the x-axis.

Important Notes

Derivatives of trigonometric functions

In the following formulae, x is in radians.

1. $\frac{d}{dx}(\sin x) = \cos x$,

$\frac{d}{dx}(\cos x) = -\sin x$,

$\frac{d}{dx}(\tan x) = \sec^2 x$,

$\frac{d}{dx}[\sin (ax + b)] = a \cos (ax + b)$,

$\frac{d}{dx}[\cos (ax + b)] = -a \sin (ax + b)$,

$\frac{d}{dx}[\tan (ax + b)] = a \sec^2 (ax + b)$,

where a and b are constants.

2. Using $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$,

$\frac{d}{dx}(\sin^n x) = n \sin^{n-1} x \cos x$,

$\frac{d}{dx}(\cos^n x) = -n \cos^{n-1} x \sin x$,

$\frac{d}{dx}(\tan^n x) = n \tan^{n-1} x \sec^2 x$.

Miscellaneous Examples

Example 13 Differentiate with respect to x.

(a) $\sqrt{3 + 2\cos 2x}$ **(b)** $\sqrt{1 + 2\sin^2 x}$

Solution: Using $\frac{d}{dx}\left(\sqrt{u}\right) = \frac{1}{2\sqrt{u}}\frac{du}{dx}$, we have:

(a)
$$\begin{aligned}\frac{d}{dx}\left(\sqrt{3 + 2\cos 2x}\right) &= \frac{1}{2\sqrt{3 + 2\cos 2x}}\frac{d}{dx}(3 + 2\cos 2x)\\ &= \frac{1}{2\sqrt{3 + 2\cos 2x}}(-4\sin 2x)\\ &= \frac{-2\sin 2x}{\sqrt{3 + 2\cos 2x}}\end{aligned}$$

(b)
$$\begin{aligned}\frac{d}{dx}\left(\sqrt{1 + 2\sin^2 x}\right) &= \frac{1}{2\sqrt{1 + 2\sin^2 x}}\frac{d}{dx}(1 + 2\sin^2 x)\\ &= \frac{1}{2\sqrt{1 + 2\sin^2 x}}(4\sin x\cos x)\\ &= \frac{2\sin x\cos x}{\sqrt{1 + 2\sin^2 x}}\end{aligned}$$

Miscellaneous Exercise 18

1. Differentiate the following with respect to x.
   (a) $3x - 2\cos x$ (b) $\sin 4x - 3\cos 2x$
   (c) $(1 - 2\tan x)^2$ (d) $\sin 2x \cos 3x$
   (e) $2x \tan 3x$ (f) $\dfrac{\sin 2x}{x}$
   (g) $\sqrt{\sin 2x - \cos x}$ (h) $\dfrac{1}{1 + \cos 4x}$

2. Differentiate the following with respect to x.
   (a) $\sin 2x - \cos^2 x$ (b) $3\sin^4 8x$
   (c) $x\cos^2 3x$ (d) $x^2 - 3\tan^2 4x$

3. Differentiate the following with respect to x.
   (a) $\sin(x^2 + 2)$ (b) $2\cos(1 - x^2)$ (c) $\tan(2x^2 + x)$

4. (a) Use the derivatives of $\sin x$ and $\cos x$ to show that $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.
   (b) Given that $t = 20\cot\theta$, find the rate of change of θ when $\theta = \dfrac{\pi}{4}$.

5. Given that $y = \sin 3x + \cos^3 x$, find the value of $\dfrac{dy}{dx}$ when $x = \dfrac{\pi}{6}$. If x decreases from $\dfrac{\pi}{6}$ to $\dfrac{\pi}{7}$, find, in terms of π, an approximate change in y.

6. Find the approximate change in $y = x - \sin x$ when x changes from $\dfrac{\pi}{3}$ to $\dfrac{\pi}{3} + 0.001$ radians. If x decreases at a constant rate of 2 radians per second, find the rate of change of y when $x = \dfrac{\pi}{3}$.

7. Find the gradient of the curve $y = 3 + \tan 2x$ at the point where $x = \dfrac{\pi}{8}$.

8. The line $y = \dfrac{1}{2}$ meets the portion of the curve $y = \sin 2x$ for which $0 < x < \dfrac{1}{2}\pi$ in points A and B. Calculate the gradients of the tangents to the curve at A and B. Show that if these tangents meet at C, then ABC is an equilateral triangle. *(C)*

9. Given the curve $y = 1 - 2\sin^2 x + \sin x$ for $0 \leqslant x \leqslant 2\pi$. Calculate the values of x for which the curve
   (a) meets the x-axis, (b) has turning points. *(C)*

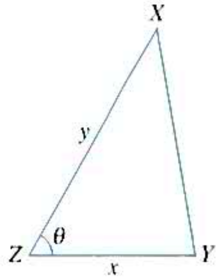
10. Find the coordinates of the turning points on the curve $y = \sin x \cos^2 x$ for $0 < x < \pi$.

11. If $y = \sin x - \dfrac{1}{3}\sin^3 x$, show that $\dfrac{dy}{dx} = \cos^3 x$.

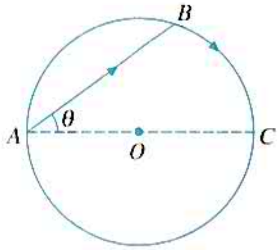
12. Given that $y = (1 - \tan^2 x)\cos^2 x$, show that $\dfrac{dy}{dx} + 4\sin x\cos x = 0$.

***13.** Find the coordinates of the two turning points on the curve $y = \sin^3 x \cos x$ for $0 < x < \pi$. Determine which point gives a maximum value of y.

14. The area A of $\triangle XYZ$ is calculated by the formula $A = \frac{1}{2}xy \sin \theta$. If x and y are measured correctly as 6 cm and 9 cm respectively, while θ is measured as $60°$ with a possible error of $1°$, show that the possible error in the area A is approximately $\frac{3\pi}{40}$.

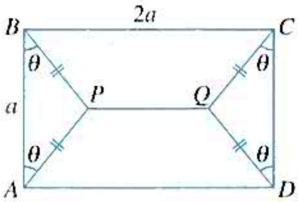


15. The figure shows a circular lake with centre O and radius 2 km. A man swims across the lake from A to B at 3 km/h and then walks along the edge of the lake from B to C at 4 km/h. If $\angle BAC = \theta$ radians and the total time taken is T hours, show that $T = \frac{1}{3}(4 \cos \theta + 3\theta)$. Find the value of θ for which T has a turning value and determine whether this gives a maximum or minimum value of T. *(C)*

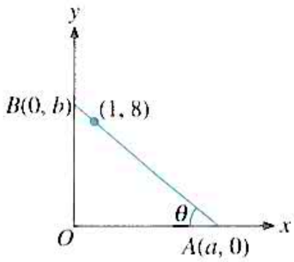


***16.** The diagram shows a rectangle $ABCD$ with $AB = a$ and $BC = 2a$. ABP and CDQ are identical isosceles triangles with $\angle BAP = \theta$. PQ is therefore parallel to AD and midway between BC and AD.
Given that $S = AP + BP + CQ + DQ + PQ$, show that $S = \frac{2a}{\cos \theta} + 2a - a \tan \theta$.

Given that $\frac{d}{d\theta}\left(\frac{1}{\cos \theta}\right) = \frac{\sin \theta}{\cos^2 \theta}$, find an expression for $\frac{dS}{d\theta}$. Hence, deduce the value of θ at which S has a stationary value. Find this value of S. *(C)*



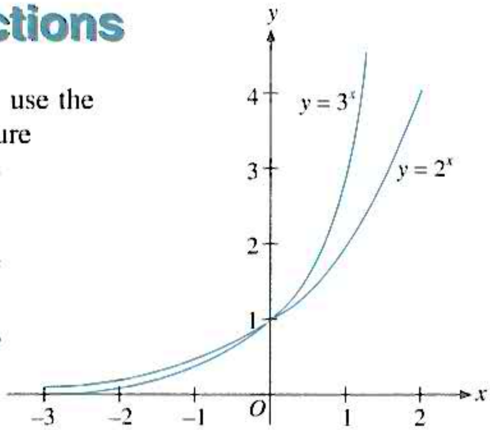
17. In the figure, $A(a, 0)$ and $B(0, b)$ are variable points on x- and y-axes respectively. The line segment AB passes through $(1, 8)$. Given that $\angle BAO = \theta$, show that $AB = \frac{1}{\cos \theta} + \frac{8}{\sin \theta}$. Find the least value of AB.



19 Exponential and Logarithmic Functions

19.1 Exponential Functions

In Chapter 8, we study the relation $y = a^x$ and use the linear law to estimate the value of a. The nature of the curve with y plotted against x depends on the value of a.



For example, the graphs of $y = 2^x$ and $y = 3^x$ are shown in the diagram. A function of the form $y = a^x$, where $a > 0$, is known as a **power function** or an **exponential function**.

Differentiation of Exponential Functions

Let us begin by differentiating the exponential function $y = a^x$ with respect to x from first principles.

Let us differentiate $y = a^x$ w.r.t. x from first principles.

$$y = a^x \quad \text{..............................} (1)$$

$$y + \delta y = a^{x+\delta x} \quad \text{..........................} (2)$$

(2) – (1):

$$\begin{aligned} \delta y &= a^{x+\delta x} - a^x \\ &= a^x a^{\delta x} - a^x \\ &= a^x(a^{\delta x} - 1) \end{aligned}$$

$$\frac{\delta y}{\delta x} = a^x\left(\frac{a^{\delta x} - 1}{\delta x}\right)$$

$$\frac{dy}{dx} = a^x \lim_{\delta x \to 0}\left(\frac{a^{\delta x} - 1}{\delta x}\right) \quad \text{..........} (3)$$

Taking $\delta x = 0.000\,1$, $0.000\,01$ and $0.000\,001$ with $a = 3$, we have, respectively,

$$\frac{a^{\delta x} - 1}{\delta x} \approx 1.098\,672,\ 1.098\,618,\ 1.098\,613.$$

In fact, we can show that when $a = 3$,

$$\lim_{\delta x \to 0}\left(\frac{a^{\delta x} - 1}{\delta x}\right) \approx 1.098\,61 \text{ (correct to 5 dec. pl.)}$$

This implies that $\lim_{\delta x \to 0}\left(\frac{a^{\delta x}-1}{\delta x}\right) = k$ and the value of k depends on the value of a.

If we want k to be equal to 1, we can find the corresponding value of a as follows:

$$\frac{a^{\delta x}-1}{\delta x} \approx 1 \quad \Rightarrow a^{\delta x} \approx 1 + \delta x$$

$$\Rightarrow a \approx (1 + \delta x)^{\frac{1}{\delta x}}$$

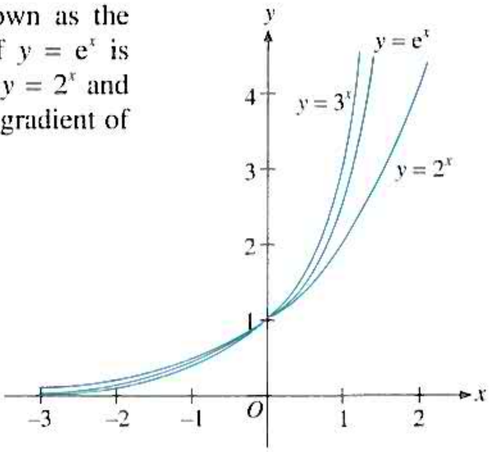
Let $\delta x = \frac{1}{n}$, we have $\frac{1}{\delta x} = n$ and

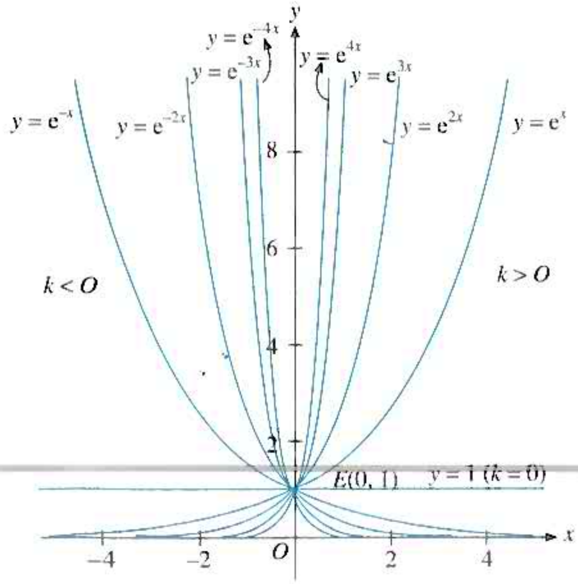
$$a = \lim_{\delta x \to 0}(1 + \delta x)^{\frac{1}{\delta x}} = \lim_{n \to +\infty}\left(1 + \frac{1}{n}\right)^n$$

which has the value e $\approx$ 2.718 28 as calculated in chapter 3.

Taking a = e in (3), we have: $\quad \boldsymbol{\frac{d}{dx}(e^x) = e^x}$

The exponential function $y = e^x$ is also known as the **natural exponential function**. The graph of $y = e^x$ is similar to the other power functions such as $y = 2^x$ and $y = 3^x$ as shown in the diagram. Note that the gradient of $y = e^x$ at (0, 1) is 1.





The diagram on the left shows a family of exponential functions:
$y = e^{kx}$ where $k \in \mathbb{R}$

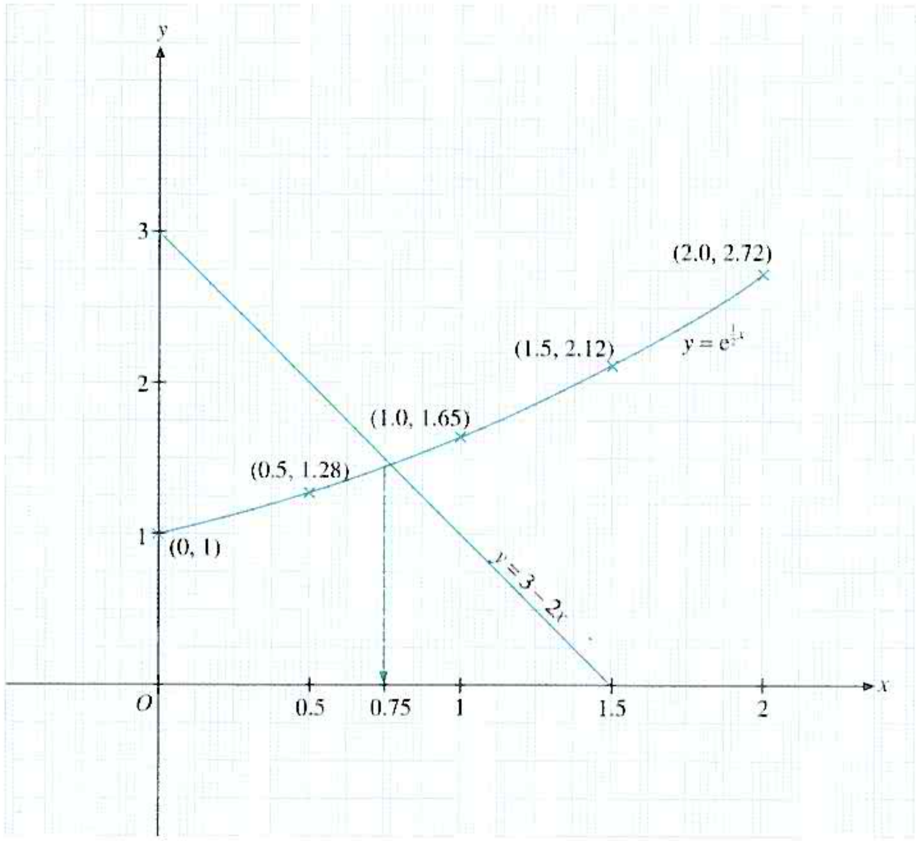
(1) The curve of $y = e^{-kx}$ is the reflection of $y = e^{kx}$ in the y-axis.

(2) All the curves pass through the point $E(0, 1)$.

(3) As $|x| \to \infty$, each curve approaches the x-axis ($y = 0$) and the x-axis is called the **asymptote** of the curve.

Example 1 Draw the graph of $y = e^{\frac{1}{2}x}$ for $0 \leqslant x \leqslant 2$, taking intervals of 0.5. By drawing the straight line $y = 3 - 2x$ on the same graph, obtain an approximate solution to the equation $e^{\frac{1}{2}x} = 3 - 2x$.

Solution: The graph of $y = e^{\frac{1}{2}x}$ is plotted as shown in the diagram.

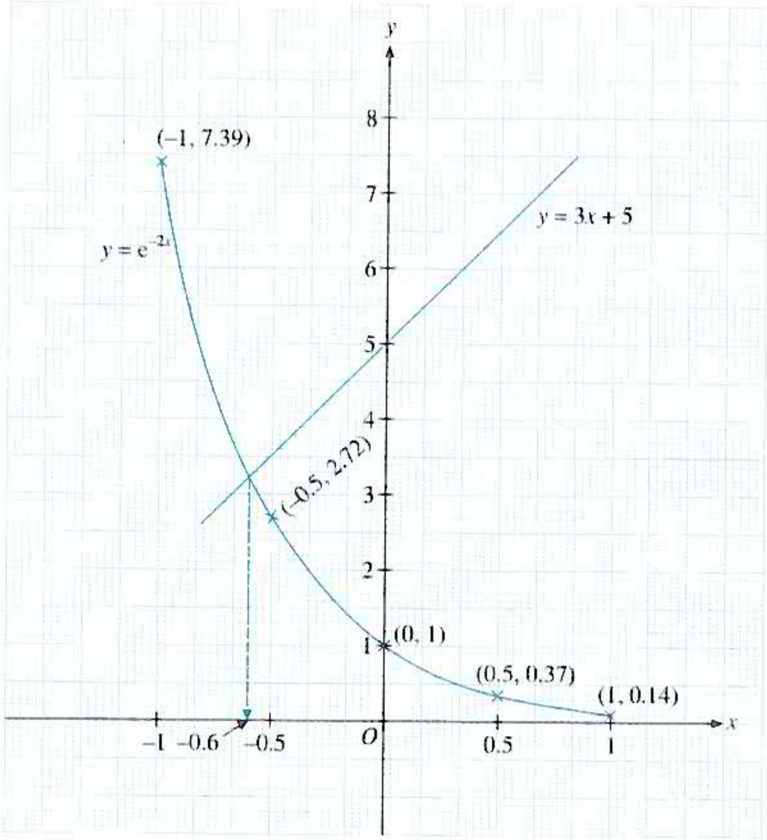


Next, we draw the line $y = 3 - 2x$ on the same diagram. This line meets the graph of $y = e^{\frac{1}{2}x}$ at the point where $x \approx 0.75$.

So, an approximate solution to $e^{\frac{1}{2}x} = 3 - 2x$ is **0.75**.

Example 2 Draw the graph of $y = e^{-2x}$ for $-1 \leqslant x \leqslant 1$, taking intervals of 0.5. By drawing a straight line on your graph, obtain an approximate solution to the equation $(3x + 5)e^{2x} = 1$.

Solution: The graph of $y = e^{-2x}$ is shown below.



Rewrite the given equation $(3x + 5)e^{2x} = 1$ as $3x + 5 = e^{-2x}$. The point of intersection of the graph of $y = e^{-2x}$ and the line $y = 3x + 5$ occurs at $x = \mathbf{-0.6}$ which gives the approximate solution to the equation $3x + 5 = e^{-2x}$.

Example 3 Given that $y = e^{2x+1}$, find $\frac{dy}{dx}$.

Solution: Let $u = 2x + 1$ and so $y = e^u$.

By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u \times 2 = 2e^{2x+1}$$

Note: In general, for $y = e^u$, where u is a function of x,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{d}{du}(e^u) \times \frac{du}{dx} = e^u \frac{du}{dx}.$$

In particular, if $u = ax + b$, where a and b are constants, we have $\frac{d}{dx}(e^{ax+b}) = ae^{ax+b}$.

Thus, we have:

(a) $\dfrac{d}{dx}(e^u) = e^u \dfrac{du}{dx}$, **where u is a function of x**

(b) $\dfrac{d}{dx}(e^{ax+b}) = ae^{ax+b}$, **where a and b are constants**

Example 4

Differentiate the following with respect to x.

(a) e^{5-2x} **(b)** e^{x^2+1} **(c)** $(e^x + 1)^3$

(d) $e^{2x} \sin x$ **(e)** $\dfrac{e^{3x}}{1+2x}$ **(f)** $\dfrac{3(e^{2x})^3}{e^{7x-2}}$

Solution:

(a) $\dfrac{d}{dx}\left(e^{5-2x}\right) = \mathbf{-2e^{5-2x}}$

(b) Using $\dfrac{d}{dx}\left(e^u\right) = e^u \dfrac{du}{dx}$

$$\begin{aligned}\frac{d}{dx}\left(e^{x^2+1}\right) &= e^{x^2+1}\frac{d}{dx}(x^2+1)\\ &= \mathbf{2xe^{x^2+1}}\end{aligned}$$

(c) Using $\dfrac{d}{dx}\left(u^n\right) = nu^{n-1}\dfrac{du}{dx}$

$$\begin{aligned}\frac{d}{dx}\left[(e^x+1)^3\right] &= 3(e^x+1)^2\frac{d}{dx}(e^x+1)\\ &= \mathbf{3e^x(e^x+1)^2}\end{aligned}$$

(d) Using $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

$$\begin{aligned}\frac{d}{dx}(e^{2x}\sin x) &= e^{2x}\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}(e^{2x})\\ &= e^{2x}\cos x + 2e^{2x}\sin x\\ &= \mathbf{e^{2x}(\cos x + 2\sin x)}\end{aligned}$$

(e) Using $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$,

$$\begin{aligned}\frac{d}{dx}\left(\frac{e^{3x}}{1+2x}\right) &= \frac{(1+2x)\frac{d}{dx}\left(e^{3x}\right) - e^{3x}\frac{d}{dx}(1+2x)}{(1+2x)^2}\\ &= \frac{(1+2x)\left(3e^{3x}\right) - e^{3x}(2)}{(1+2x)^2}\\ &= \mathbf{\frac{(1+6x)e^{3x}}{(1+2x)^2}}\end{aligned}$$

(f)
$$\frac{3(e^{2x})^3}{e^{7x-2}} = \frac{3e^{6x}}{e^{7x-2}}$$
$$= 3e^{6x-(7x-2)}$$
$$= 3e^{2-x}$$

$$\frac{d}{dx}\left(\frac{3(e^{2x})^3}{e^{7x-2}}\right) = \frac{d}{dx}(3e^{2-x})$$
$$= \mathbf{-3e^{2-x}}$$

Example 5

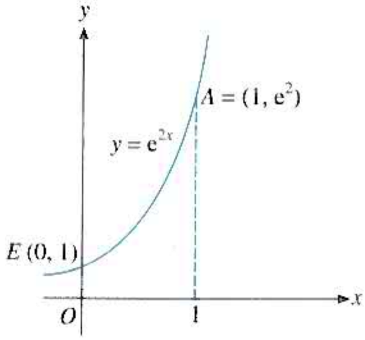
Sketch the curve defined by $y = e^{2x}$. Find

(a) the equation of the tangent at the point A where $x = 1$,

(b) the coordinates of the point where the tangent at A crosses the x-axis.

Solution:

A sketch of the curve of $y = e^{2x}$ passing through $E(0, 1)$ is shown below.



(a) $\frac{dy}{dx} = 2e^{2x}$

At A, $x = 1$, $y = e^2$ and $\frac{dy}{dx} = 2e^2$.

Equation of the tangent at A is $y - e^2 = 2e^2(x - 1)$

$\mathbf{y = e^2(2x - 1)}$

(b) At the point where the tangent crosses the x-axis, $y = 0$ and so $x = \frac{1}{2}$.

The coordinates of the point are $\mathbf{\left(\frac{1}{2}, 0\right)}$.

Exercise 19.1

1. For each of the following functions, draw the graph for $0 \leqslant x \leqslant 2$, taking intervals of 0.5 and state the coordinates of the point where the graph crosses the y-axis.

   (a) $y = e^{\frac{3}{2}x}$ **(b)** $y = e^{x+0.5}$ **(c)** $y = e^{5-2x}$

2. Draw the graph of $y = e^{2x}$ for $0 \leqslant x \leqslant 2$, taking intervals of 0.5. By drawing the straight line $y = 5 - 2x$ on the same graph, obtain an approximate solution to the equation $e^{2x} = 5 - 2x$.

3. Draw the graph of $y = e^{-x}$ for $-2 \leqslant x \leqslant 0$, taking intervals of 0.5. By drawing a straight line on the same graph, obtain an approximate solution to the equation $e^{-x} = 3 + 2x$.

4. Draw the graph of $y = e^{x+1}$ for $-1 \leqslant x \leqslant 1$, taking intervals of 0.5. By drawing suitable straight lines on your diagram, obtain an approximate solution to each of the following equations.
(a) $e^{x+1} + 5x = 5$ (b) $x + 1 = \ln(5 - 3x) - \ln 2$

5. Sketch, on separate diagrams, the graphs of the following exponential functions for the given intervals and state the coordinates of the points where these graphs cross the y-axis.

(a) $y = 2^x, -1 \leqslant x \leqslant 3$ (b) $y = e^{-\frac{1}{2}x}, -2 \leqslant x \leqslant 1$

(c) $y = e^{-x} + 2, -2 \leqslant x \leqslant 2$ (d) $y = \frac{1}{4}e^{1.2x}, 0 \leqslant x \leqslant 3$

6. Given that $y = 2^{x+1}$, find
(a) the value of y when $x = 0$, (b) the value of x when $y = 8$.
Hence sketch the graph for positive values of x.

7. Given that $y = 4^{x-1}$, find
(a) the value of y when $x = 0$, (b) the value of x when $y = 8$.
Hence sketch the graph for $x \geqslant 0$.

8. Differentiate the following with respect to x.

(a) $e^{\frac{1}{2}x}$ (b) e^{4x-1} (c) e^{-2x} (d) e^{2-7x}
(e) $e^{3x} + 7$ (f) $2e^{5x}$ (g) $4e^x - 6$ (h) $5e^{3-5x}$

(i) $e^x + \dfrac{1}{e^x}$ (j) $e^{3x} - 3e^{-x}$ (k) $8e^{\frac{1}{4}x} + \frac{1}{2}e^{6x}$ (l) $\frac{1}{3}e^{9x} - 2e^3$

9. Using the rules of indices, express the following as e^{ax+b}. Hence, or otherwise, differentiate with respect to x.

(a) $e^{3x} \cdot e^{x-1}$ (b) $\dfrac{e^{2x+3}}{e^x}$ (c) $(e^{1-x})^2$ (d) $\sqrt{e^{3x}}$

10. Differentiate the following with respect to x.
(a) $(e^{2x} + 1)^3$ (b) $(x + e^x)^5$ (c) $(x + 1)e^{2x}$
(d) $e^x \sin x$ (e) e^{x^3+2x} (f) $e^{-x} \cos 2x$

11. Differentiate the following with respect to x.

(a) $\dfrac{e^x}{1+x}$ (b) $\dfrac{x}{1+e^{-x}}$ (c) $\dfrac{e^x}{1+e^{2x}}$

(d) $\dfrac{\sin x}{e^x - 1}$ (e) $\dfrac{e^{3x}}{\cos x}$ (f) $\dfrac{x + e^{2x}}{e^x}$

12. Sketch the curve defined by $y = e^{2x-1}$ for $0 \leqslant x \leqslant 3$.
(a) Find the equation of the tangent and the equation of the normal to the curve at the point A where $x = 2$.
(b) Given that the tangent and the normal cross the x-axis at B and C respectively, find the area of the triangle ABC.

13. Find the equation of the tangent to the curve $y = 2e^{3x}$ at the point where $x = 1$. Find also the equation of the normal to the curve at this point.

14. Find the value of x for which $y = e^x - 2x - 1$ has a minimum value and calculate this minimum value.

15. The curves $y = e^{2x-3}$ and $y = e^{a-x}$ meet at P where $x = 2$. Find
(a) the value of a, **(b)** the gradient of each curve at P.

16. Given that $y = \frac{e^{-2x}}{6}$, find $\frac{dy}{dx}$. Hence find, in terms of p, the approximate change in y when x increases from 1 to $1 + p$, where p is small.

17. The area of a region A (cm^2) after t seconds is given by $A = 12 - 8e^{-\frac{t}{2}}$ where $t > 0$. Find in terms of e,
(a) the area of the region when $t = 2$,
(b) the rate of change of A when $t = 2$.
Hence, obtain the percentage change in A when t changes from 2 to 2.1, giving your answer correct to 1 decimal place.

18. Variables x and y are connected by the equation $y = 5e^{-3x}$. Given that x is increasing at the rate of 0.1 unit s^{-1}, find the rate of change of y when $x = 0.5$, giving your answer correct to 2 decimal places.

19.2 Logarithmic Functions

Let us consider the inverse of the function f defined by $f(x) = e^x$.

Let $\qquad y = f^{-1}(x)$

Then, $\qquad x = f(y)$

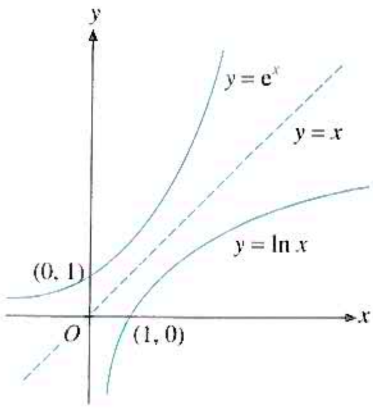
$\qquad x = e^y$

$\qquad y = \log_e x$

which is usually written as $y = \ln x$ and is known as the **natural logarithmic function** of x.

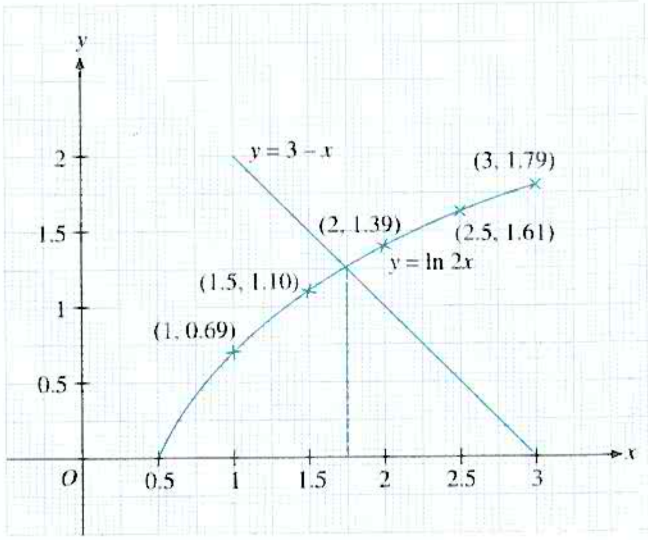
Since the inverse function of $y = \ln x$ is the exponential function $y = e^x$, the graph of $y = \ln x$ is the reflection of $y = e^x$ in the line $y = x$ as shown in the diagram. It is important to note that:

(1) The curve of $y = \ln x$ cuts the x-axis at (1, 0).
(2) The domain of $y = \ln x$ is $x > 0$.
(3) The curve has an asymptote of $x = 0$.



Example 6 Draw the graph of $y = \ln 2x$ for $0.5 \leqslant x \leqslant 3$, taking intervals of 0.5. By drawing a straight line on your diagram, obtain an approximate solution to the equation $\ln 2x = 3 - x$.

Solution: The graph of $y = \ln 2x$ is shown below.



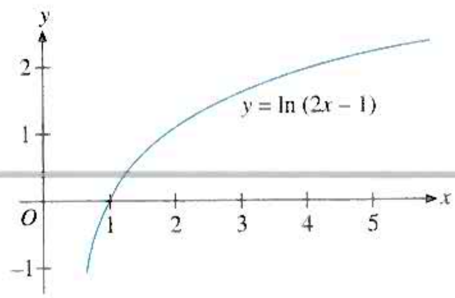
An approximate solution to $\ln 2x = 3 - x$ obtained from the point of intersection of $y = \ln 2x$ and $y = 3 - x$ is **1.75**.

Note: The curve $y = \ln 2x$ is defined for $2x > 0$, i.e. $x > 0$, and it cuts the x-axis at the point where $2x = 1$, i.e. $x = \frac{1}{2}$.

A curve defined by $y = \ln(ax + b)$ has domain $ax + b > 0$ and the curve cuts the x-axis at the point where $ax + b = 1$.

Example 7 State the domain of the curve $y = \ln(2x - 1)$ and the coordinates of the point where the curve crosses the x-axis. Hence sketch the curve.

Solution: The curve $y = \ln(2x - 1)$ has domain $2x - 1 > 0$, i.e. $x > \frac{1}{2}$.
The curve cuts the x-axis at the point where $y = 0$, i.e. $2x - 1 = 1$ and so $x = 1$. The coordinates of the required point is **(1, 0)**.
A sketch is shown below.



Differentiation of Logarithmic Functions

The function $y = \ln x$ can be written as $x = e^y$.
Differentiating w.r.t. x, we have

$$\frac{d}{dx}(x) = \frac{d}{dx}(e^y)$$
$$= \frac{d}{dy}(e^y)\frac{dy}{dx}$$
$$1 = e^y\frac{dy}{dx}$$

i.e.
$$1 = x\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{x}$$

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

If $y = \ln u$, where u is a function of x,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{d}{du}(\ln u)\frac{du}{dx} = \frac{1}{u}\frac{du}{dx}.$$

In particular, if $u = ax + b$, then $\frac{d}{dx}(\ln(ax + b)) = \frac{a}{ax + b}$.

Thus, we have:

(a) $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$, **where u is a function of x**

(b) $\frac{d}{dx}[\ln(ax + b)] = \frac{a}{ax + b}$, **where a and b are constants**

Example 8 Differentiate the following with respect to x.

(a) $\ln(3x + 2)$ **(b)** $\ln(7 - 2x)^3$ **(c)** $\ln(x^2 + 1)$

(d) $\ln(1 + \sin x)$ **(e)** $x \ln x$ **(f)** $\frac{\ln x}{x+1}$

Solution:

(a) $\frac{d}{dx}[\ln(3x + 2)] = \frac{3}{3x + 2}$

(b) $\frac{d}{dx}\left[\ln(7 - 2x)^3\right] = \frac{d}{dx}[3\ln(7 - 2x)]$

$$= 3 \times \frac{-2}{7 - 2x} = \frac{6}{2x - 7}$$

(c) Using $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$

$$\frac{d}{dx}[\ln(x^2 + 1)] = \frac{1}{x^2 + 1}\frac{d}{dx}(x^2 + 1) = \frac{2x}{x^2 + 1}$$

(d) Using $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$

$$\frac{d}{dx}[\ln(1 + \sin x)] = \frac{1}{1 + \sin x}\frac{d}{dx}(1 + \sin x)$$

$$= \frac{\cos x}{1 + \sin x}$$

(e) Using $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{d}{dx}(x \ln x) = x\frac{d}{dx}(\ln x) + \ln x\frac{d}{dx}(x)$$

$$= x \times \frac{1}{x} + \ln x$$

$$= 1 + \ln x$$

(f) Using $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{d}{dx}\left(\frac{\ln x}{x + 1}\right) = \frac{(x + 1)\frac{d}{dx}(\ln x) - \ln x\frac{d}{dx}(x + 1)}{(x + 1)^2}$$

$$= \frac{(x + 1) \times \frac{1}{x} - \ln x}{(x + 1)^2}$$

$$= \frac{(x + 1) - x \ln x}{x(x + 1)^2}$$

Example 9

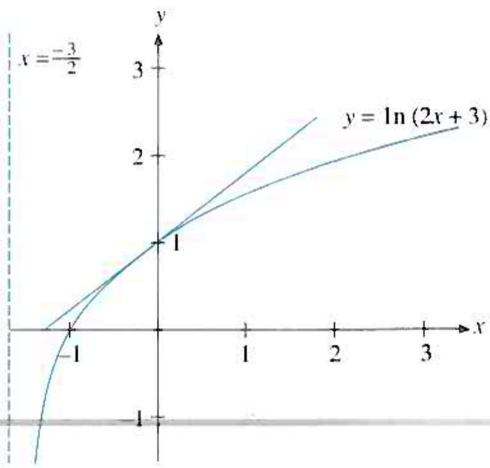
Sketch the curve of $y = \ln(2x + 3)$ for $-\frac{3}{2} < x \leqslant 3$. Find

(a) the equation of the tangent at the point where $x = 0$,

(b) the coordinates of the point where the tangent meets the x-axis.

Solution:

The curve $y = \ln(2x + 3)$ is defined for $2x + 3 > 0$, i.e. $x > -\frac{3}{2}$. It crosses the x-axis at $y = 0$ where $2x + 3 = 1$, i.e. $x = -1$. When $x = 3$, $y = \ln 9 \approx 2.2$. A sketch of the curve is shown here.



(a) $y = \ln(2x + 3)$

$$\frac{dy}{dx} = \frac{2}{2x + 3}$$

At $x = 0$, $y = \ln 3$

$$\frac{dy}{dx} = \frac{2}{0 + 3} = \frac{2}{3}$$

Equation of tangent at $x = 0$ is $y - \ln 3 = \frac{2}{3}(x - 0)$ or

$y = \frac{2}{3}x + \ln 3$

(b) When the tangent meets the x-axis, $y = 0$ and so

$-\ln 3 = \frac{2}{3}x \quad \Rightarrow \quad x = -\frac{3}{2} \ln 3$

Therefore, the required coordinates are $\left(-\frac{3}{2}\ln 3, 0\right)$

Exercise 19.2

1. For each of the following functions, draw the graph for $0 < x \leqslant 2$, taking intervals of 0.5 and state the coordinates of the point where the graph crosses the x-axis.
   (a) $y = \ln 3x$ **(b)** $y = \ln (2x + 1)$ **(c)** $y = \ln (3 - x)$

2. Sketch the graph of $y = \ln (2x + 1)$ for $-\frac{1}{2} < x \leqslant 1$, and state the coordinates of the point where the graph crosses the x-axis.

3. Draw the graph of $y = \ln (x + 3)$ for $-2 \leqslant x \leqslant 1$, taking intervals of 0.5. By drawing a straight line $y = 3 - 2x$ on the same graph, obtain an approximate solution to the equation $\ln (x + 3) = 3 - 2x$.

4. Draw the graph of $y = \ln 2x$ for $0 < x \leqslant 2$, taking intervals of 0.5. By drawing a straight line $y = 2 - x$ on the same graph, obtain an approximate solution to the equation $\ln 2x = 2 - x$.

5. Differentiate the following with respect to x.
   (a) $\ln (5x + 1)$ **(b)** $\ln (4x - 3)^2$ **(c)** $\ln (8 - x^3)$
   (d) $x \ln (5 - 2x)$ **(e)** $\ln [(x + 1)(x - 3)]$ **(f)** $\ln (\sin x + \cos x)$
   (g) $\ln (1 + e^x)$ **(h)** $\frac{\ln x}{x}$ **(i)** $\ln \left(\sqrt{x^2 + 1} - x\right)$

6. Sketch the graph of $y = \ln (x - 2)$ for $2 < x \leqslant 5$, and state the coordinates of the point where the graph crosses the x-axis. Find the equation of the tangent at the point where $x = 4$ and find the point where this tangent cuts the y-axis.

7. Show that the curve $y = \ln (2x - 1)$, $x > \frac{1}{2}$, has no stationary point.

8. The curve $y = \ln (3x - k)$ crosses the x-axis at $x = 2$. Find the value of k and the equation of the normal to the curve at this point.

9. Using a graph paper, draw the curve $y = \ln (2x + 1)$ for $0 \leqslant x \leqslant 3$, taking values of x at unit intervals. By drawing an appropriate straight line, obtain an approximate value for the root of the equation $x + 3 \ln (2x + 1) = 4.5$.

10. For each of the following curves, state the domain and the coordinates of the point where the curve crosses the x-axis.
    (a) $y = \ln (2x - 5)$ **(b)** $y = \ln (3 - 2x)$
    (c) $y = 3 \ln (x - 3)$ **(d)** $y = \ln (3x - 2)^2$

11. Given that $y = \ln(x^2 - 4)$ for $x > 2$, find $\frac{dy}{dx}$. Hence find, in terms of p, the approximate change in y when x increases from 3 to $3 + p$, where p is small.

12. The volume of water in a container V (cm^3) after t seconds is given by $V = 20e^{-\frac{t}{5}}$. Find, in terms of e,
(a) the value of V when $t = 5$, (b) the rate of change of V when $t = 5$.
Hence obtain the percentage change in V when t changes from 5 to 5.1.

13. Two variables x and y are related by the equation $y = \ln(2 + e^{-x})$ for $x > 0$. Given that x is increasing at the rate of 3 unit s^{-1}, find, in terms of e, the rate of change of y when $x = 1$.

Important Notes

1. **Differentiation of exponential functions**

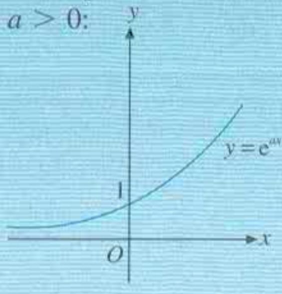
(a) $\frac{d}{dx}(e^x) = e^x$ (b) $\frac{d}{dx}(e^{ax+b}) = ae^{ax+b}$

More generally, by the chain rule, we have

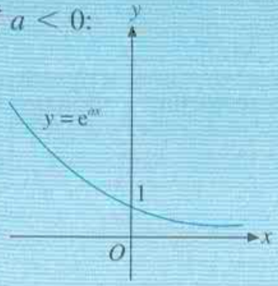
$$\frac{d}{dx}(e^u) = e^u \frac{du}{dx}, \text{ where } u \text{ is a function of } x.$$

2. **Graphs of exponential function $y = e^{ax}$**

(a) If $a > 0$:



(b) If $a < 0$:



3. **Differentiation of logarithmic functions**

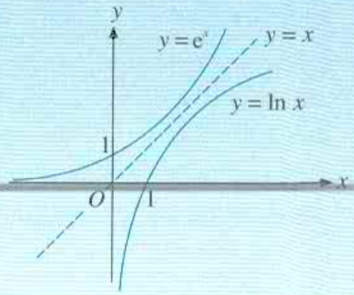
(a) $\frac{d}{dx}(\ln x) = \frac{1}{x}$ (b) $\frac{d}{dx}[\ln(ax + b)] = \frac{a}{ax + b}$

More generally,

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}, \text{ where } u \text{ is a function of } x.$$

4. **Graph of $y = \ln x$**

The functions of $y = \ln x$ and $y = e^x$ are inverse functions of each other. Hence the graph of $y = \ln x$, $(x > 0)$ is the reflection of the graph of $y = e^x$ in the line $y = x$.



5. **Graph of $y = \ln(ax + b)$**
The graph is defined for $ax + b > 0$ and it crosses the x-axis at the point where $ax + b = 1$.

Miscellaneous Examples

Example 10 Find the value of x for which the curve $y = xe^{-x}$ has a stationary point. Determine whether it is a maximum or a minimum point.

Solution:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(xe^{-x}) \\ &= e^{-x}\frac{d}{dx}(x) + x\frac{d}{dx}(e^{-x}) \\ &= e^{-x} - xe^{-x} \\ &= e^{-x}(1 - x)\end{aligned}$$

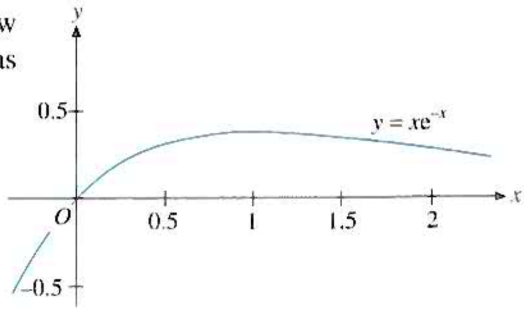
At the stationary point, $\frac{dy}{dx} = 0$,

$$\text{i.e. } 1 - x = 0 \quad (\because e^{-x} > 0)$$
$$x = 1$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}[e^{-x}(1 - x)] \\ &= -e^{-x} - e^{-x}(1 - x) \\ &= -e^{-x}(2 - x)\end{aligned}$$

At $x = 1$, $\frac{d^2y}{dx^2} < 0$ and so the stationary point is a **maximum point**.

Use a graph plotter, draw the graph of $y = xe^{-x}$ as shown:



Example 11 A function f is defined by $f : x \mapsto 2e^{\frac{1}{2}x} + 1$ for $-2 \leqslant x \leqslant 2$.

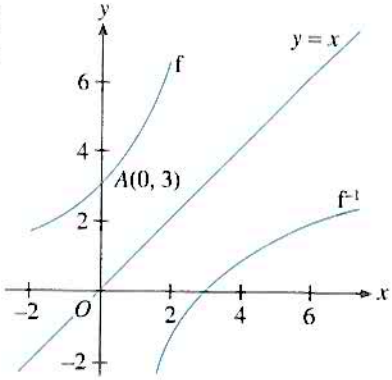
(a) Sketch the curve of f and show on the diagram the coordinates of the point A of intersection with the y-axis.

(b) State, in terms of e, the range of f and obtain an expression for f^{-1} in terms of x.

Solution: (a)

x	-2	0	2
y	$2e^{-1} + 1 \approx 1.74$	3	$2e + 1 \approx 6.44$

With the table of values a sketch of the curve of f is drawn with $A(0, 3)$.



(b) The range of f is $\frac{2}{e} + 1 \leqslant f(x) \leqslant 2e + 1$.

Let $y = f(x)$.

Then
$$y = 2e^{\frac{1}{2}x} + 1$$
$$\Rightarrow e^{\frac{1}{2}x} = \frac{y-1}{2}$$
$$\Rightarrow x = 2 \ln\left(\frac{y-1}{2}\right), \quad \frac{2}{e} + 1 \leqslant y \leqslant 2e + 1$$

Since $x = f^{-1}(y)$,
$$f^{-1}(y) = 2 \ln\left(\frac{y-1}{2}\right), \quad \frac{2}{e} + 1 \leqslant y \leqslant 2e + 1$$
$$f^{-1}(x) = 2 \ln\left(\frac{x-1}{2}\right), \quad \frac{2}{e} + 1 \leqslant x \leqslant 2e + 1$$

Miscellaneous Exercise 19

1. Differentiate the following with respect to x.
   (a) x^3e^x (b) $(x + 1)e^{-x}$ (c) $e^{2x} \sin x$
   (d) $3x^2 \ln x$ (e) $(x^2 - 1) \ln (2x + 1)$ (f) $e^{\frac{1}{2}x} \ln (5 - 4x)$
2. Differentiate the following with respect to x.
   (a) $\frac{e^x}{2x+1}$ (b) $\frac{x^2}{\ln 2x}$ (c) $\frac{e^{\frac{1}{2}x}}{\cos 2x}$
   (d) $\ln\left(x + \sqrt{x^2 + 1}\right)$ (e) $\ln (\sin^3 x)$ (f) $\ln (x + \sin x)$

3. Differentiate the following with respect to x.

(a) e^{x^3-2x} (b) $e^{1+\sin x}$ (c) $e^{\cos 2x}$

(d) $e^{\sqrt{x}}$ (e) e^{e^x} (f) $\ln(1-x^2)$

(g) $\ln(3+\tan x)$ (h) $\ln(\sin 2x)$ (i) $(\ln x)^2$

(j) $\dfrac{1}{\ln x}$ (k) $\sqrt{\ln x + e^x}$ (l) $\ln(e^{2x}+x^2)$

4. Differentiate the following with respect to x.

(a) $\ln 6x^4$ (b) $\ln\left(\dfrac{1-2x}{1+x}\right)$ (c) $\ln(5+4x)^3$

(d) $\ln(\sec x)$ (e) $\ln(x \sin 2x)$ (f) $\ln\sqrt{\dfrac{x+1}{1-x}}$

(g) $x^3 \ln(\cos^2 x)$ (h) $e^x \ln(x^2+1)$ (i) $\ln(e^{\sin x}+1)$

5. Sketch the graph of $y = e^{0.4x}$ for $-2 \leqslant x \leqslant 3$, and state the coordinates of the point where the graph crosses the y-axis.

6. Given that $y = 3(2^{x-1})$, find
(a) the value of y when $x = 0$, (b) the value of x when $y = 12$.
Hence sketch the graph of $y = 3(2^{x-1})$ for $x \geqslant 0$.

7. Given that $y = 3e^{-2x}$, find

(a) the value of y when $x = 0$, (b) the value of x and of $\dfrac{dy}{dx}$ when $y = 1$.

Sketch the graph of $y = 3e^{-2x}$ for positive values of x.

8. Sketch the graph of $y = \ln(2x-4)$ for $2 < x \leqslant 5$, and state the coordinates of the point where the graph crosses the x-axis.

9. Sketch the graph of $y = \ln x$ for $x > 0$. Express $x^2e^x = 12$ in the form $\ln x = ax + b$ and state the value of a and of b. Insert on your sketch the additional graph required to illustrate how a graphical solution of the equation $x^2e^x = 12$ may be obtained.

10. Draw the graph of $y = e^{-x}$ for $-1 \leqslant x \leqslant 0$, taking intervals of 0.25. By drawing a straight line on your diagram, obtain an approximate solution to the equation $e^x(x+1) = \dfrac{1}{2}$.

11. Given that $y = xe^{x-3}$, find $\dfrac{dy}{dx}$ and hence the gradient of the curve at the point (3, 3).

12. (a) Find the coordinates of the turning point on the curve $y = 2e^{4x} + 8e^{-4x}$ and determine the nature of this turning point.
(b) Find the coordinates of the stationary point of the curve $y = x \ln x - 2x$.

13. Find the equations of the tangents to the curve $y = xe^x$ at the points where $x = 1$ and $x = -1$.

14. Given that a curve has the equation $y = 10(x+2)e^x$, find $\dfrac{dy}{dx}$ and hence find the coordinates of the points at which the curve crosses the x- and y-axes. Sketch the curve for $-4 \leqslant x \leqslant 1$. *(C)*

15. An object is heated in an oven until it reaches a temperature of X degrees Celsius. It is then allowed to cool. Its temperature, θ degrees Celsius, when it has been cooling for time t minutes, is given by the equation

$$\theta = 18 + 62e^{-\frac{t}{8}}.$$

Find **(a)** the value of X,
(b) the value of θ when $t = 16$,
(c) the value of t when $\theta = 48$,
(d) the rate at which θ is decreasing when $t = 16$.

State the value which θ approaches as t becomes very large. *(C)*

16. The mass, m grams, of a radioactive substance, present at time t days after first being observed, is given by the formula

$$m = 24e^{-0.02t}$$

Find

(a) the value of m where $t = 30$,
(b) the value of t when the mass is half of its value at $t = 0$,
(c) the rate at which the mass is decreasing when $t = 50$. *(C)*

17. The population of a village at the beginning of the year 1800 was 240. The population increased so that, after a period of n years, the new population was $240(1.06)^n$. Find

(a) the population at the beginning of 1820,
(b) the year in which the population first reached 2500. *(C)*

Revision Exercise 13

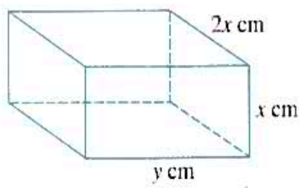
1. (a) The gradient of the curve $y = ax + \frac{b}{x^2}$ at (2, 5) is 2. Calculate the value of a and of b.
   (b) Differentiate with respect to x,
   (i) $\sqrt{2 - x^2}$, (ii) $(\sqrt{x} - 1)^2$.
   (c) Find the equation of the normal to the curve $y = \frac{3}{x^2 - 1}$ at the point where $x = 2$.

2. (a) Find the equation of the tangent to the curve $y = \frac{5}{x^2 + 1}$ at the point (2, 1).
   (b) Find the equation of the tangent to the curve $y = x^2 - 2x$ at each of the two points where $y = 3$.
   (c) Find the value of a and of b for which $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\cos x}{3 + \sin x}\right) = \frac{a + b \sin x}{(3 + \sin x)^2}$.

3. (a) Given that $y = x^3 - 3x$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$. Hence obtain an expression for the approximate increase in the value of y when x increases from 2 to $2 + p$, where p is small.
   (b) The area of a rectangle has a constant value of 50 cm^2. One side, of length x cm, is increasing at a rate of 0.2 cm s^{-1}. Find the rate at which the other side is decreasing at the instant when $x = 5$.

4. (a) Given that $y = \frac{4 + x}{x}$, find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x = 2$ and hence deduce the approximate decrease in the value of y when x increases from 2 to 2.01.
   (b) The radius, r cm, of a sphere is increasing in such a way that the rate of increase of the volume of the circle is πr^3 cm^3 s^{-1}. Calculate the rate of increase of the radius when the radius is 8 cm.

5. The diagram shows a rectangular block that has a volume of 72 cm^3. Show that the surface area, A cm^2, of the block is given by $A = 4x^2 + \frac{216}{x}$. Hence, find the minimum value of A.



6. (a) Differentiate with respect to x
   (i) $\frac{e^{2x}}{x + 1}$, (ii) $\ln\left(\frac{x + 2}{2 - x}\right)$.
   (b) Sketch the curve $y = e^{-x} - 1$, indicating the intercepts on the axes. Find the gradient of the curve at the point where $x = \ln 2$.

7. Find the coordinates of the turning point of the curve $y = 3 \sin x + \cos x$ where $0 \leqslant x \leqslant \frac{\pi}{2}$ and determine whether this is a maximum or a minimum point.

Revision Exercise 14

1. **(a)** Calculate the coordinates of the points on the curve $y = 2x^3 - 3x^2 - 9x + 1$ at which the tangents to the curve are parallel to $y - 3x = 4$.
   (b) Differentiate with respect to x
   (i) $\sqrt{x+1}\,(\sqrt{x} - 1)$, **(ii)** $\frac{x-4}{x^2}$.
   (c) Calculate the gradient of the curve $y = \sqrt{2 + \frac{1}{x}}$ at the point where $y = 2$.

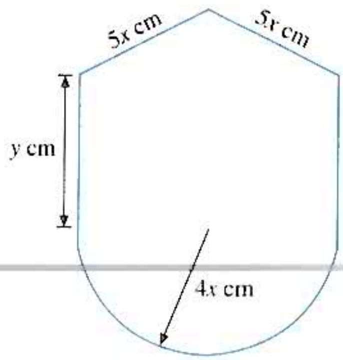
2. **(a)** Differentiate with respect to x
   (i) $\tan(3x - 1)$, **(ii)** $(x^2 - 1)\cos 2x$.
   (b) Find the equation of the normal to the curve $y = \frac{x-3}{x+1}$ at the point where $y = 3$.
   (c) Given that $y = 4\sin x - 7\cos x$, where $0 < x < 2\pi$, find the values of x for which y is stationary.

3. **(a)** For a sphere of radius r, the volume is $V = \frac{4}{3}\pi r^3$ and the surface area is $S = 4\pi r^2$. Given that when $r = 3$ cm, S is decreasing at a rate of 20 cm^2 s^{-1}, find the rate of decrease of V at this instant.
   (b) Given that $y = \sqrt[3]{8 + x}$, write down $\frac{dy}{dx}$, and hence find an expression for the approximate change in y when x changes from 0 by the small amount p.

4. Given that $y = x^3 + 2x - 5$, and that the value of x increases from 2 by a small amount $\frac{p}{100}$, find in terms of p,
   (a) the approximate change in y,
   (b) the corresponding percentage change in y.

5. Two variables x and y are connected by the equation $y = \frac{x+1}{x-1}$. Calculate the value of $\frac{dy}{dx}$ when $x = 2$. Hence find an expression for the approximate value of y when $x = 2 + p$, where p is small.

6. A piece of wire of length 80 cm is bent to form the shape shown in the diagram. This shape consists of a semicircular arc, radius $4x$ cm, 2 sides of length y cm of a rectangle and the 2 equal sides of an isosceles triangle. Express y in terms of x and hence show that the total area enclosed, A cm^2, is such that
$$A = 320x - 8\pi x^2 - 28x^2.$$
Calculate the value of x, correct to 3 significant figures, for which A has a stationary value. Determine whether this value of x makes A a maximum or a minimum.



7. (a) Differentiate with respect to x

(i) $\dfrac{xe^x}{1+x}$, (ii) $\sqrt{1+\sin 2x}$.

(b) Find the coordinates of the turning points on the curve $y = \sin^2 x \cos x$ for $0 < x < \pi$.

Revision Exercise 15

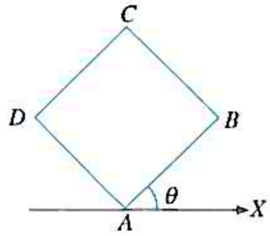
1. (a) Show that the point $P(3, 2)$ is on the curve $y = x^2 - 5x + 8$. Find the coordinates of the point of intersection of the tangent to the curve at P with the line $x + y + 3 = 0$.

(b) The equation of a curve is $y = 4x^2 - 5x + 6$. Calculate the coordinates of the point on the curve at which the gradient is 11. The curve cuts the line $y = 5$ at P and Q. Find the gradient of the curve at P and at Q.

2. (a) Find the equations of the normals to the curve $y = \dfrac{2}{\sqrt{x}}$ at the points (1, 2) and (4, 1). Find the coordinates of P, the point of intersection of these normals.

(b) The equation of a curve is $y = \dfrac{x+3}{x-1}$. Find the coordinates of the points whose gradient is $-\dfrac{4}{9}$.

3. (a) In the figure, $ABCD$ is a square of side 20 cm and $B\hat{A}X = \theta$ radians. y is the distance from C to the horizontal line AX.

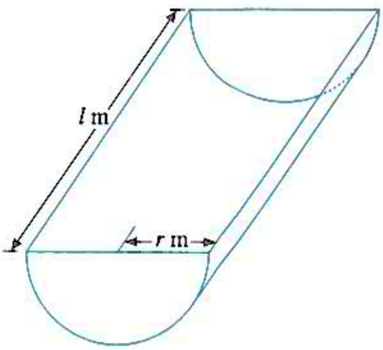


(i) Express y in terms of θ.

(ii) Find an expression for the approximate change in y as θ increases from $\dfrac{\pi}{6}$ by a small amount p.

(b) Given that $y = x - \dfrac{3}{x}$, find the positive value of x when $y = 2$. Find the approximate percentage change in y when x increases by p%.

4. The diagram shows a water trough of semicircular cross-section, radius r m. Its length is l m. The total amount of material used to make the trough is 20 m^2.



(a) Show that the volume, V m^3, of the trough is given by $V = 10r - \dfrac{1}{2}\pi r^3$.

(b) Find the maximum volume of the trough. Show that this volume is indeed the maximum.

5. (a) Differentiate $\dfrac{2x}{x^2+1}$ with respect to x.

(b) Find the value of x between 0 and π for which the curve $y = (1 - \sin x)e^x$ has a stationary point.

6. **(a)** Differentiate with respect to x

 (i) $e^{3x - \sin x}$, **(ii)** $\ln(1 + \sqrt{x})$.

 (b) Find the gradient of the curve $y = \dfrac{e^x}{x - 1}$ at the point where $x = 2$.

 (c) Show that $\dfrac{d}{dx}\left(\dfrac{1 + \sin x}{\cos x}\right) = \dfrac{1}{1 - \sin x}$.

7. **(a)** Find the coordinates of the stationary point on the curve defined by $y = \dfrac{\sqrt{x - 1}}{x}$. Determine the nature of this point. Show also that there is only one x-intercept. Hence sketch the curve for $x \geqslant 1$.

 (b) Sketch the graph of $y = \ln(2x - 1)$ for $\frac{1}{2} < x \leqslant 3$, and state the coordinates of the point where the graph meets the x-axis.

Revision Exercise 16

1. **(a)** Calculate the gradients of the tangents to the curve $y = x - \dfrac{2}{x - 1}$ at the points where it crosses the x-axis and show that there is no stationary point on the curve.

 (b) Find the equation of the normal to the curve $y = \dfrac{x + 1}{x - 1}$ at the point A where the curve crosses the line $y = 7 - 2x$. The normal meets the curve at another point B. Find the coordinates of B.

2. **(a)** Given that $y = x - \sin x$, find $\dfrac{dy}{dx}$ and the approximate change in y when x increases from $\dfrac{\pi}{3}$ by a small amount p.

 (b) Two variables x and y are connected by the equation $y = \dfrac{1}{5 - 2x}$. Given that x is increasing at the rate of 5 units s^{-1}, find the rate of increase of y when $x = 1\frac{1}{2}$.

3. **(a)** $ABCD$ is a rectangle with $AB = x$ cm and $AD = 14 - x$ cm. The sides of the rectangle vary with time such that x increases at a rate of 0.3 cm s^{-1}. Find

 (i) the rate of change of the area of the rectangle when $x = 6$,

 (ii) the rate of change of the length of the diagonal at this instant.

 (b) In a triangle ABC, $AB = 3$ cm, $AC = 6$ cm, $BC = x$ cm and $B\hat{A}C = \theta$ radians.

 (i) Show that $x = \sqrt{45 - 36\cos\theta}$.

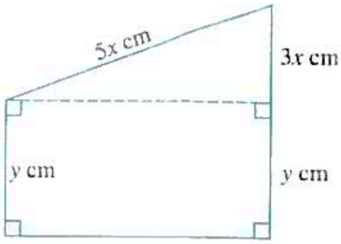
 (ii) If θ increases from $\dfrac{\pi}{3}$ by a small amount p, find the change in x.

4. **(a)** Find the coordinates of the stationary points on the curve $y = x(x - 1)^2 + 3$ and determine the nature of each of these points.

(b) A piece of wire, l cm long, is bent to form the shape shown in the diagram. Express l in terms of x and y. Given that the area enclosed is 162 cm^2, show that

$l = 9x + \dfrac{81}{x}$.

Hence, determine the value of x for which l is a minimum.



5. A circular cone has base radius r and height h. The curved surface area and volume of the cone are given by $A = \pi r\sqrt{r^2 + h^2}$ and $V = \frac{1}{3}\pi r^2 h$ respectively. Given that $V = 4\pi$, express A in terms of h. Find the value of h for which $\frac{dA}{dh} = 0$ and determine whether the corresponding value of A is a maximum or a minimum.

6. **(a)** Find the coordinates of the point where the curve $y = \ln(x^2 - 3)$ crosses the positive x-axis and the gradient of the curve at this point.

(b) Find the coordinates of the point where the curve $y = e^{\sqrt{x} - 2}$ cuts the line $y = 1$ and show that the tangent to the curve at this point passes through the origin.

7. **(a)** Differentiate with respect to x

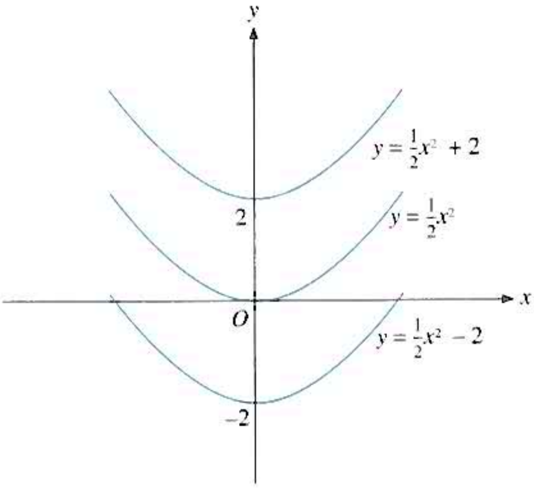
(i) $\tan^2(2x + 1)$, **(ii)** $\sin(2x^2 - 1)$.

(b) Two variables x and y are related by $y = \ln(3x^2 - 2)$. At the instant when $x = 1$, x is increasing at a rate of 3 units per second. Find the rate of change of y.

20 Integration

20.1 Integration as the Reverse Process of Differentiation and Indefinite Integrals

Meaning of Integration



In the process of differentiation, if $y = \frac{1}{2}x^2 + c$, where c is a constant, then $\frac{dy}{dx} = x$. This means that for any curve defined by $y = \frac{1}{2}x^2 + c$, we have, by the process of differentiation, the same gradient function $\frac{dy}{dx} = x$.

Conversely, if the gradient function $\frac{dy}{dx} = x$, then we know that the equation of the curve is of the form $y = \frac{1}{2}x^2 + c$. This process is the reverse of differentiation and is called **integration** and we write:

$$\frac{dy}{dx} = x \quad \Rightarrow \quad y = \int x \, dx$$

Since $y = \frac{1}{2}x^2 + c$,

$$\frac{d}{dx}\left(\frac{1}{2}x^2 + c\right) = x \quad \Rightarrow \quad \int x \, dx = \frac{1}{2}x^2 + c.$$

Indefinite Integral

Since there is an **arbitrary constant c** in the expression $\frac{1}{2}x^2 + c$, we say that this expression is an **indefinite integral**. Similarly, we have

$$\frac{d}{dx}\left(\frac{1}{3}x^3 + c\right) = x^2 \quad \Rightarrow \quad \int x^2 \, dx = \frac{1}{3}x^3 + c$$

which is an indefinite integral.

In general, if $n \neq -1$, a and n are constants,

$$\frac{d}{dx}\left(\frac{ax^{n+1}}{n+1} + c\right) = ax^n.$$

$$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c, \text{ where } c \text{ is an arbitrary constant.}$$

If $\frac{dy}{dx} = ax^n$, then $\quad y = \int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$

For the special case where $n = 0$,

$$\frac{dy}{dx} = a \quad \Rightarrow \quad y = ax + c$$

Example 1 Find y if $\frac{dy}{dx}$ is given by

(a) $3x^3$, **(b)** $\frac{1}{x^2}$, **(c)** $\sqrt{x}$, **(d)** 5.

Solution:

(a) $\frac{dy}{dx} = 3x^3$

$$\begin{aligned} y &= \int 3x^3\,dx \\ &= \frac{3x^{3+1}}{3+1} + c \\ &= \frac{3}{4}x^4 + c \end{aligned}$$

(b) $\frac{dy}{dx} = \frac{1}{x^2}$

$$\begin{aligned} y &= \int \frac{1}{x^2}\,dx \\ &= \int x^{-2}\,dx \\ &= \frac{x^{-2+1}}{-2+1} + c \\ &= -x^{-1} + c \\ &= -\frac{1}{x} + c \end{aligned}$$

(c) $\frac{dy}{dx} = \sqrt{x}$

$$\begin{aligned} y &= \int \sqrt{x}\,dx \\ &= \int x^{\frac{1}{2}}\,dx \\ &= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c \\ &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c \\ &= \frac{2}{3}x^{\frac{3}{2}} + c \end{aligned}$$

(d) $\frac{dy}{dx} = 5$

$$\begin{aligned} y &= \int 5\,dx \\ &= 5x + c \end{aligned}$$

For further integration, we consider a curve defined by $y = x^3 + x^2 + c$ where c is a constant. By the process of differentiation, the gradient function of the curve is $\frac{dy}{dx} = 3x^2 + 2x$. Conversely, if the gradient function of a curve is given by $\frac{dy}{dx} = 3x^2 + 2x$, then

$$\begin{aligned} y &= \int (3x^2 + 2x)\, dx \\ &= x^3 + x^2 + c. \end{aligned}$$

In general, if $n \neq -1$, $m \neq -1$, a, b, m and n are constants, then

$$\frac{d}{dx}\left(\frac{ax^{n+1}}{n+1} + \frac{bx^{m+1}}{m+1} + c\right) = ax^n + bx^m$$

$$\int (ax^n + bx^m)\, dx = \frac{ax^{n+1}}{n+1} + \frac{bx^{m+1}}{m+1} + c,$$

where c is an arbitrary constant.

Example 2

Find y if $\frac{dy}{dx}$ is given by

(a) $x^2 + 2$,

(b) $x(x + 3)$,

(c) $\frac{x^2 - 1}{x^2}$,

(d) $\frac{x^2 + 1}{\sqrt{x}}$.

Solution:

(a) $\frac{dy}{dx} = x^2 + 2$

$$\begin{aligned} y &= \int (x^2 + 2)\, dx \\ &= \frac{x^3}{3} + 2x + c \end{aligned}$$

(b) $\frac{dy}{dx} = x(x + 3)$

$$\begin{aligned} y &= \int x(x + 3)\, dx \\ &= \int (x^2 + 3x)\, dx \\ &= \frac{x^3}{3} + \frac{3x^2}{2} + c \end{aligned}$$

(c) $\frac{dy}{dx} = \frac{x^2 - 1}{x^2}$

$$y = \int \frac{x^2 - 1}{x^2}\, dx$$
$$= \int \left(1 - \frac{1}{x^2}\right) dx$$
$$= \int (1 - x^{-2})\, dx$$
$$= x - \frac{x^{-2+1}}{-2+1} + c$$
$$= \boldsymbol{x + \frac{1}{x} + c}$$

(d) $\frac{dy}{dx} = \frac{x^2 + 1}{\sqrt{x}}$

$$y = \int \frac{x^2 + 1}{\sqrt{x}}\, dx$$
$$= \int \frac{x^2 + 1}{x^{\frac{1}{2}}}\, dx$$
$$= \int \left(x^{\frac{3}{2}} + x^{-\frac{1}{2}}\right)$$
$$= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$
$$= \boldsymbol{\frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{1}{2}} + c}$$

Example 3 Find the equation of the curve which passes through the point (2, 3) and for which $\frac{dy}{dx} = 3x^2 + x$.

Solution: With the gradient function $\frac{dy}{dx} = 3x^2 + x$, the equation of the curve is:

$$y = \int (3x^2 + x)\, dx$$
$$= x^3 + \frac{x^2}{2} + c$$

Since (2, 3) lies on the curve,

$$3 = 8 + 2 + c$$
$$c = -7$$

The equation of the curve is $\boldsymbol{y = x^3 + \frac{x^2}{2} - 7}$.

Example 4 Given that $\frac{dy}{dx}$ is inversely proportional to x^2 and that $y = 1$ and $\frac{dy}{dx} = 3$ when $x = 3$, find the value of y when $x = 2$.

Solution: $\frac{dy}{dx} \propto \frac{1}{x^2} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{k}{x^2}$, where k is a constant

When $x = 3$, $\frac{dy}{dx} = 3$, we have $3 = \frac{k}{3^2}$ and so, $k = 27$

$$\therefore \quad \frac{dy}{dx} = \frac{27}{x^2}$$

$$y = \int \frac{27}{x^2}\, dx$$

$$= 27 \int x^{-2}\, dx$$

$$= -\frac{27}{x} + c$$

When $x = 3$, $y = 1$ and so,

$$1 = -9 + c$$

$\therefore \quad c = 10$

Hence $y = -\frac{27}{x} + 10$.

When $x = 2$, $y = -\frac{27}{2} + 10 = \mathbf{-3\frac{1}{2}}$

Example 5 Find A as a function of x given that $\frac{dA}{dx} = x(x - 1)$ and that $A = 5$ when $x = 3$.

Solution:

$$\frac{dA}{dx} = x(x - 1)$$

$$A = \int x(x - 1)\, dx$$

$$= \int (x^2 - x)\, dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + c$$

Given that $A = 5$ when $x = 3$, we have

$$5 = 9 - \frac{9}{2} + c$$

$\therefore \quad c = \frac{1}{2}$

The required function is $A = \mathbf{\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{2}}$.

Integration of $(ax + b)^n$, $a \neq 0$, $n \neq -1$

In the process of differentiation, we have

$$\frac{d}{dx}\left[\frac{(ax + b)^{n+1}}{a(n + 1)} + c\right] = (ax + b)^n$$

In the reverse process, by integration:

$$\int (ax + b)^n\, dx = \frac{(ax + b)^{n+1}}{a(n + 1)} + c$$

Example 6 Integrate $(2x + 1)^3$ with respect to x.

Solution:

$$\int (2x+1)^3 \, dx = \frac{(2x+1)^4}{2 \times 4} + c$$
$$= \frac{1}{8}(2x+1)^4 + c$$

Example 7 Find $\int \frac{1}{(2x-1)^2} \, dx$.

Solution:

$$\int \frac{1}{(2x-1)^2} \, dx = \int (2x-1)^{-2} \, dx$$
$$= \frac{(2x-1)^{-2+1}}{2(-2+1)} + c$$
$$= -\frac{1}{2(2x-1)} + c$$

Exercise 20.1

1. Find an expression for y if $\frac{dy}{dx}$ is each of the following:

   (a) $2x^3$ (b) -5 (c) $\sqrt{x}$ (d) $-\frac{1}{x^2}$ (e) $\frac{2}{\sqrt{x}}$ (f) $\frac{1}{2x^3}$

2. Find an expression for y if $\frac{dy}{dx}$ is each of the following:

   (a) $6x + 3$ (b) 4 (c) $3x(x + 2)$

   (d) $(x - 1)(x + 2)$ (e) $x(2 + \frac{1}{x})$ (f) $\frac{2x^2 + 3}{x^2}$

3. Integrate with respect to x.

   (a) $x^2 + \frac{1}{x^2}$ (b) $\frac{x^2 + 1}{2x^2}$

   (c) $3 - \sqrt{x}$ (d) $\sqrt{x}(\sqrt{x} + 3)$

4. Given that a and b are constants, integrate with respect to x.

   (a) $ax + b$ (b) $a - bx^2$

5. Find

   (a) $\int (2 + 4x - 3x^2) \, dx$, (b) $\int \left(x^4 - \frac{1}{x^2}\right) dx$,

   (c) $\int (2x - \sqrt{x})^2 \, dx$, (d) $\int \frac{x+1}{\sqrt{x}} \, dx$.

6. Find the equation of the curve which passes through the point (2, 4) and for which $\frac{dy}{dx} = x(3x - 1)$.

7. Find the equation of the curve which passes through the points (2, –2) and (4, 2) and for which $\frac{dy}{dx} = x^2(x - k)$ where k is a constant.

8. Given that the gradient of a curve is $2x + \frac{3}{x^2}$ and that the curve passes through the point (–1, 5), determine the equation of the curve.

9. Find the equation of the curve which passes through the points (1, 3) and (2, 9) and whose gradient is proportional to $x(2x^2 - 3)$.

10. Given that the gradient of a curve is $x(2 - 3x)$ and that the curve passes through the points (1, 2) and (–2, p), find the value of p.

11. Given that the gradient of a curve is $ax - 3$ and that the curve passes through the points (–1, 8) and (3, 4), find the equation of the curve.

12. Given that $\frac{dy}{dx}$ is directly proportional to $x^2 - 1$, and that $y = 3$ and $\frac{dy}{dx} = 9$ when $x = 2$, find the value of y when $x = 3$.

13. Find x as a function of t given that $\frac{dx}{dt} = 3t^2 + 2$ and that $x = 1$ when $t = 0$.

14. The rate of change of the area, A cm^2, of a circle is $6t^2 - 2t + 1$. Find A in terms of t if the area of the circle is 11 cm^2 when $t = 2$.

15. Integrate with respect to x.

(a) $(3x + 1)^4$ (b) $(1 - x)^3$ (c) $(2x + 5)^{-3}$

(d) $\sqrt{6x - 1}$ (e) $\frac{2}{(2x - 7)^2}$ (f) $\frac{1}{\sqrt{3 - 2x}}$

(g) $\frac{3}{5(3x - 1)^6}$ (h) $\frac{4}{3\sqrt{6x - 1}}$ (i) $\left(\frac{4}{1 - 2x}\right)^2$

16. Find

(a) $\int (1 - x)^6 \, dx$, (b) $\int 3(2x - 5)^2 \, dx$, (c) $\int \frac{2}{(1 - x)^2} \, dx$, (d) $\int \sqrt{4t - 1} \, dt$.

17. Given that $\frac{dy}{dx} = (3x - 2)^2$ and that $y = 0$ when $x = 1$, calculate the value of y when $x = 1.5$.

18. The gradient of a curve is $6(4x - 1)^2$ and the curve passes through the origin. Find the equation of the curve.

19. Find A as a function of x given that $\frac{dA}{dx} = 3(x - 1)^2$ and that $A = 10$ when $x = 3$.

20. Find s as a function of t given that $\frac{ds}{dt} = 6(2t - 1)^2 + 1$ and that $s = 4$ when $t = 1$.

20.2 Definite Integrals

Consider an expression $F(x)$ such that $\frac{d}{dx}(F(x)) = 3x^2$. What is the value of $F(b) - F(a)$?

Since $\frac{d}{dx}(F(x)) = 3x^2$, $F(x) = \int 3x^2 \, dx = x^3 + c$

Then $F(b) - F(a) = (b^3 + c) - (a^3 + c) = b^3 - a^3$ (c is cancelled out)

or simply $F(b) - F(a) = \left[x^3\right]_a^b$

which can be written as

$$F(b) - F(a) = \left[\int 3x^2 \, dx\right]_a^b$$

or

$$F(b) - F(a) = \int_a^b 3x^2 \, dx \quad \text{.......................... (1)}$$

Note that the constant c is cancelled out and so we ignore the constant and write $\int 3x^2 \, dx = x^3$ when computing the difference $F(b) - F(a)$.

Let $f(x) = 3x^2$.

Then $\frac{d}{dx}[F(x)] = f(x)$ and (1) becomes

$$F(b) - F(a) = \int_a^b f(x) \, dx.$$

In general, if $\frac{d}{dx}(F(x)) = f(x)$, then the change in $F(x)$ when x changes from a to b is

$$F(b) - F(a) = \left[F(x)\right]_a^b = \int_a^b f(x) \, dx.$$

$$\frac{d}{dx}[F(x)] = f(x) \quad \Rightarrow \quad \int_a^b f(x) \, dx = F(b) - F(a)$$

The integral $\int_a^b f(x) \, dx$ is known as a **definite integral**.

For example, we have

$$\frac{d}{dx}(3x^2 + 2x) = 6x + 2$$

and so

$$\begin{aligned}\int_2^3 (6x + 2) \, dx &= \left[3x^2 + 2x\right]_2^3 \\ &= [3(3)^2 + 2(3)] - [3(2)^2 + 2(2)] \\ &= 17\end{aligned}$$

Note that we ignore the constant c in the integral since it is cancelled out as mentioned earlier.

Example 8 Evaluate

(a) $\int_1^8 x^{-\frac{1}{3}}\,dx$, (b) $\int_0^4 \sqrt{2x+1}\,dx$.

Solution:

(a) $$\int_1^8 x^{-\frac{1}{3}}\,dx = \left[\frac{x^{\frac{2}{3}}}{\frac{2}{3}}\right]_1^8$$
$$= \frac{3}{2}\left[x^{\frac{2}{3}}\right]_1^8$$
$$= \frac{3}{2}(4-1)$$
$$= \frac{9}{2}$$

(b) $$\int_0^4 \sqrt{2x+1}\,dx = \int_0^4 (2x+1)^{\frac{1}{2}}\,dx$$
$$= \left[\frac{(2x+1)^{\frac{3}{2}}}{2\left(\frac{3}{2}\right)}\right]_0^4$$
$$= \frac{1}{3}(3^3 - 1^3)$$
$$= \frac{26}{3}$$

Example 9 Evaluate $\int_1^4 \frac{2x-1}{\sqrt{x}}\,dx$.

Solution:

$$\frac{2x-1}{\sqrt{x}} = \frac{2x}{\sqrt{x}} - \frac{1}{\sqrt{x}}$$
$$= 2x^{\frac{1}{2}} - x^{-\frac{1}{2}}$$

$$\int_1^4 \frac{2x-1}{\sqrt{x}}\,dx = \int_1^4 \left(2x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)dx$$
$$= \left[\frac{2x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4$$
$$= \left[\frac{4}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}}\right]_1^4$$
$$= \left[\frac{4}{3}(8) - 2(2)\right] - \left[\frac{4}{3} - 2\right]$$
$$= \frac{22}{3}$$

Example 10 Given that $\frac{d}{dx}[A(x)] = 2x - 1$, find the value of $A(3) - A(1)$.

Solution:

$$\frac{d}{dx}[A(x)] = 2x - 1$$
$$A(3) - A(1) = \int_1^3 (2x-1)\,dx$$
$$= \left[x^2 - x\right]_1^3$$
$$= 6$$

Example 11 Given that $y = (x-2)\sqrt{x+1}$, show that $\dfrac{dy}{dx} = \dfrac{3x}{2\sqrt{x+1}}$.

Hence evaluate $\displaystyle\int_3^8 \frac{x}{\sqrt{x+1}}\, dx$.

Solution: $y = (x-2)\sqrt{x+1}$

By the product rule,

$$\begin{aligned}\frac{dy}{dx} &= (x-2)\frac{d}{dx}(\sqrt{x+1}) + \sqrt{x+1}\,\frac{d}{dx}(x-2)\\ &= (x-2)\left(\frac{1}{2\sqrt{x+1}}\right) + \sqrt{x+1}\\ &= \frac{(x-2)+2(x+1)}{2\sqrt{x+1}}\\ &= \frac{3x}{2\sqrt{x+1}}\end{aligned}$$

Reversing the process of integration, we have:

$$\int_3^8 \frac{3x}{2\sqrt{x+1}}\, dx = \left[(x-2)\sqrt{x+1}\right]_3^8$$

$$\begin{aligned}\frac{3}{2}\int_3^8 \frac{x}{\sqrt{x+1}}\, dx &= 6\sqrt{9} - \sqrt{4}\\ &= 16\end{aligned}$$

Hence $\displaystyle\int_3^8 \frac{x}{\sqrt{x+1}}\, dx = \frac{32}{3}$.

(a) $\dfrac{d}{dx}[F(x)] = f(x) \Rightarrow \displaystyle\int f(x)\, dx = F(x) + c$, **for indefinite integral**

$\dfrac{d}{dx}[F(x)] = f(x) \Rightarrow \displaystyle\int_a^b f(x)\, dx = F(b) - F(a)$, **for definite integral**

(b) **If α and β are constants, $\dfrac{d}{dx}[F(x)] = f(x)$ and $\dfrac{d}{dx}[G(x)] = g(x)$, then**

$$\frac{d}{dx}[\alpha F(x) + \beta G(x)] = \alpha f(x) + \beta g(x)$$

and so

$\displaystyle\int[\alpha f(x) + \beta g(x)]\, dx = \alpha\int f(x)\, dx + \beta\int g(x)\, dx$, **for indefinite integral,**

$\displaystyle\int_a^b [\alpha f(x) + \beta g(x)]\, dx = \alpha\int_a^b f(x)\, dx + \beta\int_a^b g(x)\, dx$, **for definite integral.**

Some Results Regarding Definite Integrals

(a) $\int_a^a f(x)\,dx = 0$

(b) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(c) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

Proof: If $\frac{d}{dx}[F(x)] = f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

(a) $\int_a^a f(x)\,dx = F(a) - F(a) = 0$

(b) $\int_a^b f(x)\,dx = F(b) - F(a) = -[F(a) - F(b)] = -\int_b^a f(x)\,dx$

(c)
$$\begin{aligned}\int_a^b f(x)\,dx + \int_b^c f(x)\,dx &= [F(b) - F(a)] + [F(c) - F(b)]\\ &= F(c) - F(a)\\ &= \int_a^c f(x)\,dx\end{aligned}$$

Example 12 Given that $\int_1^3 f(x)\,dx = 3$ and $\int_3^4 f(x)\,dx = 2$, find

(a) $\int_3^1 f(x)\,dx$, **(b)** $\int_1^4 f(x)\,dx$,

(c) $\int_1^3 f(x)\,dx + \int_4^3 f(x)\,dx$.

Solution:

(a) $\int_3^1 f(x)\,dx = -\int_1^3 f(x)\,dx = -3$

(b) $\int_1^4 f(x)\,dx = \int_1^3 f(x)\,dx + \int_3^4 f(x)\,dx = 3 + 2 = 5$

(c)
$$\begin{aligned}\int_1^3 f(x)\,dx + \int_4^3 f(x)\,dx &= \int_1^3 f(x)\,dx - \int_3^4 f(x)\,dx\\ &= 3 - 2\\ &= 1\end{aligned}$$

Example 13 Given that $\int_1^3 f(x)\,dx = 3$ and $\int_3^6 f(x)\,dx = 5$, find

(a) $\int_1^3 2f(x)\,dx$, (b) $\int_1^3 [f(x) + x]\,dx$,

(c) $\int_1^6 [2f(x) + x]\,dx$.

Solution:

(a) $\int_1^3 2f(x)\,dx = 2\int_1^3 f(x)\,dx = 2 \times 3 = \mathbf{6}$

(b) $\int_1^3 [f(x) + x]\,dx = \int_1^3 f(x)\,dx + \int_1^3 x\,dx = 3 + \left[\frac{x^2}{2}\right]_1^3 = \mathbf{7}$

(c) $$\int_1^6 [2f(x) + x]\,dx = 2\int_1^6 f(x)\,dx + \int_1^6 x\,dx$$
$$= 2\left[\int_1^3 f(x)\,dx + \int_3^6 f(x)\,dx\right] + \left[\frac{x^2}{2}\right]_1^6$$
$$= 2(3 + 5) + \left(\frac{36}{2} - \frac{1}{2}\right)$$
$$= \mathbf{33\frac{1}{2}}$$

Exercise 20.2

1. Evaluate the following definite integrals.

(a) $\int_2^5 3x\,dx$ (b) $\int_1^9 x^{\frac{1}{2}}\,dx$ (c) $\int_1^8 \frac{1}{2}x^{-\frac{1}{3}}\,dx$

(d) $\int_2^3 \frac{1}{3x^2}\,dx$ (e) $\int_4^9 \frac{1}{\sqrt{x}}\,dx$ (f) $\int_1^4 x\sqrt{x}\,dx$

2. Evaluate the following definite integrals.

(a) $\int_{-1}^1 (8x - 4)\,dx$ (b) $\int_{-1}^0 (3x^2 - 2x + 5)\,dx$

(c) $\int_1^4 (6x - 3\sqrt{x})\,dx$ (d) $\int_1^4 \left(\sqrt{x} - \frac{2}{\sqrt{x}}\right)dx$

(e) $\int_1^2 \left(x^2 - \frac{4}{x^2}\right)dx$ (f) $\int_1^2 \left(8x^3 - 2 + \frac{1}{2x^2}\right)dx$

3. Evaluate the following definite integrals.

(a) $\int_0^2 x(x^2 - 2)\,dx$ (b) $\int_1^2 (x + 1)(x - 2)\,dx$

(c) $\int_{-1}^0 x(x - 2)(x + 2)\,dx$ (d) $\int_0^4 \sqrt{x}(1 - \sqrt{x})\,dx$

(e) $\int_1^3 \frac{1}{x^2}(4x^2 - 9)\,dx$ (f) $\int_0^1 x^2(2 - 3\sqrt{x})\,dx$

4. Evaluate the following definite integrals.

(a) $\int_1^4 \frac{x^2 + 1}{x^2}\,dx$ (b) $\int_1^2 \frac{1 - 2x^3}{x^2}\,dx$

(c) $\int_1^4 \frac{2x - 1}{\sqrt{x}}\,dx$ (d) $\int_1^9 \frac{3 - 2\sqrt{x}}{x^2}\,dx$

(e) $\int_1^3 \frac{1 - 4x + x^3}{2x^3}\,dx$ (f) $\int_1^2 \frac{(x + 3)(x - 3)}{x^2}\,dx$

5. Evaluate

(a) $\int_0^2 (4t^2 - t)\,dt$, (b) $\int_1^2 \frac{2t^2 + 1}{t^2}\,dt$,

(c) $\int_1^3 2r(r - 2)\,dr$, (d) $\int_2^3 \frac{(r - 1)(r + 1)}{r^2}\,dr$.

6. Find the value of the constant k if $\int_1^2 (4x + k)\,dx = 1$.

7. Given that $\frac{d}{dx}[F(x)] = \sqrt{x} - 1$, evaluate $F(4) - F(1)$.

8. Given that $\frac{d}{dx}[V(x)] = \pi x^2$, evaluate $V(3) - V(1)$.

9. Given that $\frac{d}{dt}[s(t)] = 2t^2 - 1$, find the value of $s(5) - s(0)$.

10. Show that $\frac{d}{dx}(x\sqrt{x + 1}) = \frac{3x + 2}{2\sqrt{x + 1}}$. Hence evaluate $\int_3^8 \frac{3x + 2}{\sqrt{x + 1}}\,dx$.

11. Show that $\frac{d}{dx}\left(\frac{x}{1 + 2x}\right) = \frac{1}{(1 + 2x)^2}$. Hence or otherwise, evaluate $\int_1^4 \left(\frac{2}{1 + 2x}\right)^2 dx$.

12. Show that $\frac{d}{dx}\left(\frac{2x}{\sqrt{x + 1}}\right) = \frac{x + 2}{(x + 1)^{\frac{3}{2}}}$. Hence evaluate $\int_0^8 \frac{x + 2}{(x + 1)^{\frac{3}{2}}}\,dx$.

13. Given that $y = x\sqrt{1 + 2x^2}$, show that $\frac{dy}{dx} = \frac{1 + 4x^2}{\sqrt{1 + 2x^2}}$.

Hence evaluate $\int_0^2 \frac{1 + 4x^2}{\sqrt{1 + 2x^2}}\,dx$.

14. Given that $\int_2^5 f(x)\,dx = 8$ and $\int_2^3 f(x)\,dx = 3$, find

(a) $\int_2^5 [3f(x) - 2x]\,dx$, **(b)** $\int_3^5 (2f(x) - 1)\,dx$.

15. Given that $\int_0^3 f(x)\,dx = \int_3^8 f(x)\,dx = 12$, find

(a) $\int_0^8 f(x)\,dx$, **(b)** $\int_0^3 f(x)\,dx + \int_8^3 f(x)\,dx$.

Find the value of the constant m for which $\int_3^8 [f(x) - mx]\,dx = 0$.

16. Given that $\int_1^3 h(x)\,dx = 5$, evaluate

(a) $\int_1^3 [h(x) + x]\,dx$, **(b)** $\int_3^1 [h(x) - x]\,dx$.

Find the value of the constant k for which $\int_1^3 [h(x) + kx^2]\,dx = 31$.

17. Given that $\int_0^5 g(x)\,dx = 10$, find the value of k for which $\int_0^5 [g(x) - kx^2]\,dx = 0$.

20.3 Integration of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \quad \Rightarrow \quad \int \cos x\,dx = \sin x + c$$

$$\frac{d}{dx}(-\cos x) = \sin x \quad \Rightarrow \quad \int \sin x\,dx = -\cos x + c$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad \Rightarrow \quad \int \sec^2 x\,dx = \tan x + c$$

Example 14 Find

(a) $\int (1 + \cos x)\,dx$, **(b)** $\int_0^{\frac{\pi}{2}} (2 + \sin x)\,dx$.

Solution:

(a) $\int (1 + \cos x)\,dx = x + \sin x + c$

(b)
$$\begin{aligned}\int_0^{\frac{\pi}{2}} (2 + \sin x)\,dx &= [2x - \cos x]_0^{\frac{\pi}{2}}\\ &= \left(2 \times \frac{\pi}{2} - \cos\frac{\pi}{2}\right) - (0 - \cos 0)\\ &= (\pi - 0) - (0 - 1)\\ &= \pi + 1\end{aligned}$$

Example 15 Find $\int \tan^2 x \, dx$.

Solution: Using the identity $\tan^2 x \equiv \sec^2 x - 1$,

$$\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \mathbf{\tan x - x + c}$$

If $a \neq 0$, we have:

$$\frac{d}{dx}\left[\frac{1}{a}\sin(ax+b)\right] = \cos(ax+b) \Rightarrow \int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$$

$$\frac{d}{dx}\left[-\frac{1}{a}\cos(ax+b)\right] = \sin(ax+b) \Rightarrow \int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$$

$$\frac{d}{dx}\left[\frac{1}{a}\tan(ax+b)\right] = \sec^2(ax+b) \Rightarrow \int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + c$$

Example 16 Find the following integrals.

(a) $\int 3\cos(2x+1)\,dx$ **(b)** $\int \sin 2x \, dx$

(c) $\int \cos\left(\frac{\pi}{4} - 2x\right) dx$ **(d)** $\int 2\sin(3x-1)\,dx$

Solution:

(a) $$\int 3\cos(2x+1)\,dx = 3\left[\frac{1}{2}\sin(2x+1)\right] + c$$
$$= \mathbf{\frac{3}{2}\sin(2x+1) + c}$$

(b) $$\int \sin 2x \, dx = \mathbf{-\frac{1}{2}\cos 2x + c}$$

(c) $$\int \cos\left(\frac{\pi}{4} - 2x\right) dx = \frac{1}{(-2)}\sin\left(\frac{\pi}{4} - 2x\right) + c$$
$$= \mathbf{-\frac{1}{2}\sin\left(\frac{\pi}{4} - 2x\right) + c}$$

(d) $$\int 2\sin(3x-1)\,dx = 2\left[-\frac{1}{3}\cos(3x-1)\right] + c$$
$$= \mathbf{-\frac{2}{3}\cos(3x-1) + c}$$

Example 17 Find the following integrals.

(a) $\int(2 + \sin 2x)\ dx$, (b) $\int(2\cos 2x + 1)\ dx$,

(c) $\int_0^{\frac{\pi}{4}} \cos x\ dx$.

Solution:

(a) $\int(2 + \sin 2x)\ dx = \mathbf{2x - \frac{1}{2}\cos 2x + c}$

(b) $\int(2\cos 2x + 1)\ dx = 2\left(\frac{1}{2}\sin 2x\right) + x + c = \mathbf{\sin 2x + x + c}$

(c) $\int_0^{\frac{\pi}{4}} \cos x\ dx = \left[\sin x\right]_0^{\frac{\pi}{4}} = \sin\frac{\pi}{4} - \sin 0 = \mathbf{\frac{1}{\sqrt{2}}}$

Exercise 20.3

1. Integrate with respect to x.

(a) $\sin x + 2$ **(b)** $1 - 3\cos x$ **(c)** $\cos x - \sin x$

(d) $\sec^2 x - 4\sin x$ **(e)** $3\cos x - 2\sin x$ **(f)** $4\cos x + 3\sec^2 x$

2. Integrate with respect to x.

(a) $\cos 2x$ **(b)** $\sin 3x$

(c) $2\cos 4x$ **(d)** $\cos\frac{1}{2}x$

(e) $\frac{1}{2}\sin\frac{1}{4}x$ **(f)** $\cos 3x$

(g) $-3\sin\frac{1}{2}x$ **(h)** $2\cos(1 - x)$

(i) $6\sin(3x + 2)$ **(j)** $\cos(1 - 2x)$

(k) $-\sin(2x + 1)$ **(l)** $3\sin(2 - x)$

(m) $\cos\left(2x + \frac{\pi}{4}\right)$ **(n)** $4\sin\left(x - \frac{\pi}{4}\right)$

3. Evaluate the following definite integrals:

(a) $\int_0^{\frac{\pi}{6}} \cos x\ dx$ **(b)** $\int_0^{\frac{\pi}{2}} \sin x\ dx$ **(c)** $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\cos x\ dx$

(d) $\int_0^{\frac{\pi}{4}} \sec^2 x\ dx$ **(e)** $\int_0^{\frac{\pi}{2}} (1 - 2\sin x)\ dx$ **(f)** $\int_0^{\frac{\pi}{6}} (3\cos x - 2)\ dx$

(g) $\int_0^{\frac{\pi}{4}} \sin 2x\ dx$ **(h)** $\int_{-\frac{\pi}{3}}^{\frac{\pi}{6}} \cos 3x\ dx$ **(i)** $\int_0^{\frac{\pi}{8}} \sec^2 2x\ dx$

(j) $\int_{\frac{\pi}{12}}^{\frac{\pi}{3}} \cos\left(2x + \frac{\pi}{3}\right) dx$ (k) $\int_{\frac{\pi}{2}}^{\pi} 2\sin(\pi - x)\, dx$ (l) $\int_0^{\frac{\pi}{4}} (\cos 2x - \sin x)\, dx$

4. Evaluate $\int_0^{\frac{\pi}{4}} (1 + \tan^2 x)\, dx$.

5. Show that $\int_0^{\frac{\pi}{4}} \frac{1 + \cos^2 x}{\cos^2 x}\, dx = \frac{\pi}{4} + 1$.

20.4 Integration of Exponential Functions

For exponential functions, we have:

$$\frac{d}{dx}(e^x) = e^x \quad \Rightarrow \quad \int e^x\, dx = e^x + c$$

and

$$\frac{d}{dx}(-e^{-x}) = e^{-x} \quad \Rightarrow \quad \int e^{-x}\, dx = -e^{-x} + c$$

In general, if $a \neq 0$,

$$\frac{d}{dx}\left(\frac{1}{a}e^{ax+b}\right) = e^{ax+b} \quad \Rightarrow \quad \int e^{ax+b}\, dx = \frac{1}{a}e^{ax+b} + c$$

Example 18 Find

(a) $\int e^{2x+1}\, dx$, (b) $\int e^{3-2x}\, dx$.

Solution:

(a) $\int e^{2x+1}\, dx = \frac{1}{2}e^{2x+1} + c$

(b) $\int e^{3-2x}\, dx = \frac{1}{-2}e^{3-2x} + c = -\frac{1}{2}e^{3-2x} + c$

Example 19 Evaluate

(a) $\int_0^2 e^{2x}\, dx$, (b) $\int_1^{\ln 2} e^x\, dx$.

Solution:

(a) $\int_0^2 e^{2x}\, dx = \left[\frac{1}{2}e^{2x}\right]_0^2 = \frac{1}{2}(e^4 - e^0) = \frac{1}{2}(e^4 - 1)$

(b) $\int_1^{\ln 2} e^x\, dx = \left[e^x\right]_1^{\ln 2} = e^{\ln 2} - e^1 = 2 - e$

Note: $e^{\ln x} = x$ for $x > 0$.

Example 20 Find y as a function of x given that $\frac{dy}{dx} = 2e^x + 1$ and $y = 3$ when $x = 0$.

Solution:

$$y = \int \left(2e^x + 1\right) dx = 2e^x + x + c$$

$y = 3$ when $x = 0 \Rightarrow$ $3 = 2e^0 + c$

$3 = 2 + c$

$c = 1$

Hence $y = 2e^x + x + 1$.

Example 21 Differentiate $(x - 1)e^x$ with respect to x and hence integrate xe^x with respect to x.

Solution:

By the product rule,

$$\frac{d}{dx}[(x-1)e^x] = (x-1)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x-1) = (x-1)e^x + e^x = xe^x$$

Hence $\int xe^x\, dx = (x - 1)e^x + c$.

Exercise 20.4

1. Integrate with respect to x.
   (a) $e^x + 1$ (b) e^{2x} (c) $2e^{3x}$ (d) $e^{-x} - e^x$
   (e) e^{-2x} (f) $2e^{\frac{1}{2}x}$ (g) e^{2x+1} (h) $3e^{1-x}$
   (i) $\frac{1}{2}e^{3x+2}$ (j) $4e^{\frac{1}{2}(1-x)}$

2. Evaluate the following, giving your answers in terms of e, where appropriate.
   (a) $\int_0^2 e^x\, dx$ (b) $\int_0^1 e^{2x}\, dx$ (c) $\int_0^2 e^{-\frac{1}{2}x}\, dx$
   (d) $\int_1^2 e^{1-x}\, dx$ (e) $\int_0^{\ln 2} e^{3x}\, dx$ (f) $\int_0^{\ln 3} e^x\, dx$

3. Find y as a function of x given that $\frac{dy}{dx} = 1 - 3e^x$ and that $y = 4$ when $x = 0$.

4. Express y in terms of x given that $\frac{dy}{dx} = 2e^{-x}$ and that $y = -1$ when $x = 0$.

5. Find y as a function of x given that $\frac{dy}{dx} = e^{2x}$ and that $y = 6$ when $x = \ln 3$.

6. Find $\frac{d}{dx}(e^{\sin x})$ and hence find $\int \cos x \, e^{\sin x} \, dx$.

7. Find $\frac{d}{dx}(e^{x^2})$ and hence evaluate $\int_0^1 x e^{x^2} \, dx$.

Important Notes

1. **Process of integration — Reverse process of differentiation**

   (a) $\frac{d}{dx}[F(x)] = f(x) \Rightarrow \int f(x) \, dx = F(x) + c$,
   where c is an arbitrary constant.

   (b) If $\frac{d}{dx}[F(x)] = f(x)$ and $\frac{d}{dx}[G(x)] = g(x)$, then
   $\int [\alpha f(x) + \beta g(x)] \, dx = \alpha \int f(x) \, dx + \beta \int g(x) \, dx$, where α and β are constants.

2. **Indefinite integrals of some standard functions**

   (a) ***Simple algebraic functions***

$$\int ax^n \, dx = \frac{ax^{n+1}}{n+1} + c, \ n \neq -1$$

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c, \ n \neq -1, \ a \neq 0$$

   (b) ***Trigonometric functions***

$$\int \cos x \, dx = \sin x + c$$

$$\int \sin x \, dx = -\cos x + c$$

$$\int \sec^2 x \, dx = \tan x + c$$

   For $a \neq 0$,

$$\int \cos (ax+b) \, dx = \frac{1}{a} \sin (ax+b) + c$$

$$\int \sin (ax+b) \, dx = -\frac{1}{a} \cos (ax+b) + c$$

$$\int \sec^2 (ax+b) \, dx = \frac{1}{a} \tan (ax+b) + c$$

(c) *Exponential functions*

$$\int e^x \, dx = e^x + c$$

$$\int e^{ax+b} \, dx = \frac{1}{a}e^{ax+b} + c, \ a \neq 0$$

3. Definite integrals

(a) $\frac{d}{dx}[F(x)] = f(x) \Rightarrow \int_a^b f(x)\,dx = F(b) - F(a)$

(b) $\int_a^b [af(x) + bg(x)]\,dx = a\int_a^b f(x)\,dx + b\int_a^b g(x)\,dx$

(c) **(i)** $\int_a^a f(x)\,dx = 0$

(ii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(iii) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

Miscellaneous Examples

Example 22 Show that $\frac{d}{dx}(x \ln x - x) = \ln x$ and hence evaluate $\int_1^3 \ln x \, dx$.

Solution:

$$\frac{d}{dx}(x \ln x - x) = \frac{d}{dx}(x \ln x) - 1 = x\left(\frac{1}{x}\right) + \ln x - 1 = \ln x$$

Hence $\int_1^3 \ln x \, dx = [x \ln x - x]_1^3$

$= (3 \ln 3 - 3) - (\ln 1 - 1)$

$= \mathbf{3 \ln 3 - 2}$

Example 23 Integrate with respect to x.

(a) $(1 - e^x)^2$

(b) $\frac{2 - e^{4x}}{e^x}$

Solution:

(a) $(1 - e^x)^2 = 1 - 2e^x + (e^x)^2$

$= 1 - 2e^x + e^{2x}$

$$\int (1 - e^x)^2 \, dx = \int (1 - 2e^x + e^{2x}) \, dx = x - 2e^x + \frac{1}{2}e^{2x} + c$$

(b) $\dfrac{2 - e^{4x}}{e^x} = \dfrac{2}{e^x} - \dfrac{e^{4x}}{e^x} = 2e^{-x} - e^{3x}$

$$\int \frac{2 - e^{4x}}{e^x} \, dx = \int (2e^{-x} - e^{3x}) \, dx = -2e^{-x} - \frac{1}{3}e^{3x} + c$$

Miscellaneous Exercise 20

1. Find

(a) $\displaystyle\int x^2(x - 3) \, dx$, (b) $\displaystyle\int \frac{2x^2 - \sqrt{x}}{x} \, dx$.

2. Evaluate $\displaystyle\int_0^4 \sqrt{2x + 1} \, dx$.

3. Evaluate $\displaystyle\int_1^4 (2x - \sqrt{x})^2 \, dx$.

4. Evaluate $\displaystyle\int_0^1 \frac{(4x + 1)^4 - 7}{2(4x + 1)^2} \, dx$.

5. Find the values of a if $\displaystyle\int_1^a (3 - 2x) \, dx = \int_2^1 4x \, dx$.

6. Find the value of k if $\displaystyle\int_2^k 6(1 - x)^2 \, dx = 52$.

7. The curve for which $\dfrac{dy}{dx} = 3x^2 + k$, where k is a constant, has a turning point at $(-2, 6)$. Find
   (a) the value of k,
   (b) the equation of the curve and the coordinates of the point at which the curve meets the y-axis.

8. The curve for which $\dfrac{dy}{dx} = 2x + a$ where a is a constant is such that the normal at $(1, 5)$ cuts the x-axis at $(6, 0)$. Find the value of a and the equation of the curve.

9. The curve for which $\frac{dy}{dx} = kx - 5$, where k is a constant is such that the tangent at (2, 2) passes through the origin. Determine

(a) the value of k,

(b) the equation of the curve.

10. Given that $\frac{dy}{dx} = k\sqrt{x}$, where k is a constant, and that when $x = 4$, y and $\frac{dy}{dx}$ are equal to 19 and 6 respectively. Find the numerical value of y when $x = 1$.

11. Given that $\frac{d}{dx}[F(x)] = x^2 - \sqrt{x}$, evaluate $F(4) - F(2)$. If $F(1) = 3$, find $F(x)$.

12. Show that $\frac{d}{dx}\left(\frac{x}{1 + 2x}\right) = \frac{1}{(1 + 2x)^2}$. Hence or otherwise, evaluate $\int_1^4 \left(\frac{3}{1 + 2x}\right)^2 dx$.

13. Given that $y = x^2\sqrt{2x - 1}$, show that $\frac{dy}{dx} = \frac{x(5x - 2)}{\sqrt{2x - 1}}$.

Hence evaluate $\int_1^5 \frac{x(5x - 2)}{\sqrt{2x - 1}} dx$.

14. Given that $\int_0^1 f(x)\, dx = \int_1^3 f(x)\, dx = 3$, evaluate

(a) $\int_0^1 f(x)\, dx + \int_3^1 f(x)\, dx$, **(b)** $\int_3^0 f(x)\, dx$,

(c) $\int_1^3 [2f(x) + 5]\, dx$.

15. Evaluate

(a) $\int_0^{\frac{\pi}{4}} (\cos x - 3 \sin x)\, dx$, **(b)** $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin\left(2x - \frac{\pi}{3}\right) dx$.

16. Show that $\frac{d}{dx}\left(\frac{1 + \cos x}{\sin x}\right) = -\frac{1}{1 - \cos x}$ and evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{1 - \cos x} dx$.

17. Show that $\frac{d}{dx}(\tan^3 x) = 3 \tan^2 x \sec^2 x$ and hence evaluate $\int_0^{\frac{\pi}{4}} \left(\frac{\tan x}{\cos x}\right)^2 dx$.

18. Show that $\int_1^2 e^{3x - 2}\, dx \approx 17.3$.

19. Find

(a) $\int (2 + e^x)^2 \, dx$, **(b)** $\int \frac{e^{3x} - 2}{e^x} \, dx$.

20. Find $\frac{d}{dx}\left(e^{\sqrt{x}}\right)$ and hence evaluate $\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx$.

21. Show that $\frac{d}{dx}[\ln(x^2 + 1)] = \frac{2x}{x^2 + 1}$ and hence evaluate $\int_0^1 \frac{x}{x^2 + 1} \, dx$.

22. Show that $\frac{d}{dx}[\ln(\cos x)] = -\tan x$ and hence evaluate $\int_0^{\frac{\pi}{3}} \tan x \, dx$.

23. Show that $\frac{d}{dx}[\ln(e^x + 1)] = \frac{e^x}{e^x + 1}$ and hence evaluate $\int_0^{\ln 2} \frac{e^x}{e^x + 1} \, dx$.

24. Find $\frac{d}{dx}(\sin^3 2x)$ and hence evaluate

(a) $\int_0^{\frac{\pi}{4}} \sin^2 2x \cos 2x \, dx$, **(b)** $\int_0^{\frac{\pi}{4}} \cos^3 2x \, dx$.

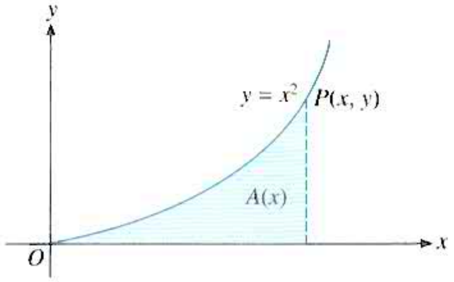
21 Applications of Integration

In Chapter 20, we studied integration as the reverse of differentiation and the indefinite and definite integrals of some basic functions. Now we shall apply this knowledge to find areas of regions enclosed by curves and lines. In Chapter 22, we will see the applications of integration to some kinematic problems.

21.1 Area between a Curve and an Axis

Area between a Curve and the *x*-axis

Let us look at an example to illustrate the application of definite integral to find the area function $A(x)$ of a shaded region under a curve $y = x^2$ as shown on the right:



The area function $A(x)$ and the curve $y = x^2$ are related by $\frac{\mathrm{d}}{\mathrm{d}x}(A(x)) = x^2$ as shown below. Corresponding to a small change δx in x, a change in A is δA as shown.

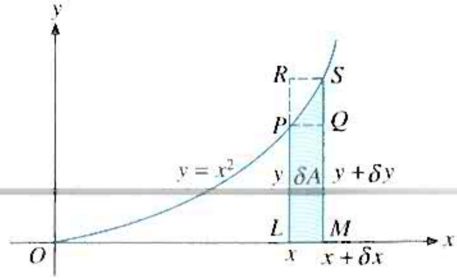
Then,

$$\begin{aligned} \text{area of } PQML &< \delta A < \text{area of } RSML. \\ PL \times LM &< \delta A < SM \times LM \\ y\delta x &< \delta A < (y + \delta y)\delta x \\ y &< \frac{\delta A}{\delta x} < (y + \delta y) \end{aligned}$$

Let $\delta x \to 0$, then $\delta y \to 0$ and so

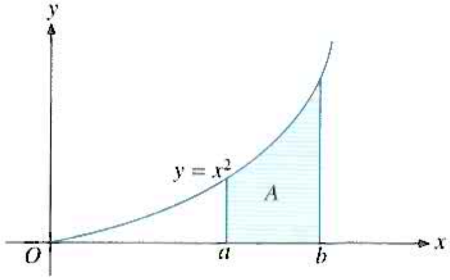
$$\frac{\delta A}{\delta x} \to y,$$

i.e. $$\frac{\mathrm{d}A}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta A}{\delta x} = y.$$



Hence for the curve $y = x^2$, the derivative of the corresponding area function A is $\frac{dA}{dx} = x^2$.

By the formula of definite integral,



$$\frac{dA}{dx} = y \Rightarrow A(b) - A(a) = \int_a^b y\,dx$$

or $$\frac{dA}{dx} = x^2 \Rightarrow A(b) - A(a) = \int_a^b x^2\,dx.$$

$\therefore$ the area under the curve $y = x^2$ and between $x = a$ and $x = b$ is given by

$$A = \int_a^b x^2\,dx.$$

The area under the curve $y = x^2$ and between $x = 2$ and $x = 3$ is

$$A(3) - A(2) = \int_2^3 x^2\,dx = \left[\frac{x^3}{3}\right]_2^3 = 6\frac{1}{3} \text{ sq. units.}$$

In general, if $A(x)$ is an area function under the curve $y = f(x)$, then,

$$\frac{dA}{dx} = y \text{ or } \frac{dA}{dx} = f(x)$$

and so:

Area under the curve $y = f(x)$ from $x = a$ to $x = b$ is given by

$$A = \int_a^b y\,dx \text{ or } A = \int_a^b f(x)\,dx,$$

where $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$.

Example 1 The diagram shows part of the curve $y = \frac{4}{x^2}$. Find

(a) the area of the region A,

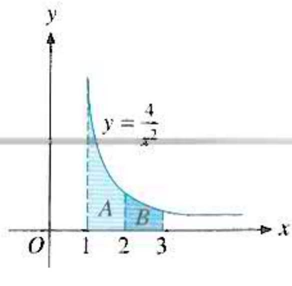
(b) the area of the region B.

Solution: **(a)** Area of the region $A = \int_1^2 \frac{4}{x^2}\,dx$

$= 4\left[-\frac{1}{x}\right]_1^2$

$= 4\left[\left(-\frac{1}{2}\right) - \left(-\frac{1}{1}\right)\right]$

$=$ **2 sq. units**



(b) Area of the region $B = \int_2^3 \frac{4}{x^2}\,dx$

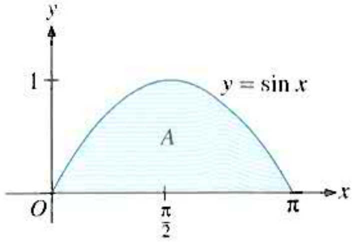
$$= 4\left[-\frac{1}{x}\right]_2^3$$

$$= 4\left[\left(-\frac{1}{3}\right) - \left(-\frac{1}{2}\right)\right]$$

$$= \frac{2}{3} \textbf{ sq. unit}$$

Example 2

The diagram shows part of the curve $y = \sin x$. Calculate the area enclosed by the curve and the x-axis.



Solution:

The area enclosed by the curve and the x-axis is

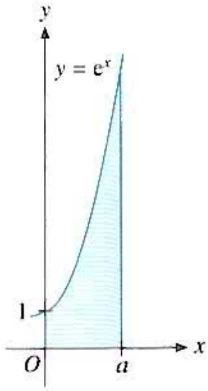
$$A = \int_0^{\pi} \sin x\,dx$$
$$= [-\cos x]_0^{\pi}$$
$$= -(\cos \pi - \cos 0)$$
$$= -(-1 - 1)$$
$$= \textbf{2 sq. units}$$

Example 3

The diagram shows the region bounded by the curve $y = e^x$, the axes and the line $x = a$, where $a > 0$.

(a) Find, in terms of a, the area of the region.

(b) Given that the area of the region is 3.5 unit^2, calculate, to two decimal places, the value of a.



Solution:

(a) Area of the region $= \int_0^a e^x\,dx$

$$= \left[e^x\right]_0^a$$
$$= e^a - e^0$$
$$= \mathbf{(e^a - 1)} \textbf{ sq. units}$$

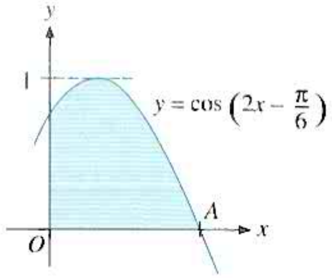
(b) Since $e^a - 1 = 3.5$

$$e^a = 4.5$$
$$a = \ln 4.5$$
$$\approx \mathbf{1.50} \text{ (to 2 dec. pl.)}$$

Example 4 The diagram shows part of the curve $y = \cos\left(2x - \frac{\pi}{6}\right)$. Find

(a) the x-coordinate of the point A where the curve cuts the x-axis,

(b) the area of the shaded region.



Solution: (a) At A, $\cos\left(2x - \frac{\pi}{6}\right) = 0$ and so $2x - \frac{\pi}{6} = \frac{\pi}{2}$

$$x = \frac{\pi}{3}$$

(b) Area of the shaded region $= \displaystyle\int_0^{\frac{\pi}{3}} \cos\left(2x - \frac{\pi}{6}\right) dx$

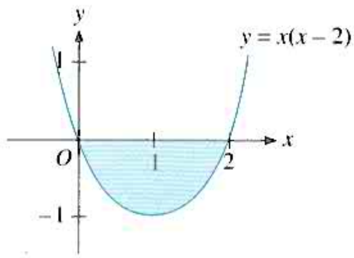
$$= \left[\frac{1}{2}\sin\left(2x - \frac{\pi}{6}\right)\right]_0^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left[\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{6}\right)\right]$$

$$= \frac{1}{2}(1 + 0.5)$$

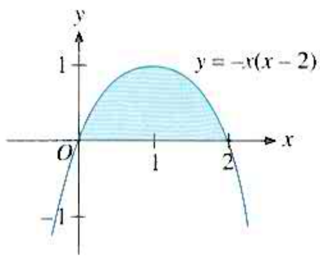
$$= \mathbf{0.75 \text{ sq. unit}}$$

Example 5 The diagram shows part of the curve $y = x(x - 2)$. Find the area of the shaded region.



Solution: Area of shaded region

= Area under the curve $y = -x(x - 2)$ from $x = 0$ to $x = 2$ as shown.



$$= \int_0^2 -x(x-2)\, dx = \int_0^2 (-x^2 + 2x)\, dx$$

$$= \left[-\frac{x^3}{3} + x^2\right]_0^2$$

$$= \left[-\frac{8}{3} + 4\right]$$

$$= \mathbf{\frac{4}{3} \text{ sq. units}}$$

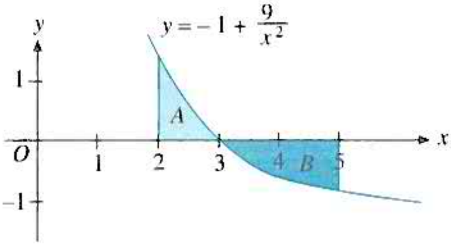
In example 5, the curve $y = -x(x - 2)$ is the reflection of the curve $y = x(x - 2)$ in the x-axis. In general, the curve $y = -f(x)$ is the reflection of the curve $y = f(x)$ in the x-axis and we have the following result for an enclosed area below the x-axis.

Area enclosed by the curve $y = f(x)$ from $x = a$ to $x = b$ is given by

$$A = \int_a^b -f(x)\,dx,$$

where $f(x) \leqslant 0$ for $a \leqslant x \leqslant b$.

Example 6 The diagram shows part of the curve $y = -1 + \frac{9}{x^2}$. Find the area of each of the two shaded regions A and B.



Solution:

Area of the region $A = \int_2^3 \left(-1 + \frac{9}{x^2}\right) dx$

$$= \left[-x - \frac{9}{x}\right]_2^3$$

$$= \left[-3 - \frac{9}{3}\right] - \left[-2 - \frac{9}{2}\right]$$

$$= \frac{1}{2} \text{ sq. unit}$$

Area of the region $B = \int_3^5 -\left(-1 + \frac{9}{x^2}\right) dx$

$$= \int_3^5 \left(1 - \frac{9}{x^2}\right) dx$$

$$= \left[x + \frac{9}{x}\right]_3^5$$

$$= \left[5 + \frac{9}{5}\right] - \left[3 + \frac{9}{3}\right]$$

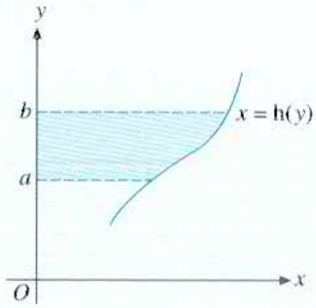
$$= \frac{4}{5} \text{ sq. unit}$$

Area between a Curve and the y-axis

Area bounded by the curve $x = h(y)$, the y-axis and the lines $y = a$ and $y = b$ is given by

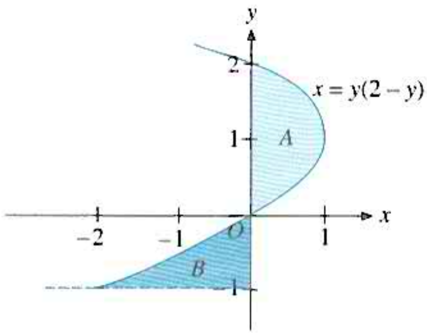
$$A = \int_a^b x \, dy,$$

where $x \geq 0$ for $a \leq y \leq b$.



Example 7

The diagram shows the shaded regions A and B enclosed by the curve $x = y(2 - y)$ and the y-axis. Find the area of each of the regions.



Solution

$$\text{Area of the shaded region } A = \int_0^2 x \, dy$$

$$= \int_0^2 (2y - y^2) \, dy$$

$$= \left[y^2 - \frac{y^3}{3} \right]_0^2$$

$$= 4 - \frac{8}{3}$$

$$= \mathbf{\frac{4}{3}} \textbf{ sq. units}$$

$$\text{Area of the shaded region } B = \int_{-1}^0 -x \, dy$$

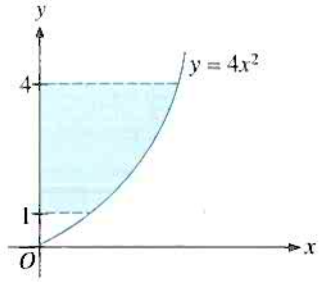
$$= \int_{-1}^0 (y^2 - 2y) \, dy$$

$$= \left[\frac{y^3}{3} - y^2 \right]_{-1}^0$$

$$= 0 - \left[\frac{-1}{3} - 1 \right]$$

$$= \mathbf{\frac{4}{3}} \textbf{ sq. units}$$

Example 8 Calculate the area of the shaded region shown on the right.



Solution

$y = 4x^2 \quad \Rightarrow \quad x^2 = \frac{y}{4}$

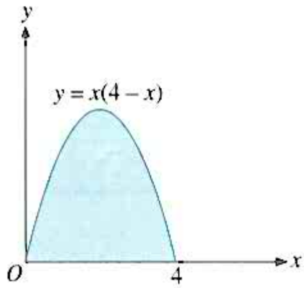
As $x \geqslant 0$, $x = \frac{1}{2}y^{\frac{1}{2}}$

$$\text{The required area} = \int_1^4 x \, dy$$
$$= \int_1^4 \frac{1}{2}y^{\frac{1}{2}} \, dy$$
$$= \frac{1}{2}\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$$
$$= \frac{1}{3}(8 - 1)$$
$$= \mathbf{2\frac{1}{3}} \textbf{ sq. units}$$

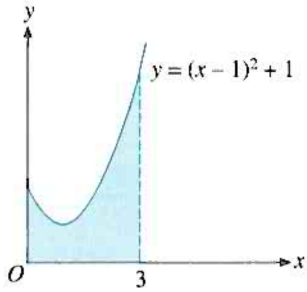
Exercise 21.1

1. For each of the following, find the shaded area.

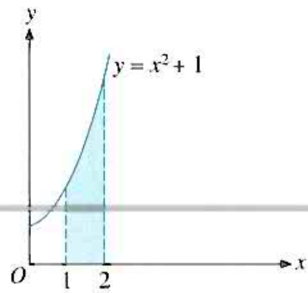
(a)



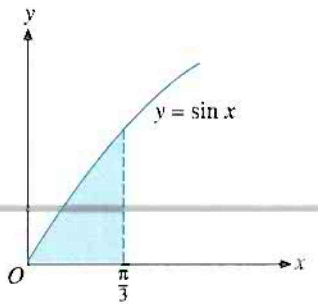
(b)



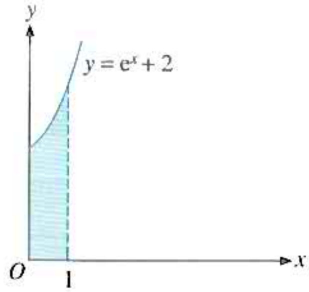
(c)



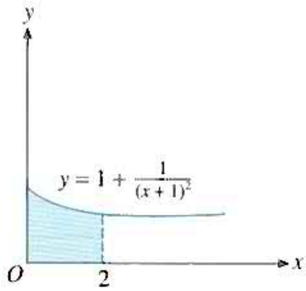
(d)



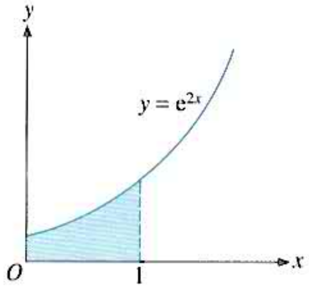
(e)



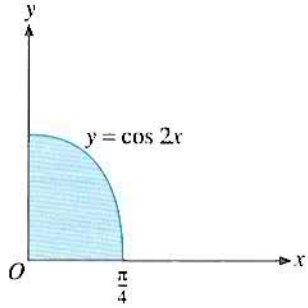
(f)



(g)

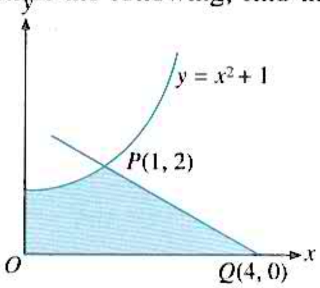


(h)

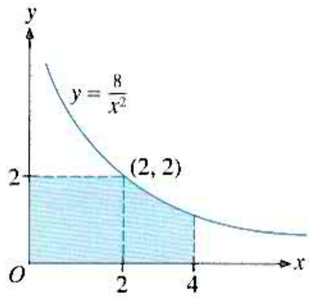


2. For each of the following, find the shaded area.

(a)



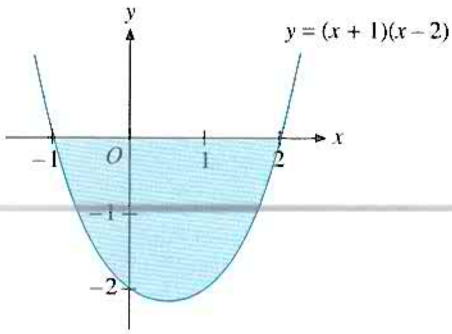
(b)



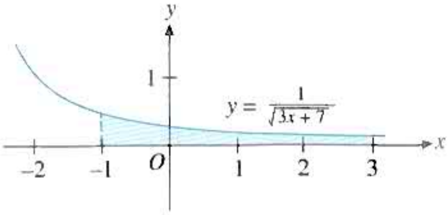
3. Find the area bounded by the following:

(a) $y = x^3$; $x = 2$, $x = 3$, x-axis **(b)** $y = \dfrac{4}{x^2}$; $x = 1$, $x = 3$, x-axis.

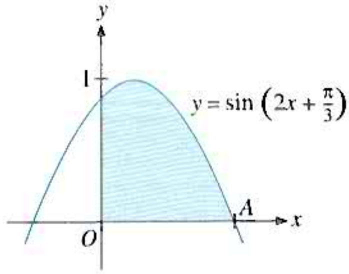
4. The diagram shows part of the curve $y = (x + 1)(x - 2)$. Find the area of the shaded region.



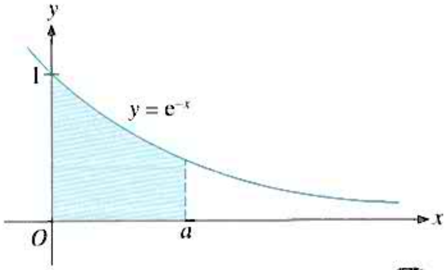
5. The diagram shows part of the curve $y = \dfrac{1}{\sqrt{3x+7}}$.
Find the area of the shaded region.



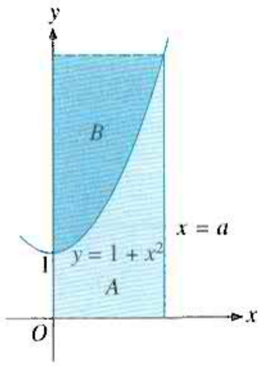
6. The diagram shows part of the curve $y = \sin\left(2x + \frac{\pi}{3}\right)$. Find
   (a) the x-coordinate of the point A where the curve cuts the x-axis,
   (b) the area of the shaded region.



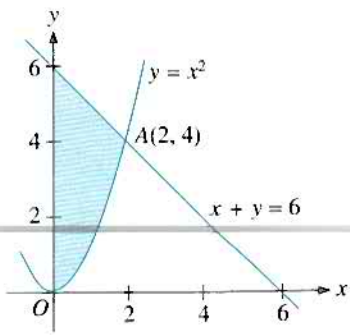
7. The diagram shows part of the curve $y = e^{-x}$.
   (a) Find, in terms of a, the area of the shaded region.
   (b) Given that the area of the shaded region is 0.5 unit^2, find the value of a.



8. The diagram shows part of the curve $y = 1 + x^2$ and of the line $x = a$. Find, in terms of a, the area of the shaded region A. Given that the area of the shaded regions A and B are equal, find, to two decimal places, the value of a.

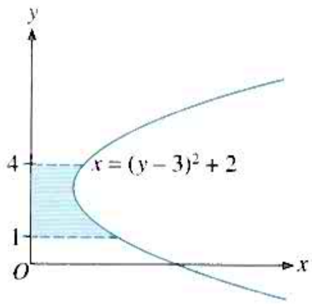


9. The diagram shows part of the curve $y = x^2$ and of the line $x + y = 6$ intersecting at A. Calculate the area of the shaded region.

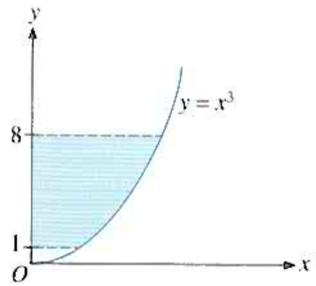


10. For each of the following, find the shaded area.

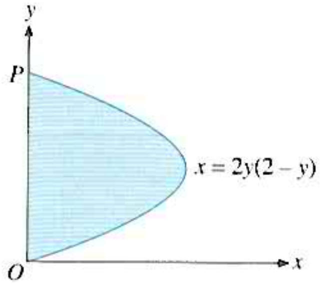
(a)



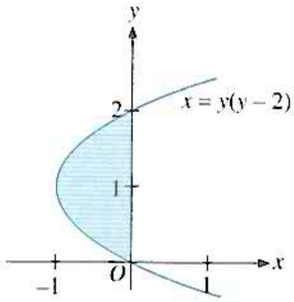
(b)



(c)



(d)



21.2 Area Bounded by Two Curves

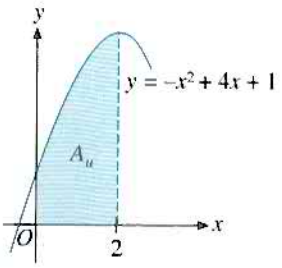


Fig. (a)

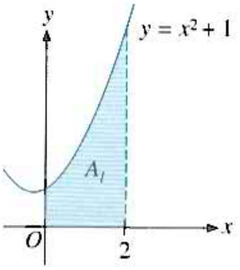


Fig. (b)

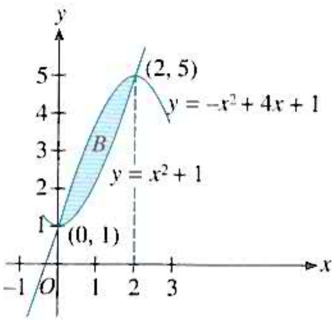


Fig. (c)

Fig. (c) shows the region B bounded by the two curves $y = x^2 + 1$ and $y = -x^2 + 4x + 1$ which intersect at (0, 1) and (2, 5). Area bounded by the curves $y = -x^2 + 4x + 1$ and $y = x^2 + 1$ is given by:

Area of region $B = A_u$ (area under the upper curve as shown in Fig. (a))
$\quad - A_l$ (area under the lower curve as shown in Fig. (b))

$$= \int_0^2 (-x^2 + 4x + 1)\ dx - \int_0^2 (x^2 + 1)\ dx$$

$$= \int_0^2 [(-x^2 + 4x + 1) - (x^2 + 1)]\ dx$$

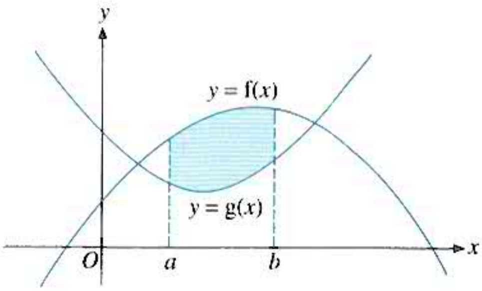
$$= \int_0^2 (-2x^2 + 4x)\, dx$$

$$= \left[-\frac{2}{3}x^3 + 2x^2\right]_0^2$$

$$= \left[-\frac{2}{3} \times 8 + 2 \times 4\right]$$

$$= 2\frac{2}{3} \text{ sq. units}$$

The diagram on the right shows the area bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$.



Area between two curves
= [area under $y = f(x)$ between $x = a$ and $x = b$]
− [area under $y = g(x)$ between $x = a$ and $x = b$]

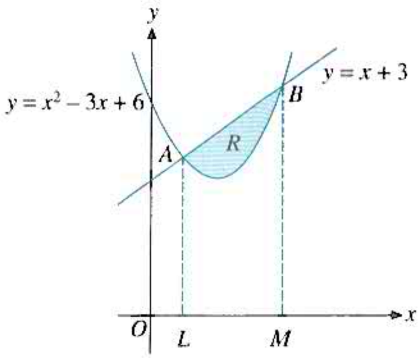
$$= \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$

$$= \int_a^b [f(x) - g(x)]\, dx, \text{ where } f(x) \geqslant g(x) \text{ for the interval } a \leqslant x \leqslant b,$$

i.e. $y = f(x)$ is the upper curve
and $y = g(x)$ is the lower curve.

Example 9

The diagram shows part of the curve $y = x^2 - 3x + 6$ and part of the line $y = x + 3$. Find
(a) the coordinates of the points A and B,
(b) the area of the region R.



Solution:

(a) For the points of intersection A and B,

$x^2 - 3x + 6 = x + 3$

i.e. $x^2 - 4x + 3 = 0$

$(x - 1)(x - 3) = 0$

$\Rightarrow \quad x = 1, y = 4$ and $x = 3, y = 6.$

Therefore, the coordinates of A and B are **(1, 4)** and **(3, 6)** respectively.

(b) Area of the region $R = \int_1^3 [(x + 3) - (x^2 - 3x + 6)]\, dx$

$$= \int_1^3 (-x^2 + 4x - 3)\, dx$$

$$= \left[-\frac{x^3}{3} + 2x^2 - 3x\right]_1^3$$

$$= \left(\frac{-27}{3} + 18 - 9\right) - \left(-\frac{1}{3} + 2 - 3\right)$$

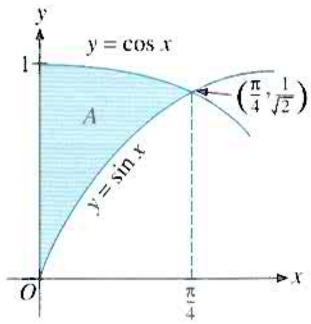
$$= \mathbf{1\frac{1}{3}\ sq.\ units}$$

Note: **(1)** The line $y = x + 3$ is the upper curve for the interval from $x = 1$ to $x = 3$.

(2) *Alternative method*:
Area of the region R
= area of the trapezium $ABML$
− area under the curve $y = x^2 - 3x + 6$

Example 10

The diagram shows part of the curves $y = \sin x$ and $y = \cos x$. Calculate the area enclosed by the curves and the y-axis.



Solution:

Area enclosed,

$$A = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx$$

$$= [\sin x + \cos x]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0 + 1)$$

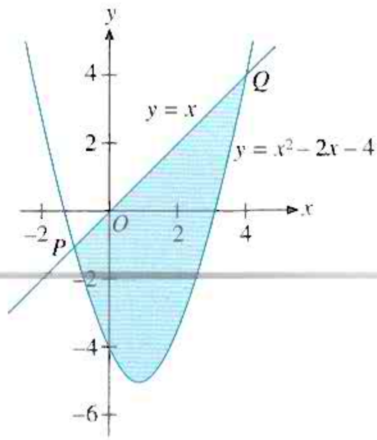
$$= \mathbf{\left(\sqrt{2} - 1\right)\ sq.\ units}$$

Example 11

The diagram shows part of the curve $y = x^2 - 2x - 4$ and part of the line $y = x$. Find

(a) the coordinates of the points P and Q,

(b) the area of the shaded region.



Solution: (a) Solving the equations $y = x^2 - 2x - 4$ and $y = x$, we have

$$x^2 - 2x - 4 = x$$

i.e. $$x^2 - 3x - 4 = 0$$

$$(x + 1)(x - 4) = 0$$

At P, $x = -1$ and $y = -1$.

At Q, $x = 4$ and $y = 4$

The coordinates of P and Q are **(−1, −1)** and **(4, 4)** respectively.

(b) Area of the shaded region $= \int_{-1}^{4} [x - (x^2 - 2x - 4)]\, dx$

$$= \int_{-1}^{4} (-x^2 + 3x + 4)\, dx$$

$$= \left[-\frac{x^3}{3} + \frac{3}{2}x^2 + 4x\right]_{-1}^{4}$$

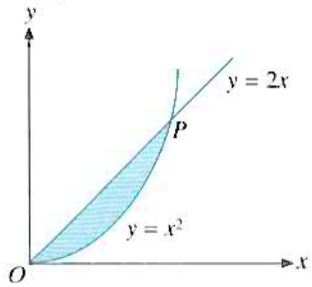
$$= \left[\left(-\frac{64}{3} + \frac{48}{2} + 16\right) - \left(\frac{1}{3} + \frac{3}{2} - 4\right)\right]$$

$$= \mathbf{20\frac{5}{6}}\ \textbf{sq. units}$$

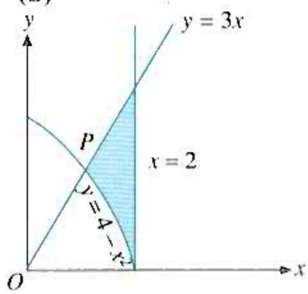
Exercise 21.2

1. For each of the following, find the x-coordinate of the point P and calculate the area of the shaded region.

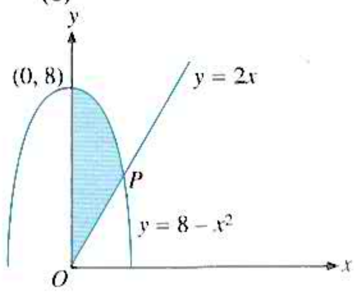
(a)



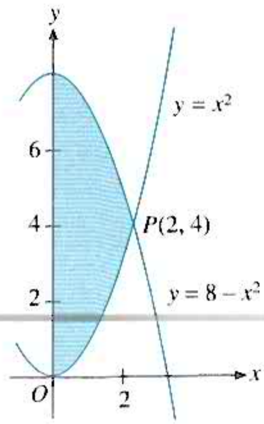
(b)



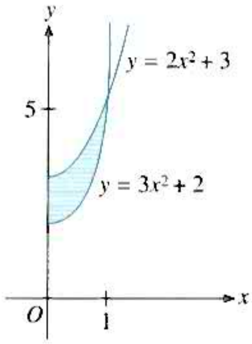
(c)



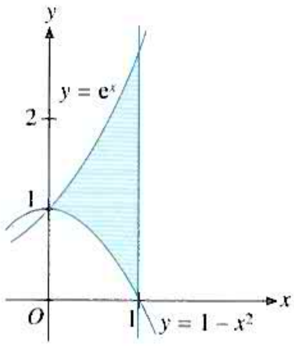
2. The diagram shows a shaded region bounded by the curves $y = x^2$, $y = 8 - x^2$ and the y-axis. Find the area of the shaded region.



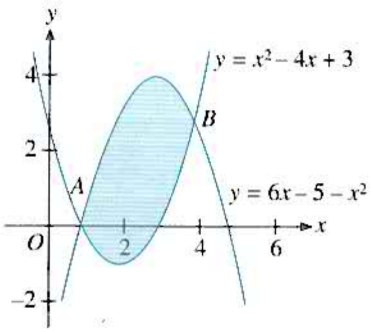
3. The diagram shows part of the curves $y = 2x^2 + 3$ and $y = 3x^2 + 2$, intersecting at (1, 5). Find the area of the shaded region.



4. The diagram shows part of the graphs of $y = 1 - x^2$ and $y = e^x$. Find, to two decimal places, the area of the shaded region.

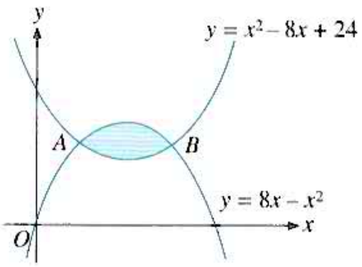


5. The diagram shows part of the curves $y = x^2 - 4x + 3$ and $y = 6x - 5 - x^2$, intersecting at the points A and B. Calculate
 (a) the coordinates of A and B,
 (b) the area of the shaded region.

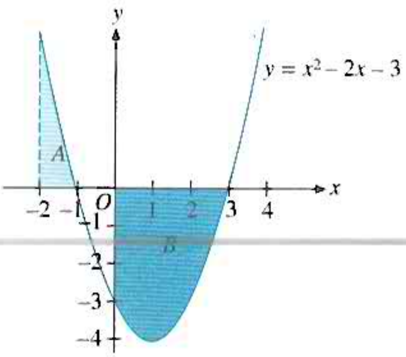


6. The diagram shows part of the curves $y = x^2 - 8x + 24$ and $y = 8x - x^2$, intersecting at the points A and B. Calculate
 (a) the coordinates of A and B,
 (b) the area of the shaded region.

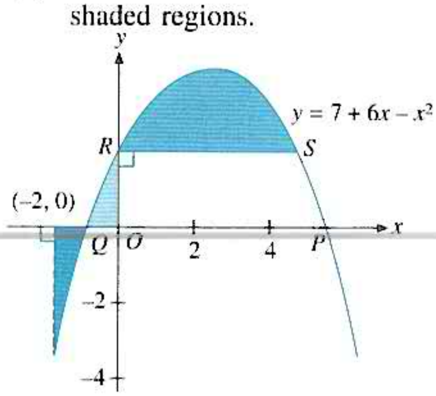
(C)



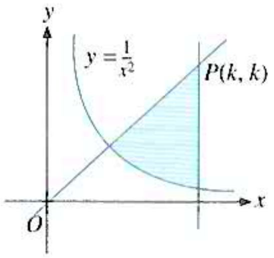
7. The diagram shows part of the curve $y = x^2 - 2x - 3$. Find the area of each of the two shaded regions.



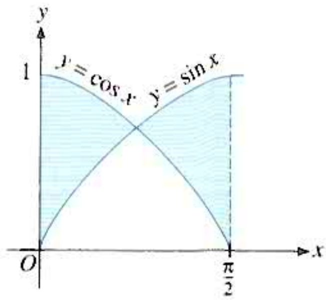
8. The diagram shows part of the curve $y = 7 + 6x - x^2$. Find
 (a) the coordinates of the points P, Q, R and S,
 (b) the area of each of the three shaded regions.



9. The diagram shows part of the curve $y = \frac{1}{x^2}$ and the point $P(k, k)$. What condition does this impose upon k? Find, in terms of k, the area of the shaded region.



10. Find the total area of the shaded regions bounded by the curves $y = \sin x$ and $y = \cos x$ for $0 \leqslant x \leqslant \frac{\pi}{2}$ as shown in the figure. .



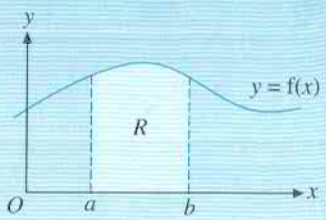
Important Notes

1. **Area**

(a) For a region R above the x-axis, enclosed by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$, the area of R is

$$A = \int_a^b y \, dx \text{ or } A = \int_a^b f(x) \, dx,$$

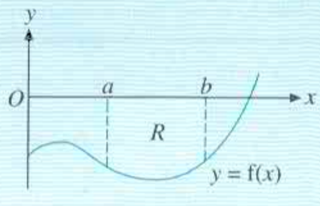
where $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$.



For a region R below the x-axis, enclosed by the curve $y = f(x)$, the x-axis, the lines $x = a$ and $x = b$, the area of R is

$$A = \int_a^b - f(x) \, dx$$

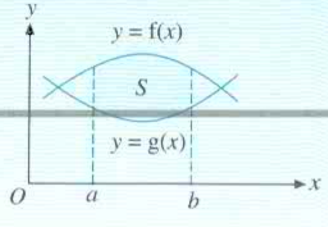
where $f(x) \leqslant 0$ for $a \leqslant x \leqslant b$.



(b) For a region S enclosed by the curves $y = g(x)$, $y = f(x)$ and the lines $x = a$ and $x = b$, the area of S is

$$A = \int_a^b [f(x) - g(x)] \, dx,$$

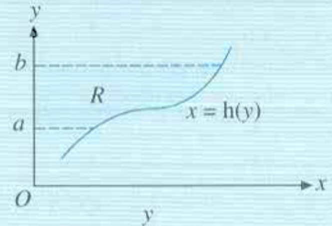
where $f(x) - g(x) \geqslant 0$ for $a \leqslant x \leqslant b$.



(c) For a region R enclosed by the curve $x = \text{h}(y)$, the y-axis and the lines $y = a$ and $y = b$, the area of R is

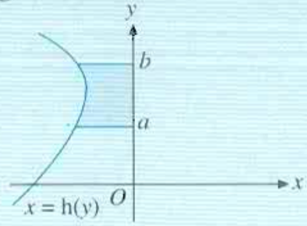
(i) $A = \displaystyle\int_a^b \text{h}(y)\ \text{d}y$,

where $\text{h}(y) \geqslant 0$ for $a \leqslant y \leqslant b$.



(ii) $A = \displaystyle\int_a^b -\text{h}(y)\ \text{d}y$,

where $\text{h}(y) \leqslant 0$ for $a \leqslant y \leqslant b$.



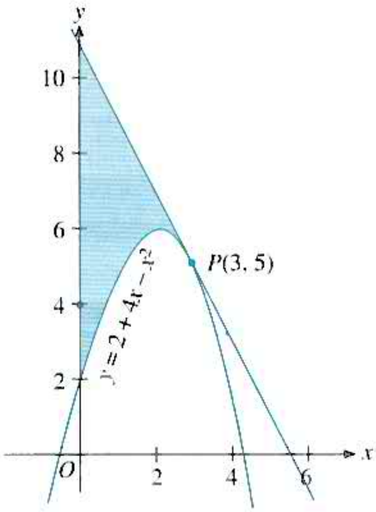
Miscellaneous Examples

Example 12

The diagram shows part of the curve $y = 2 + 4x - x^2$ and the tangent to the curve at the point $P(3, 5)$.

(a) Find the equation of the tangent line.

(b) Calculate the area of the shaded region.



Solution:

(a) $y = 2 + 4x - x^2$

$\Rightarrow \dfrac{\text{d}y}{\text{d}x} = 4 - 2x$

At the point $P(3, 5)$,

$\dfrac{\text{d}y}{\text{d}x} = 4 - 6 = -2.$

The equation of the tangent at P is $y - 5 = -2(x - 3)$, i.e. $\mathbf{y = -2x + 11}$.

(b) Hence, the area of the shaded region

$$= \int_0^3 [(-2x + 11) - (2 + 4x - x^2)]\ \text{d}x$$

$$= \int_0^3 (x^2 - 6x + 9)\ \text{d}x$$

$$= \left[\frac{1}{3}x^3 - 3x^2 + 9x\right]_0^3$$

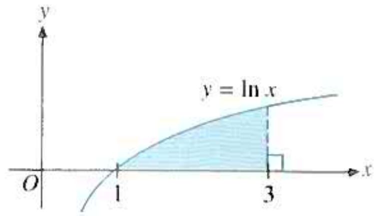
$$= (9 - 27 + 27)$$

$= \mathbf{9}$ **sq. units**

Example 13

Show that $\frac{d}{dx}[x \ln x - x] = \ln x$.

The diagram shows part of the curve $y = \ln x$. Find, correct to two decimal places, the area of the shaded region.



Solution:

$$\frac{d}{dx}[x \ln x - x] = \frac{d}{dx}(x \ln x) - \frac{d}{dx}(x)$$
$$= \left[x\frac{d}{dx}(\ln x) + \ln x\frac{d}{dx}(x)\right] - 1$$
$$= \left(x \times \frac{1}{x} + \ln x\right) - 1$$
$$= \ln x$$

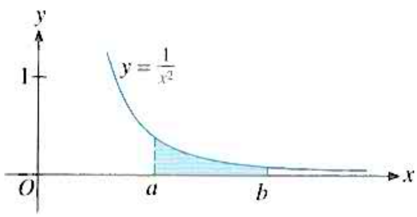
$$\text{Area of the shaded region} = \int_1^3 \ln x \, dx$$
$$= [x \ln x - x]_1^3$$
$$= [3 \ln 3 - 3] - [\ln 1 - 1]$$
$$= 3 \ln 3 - 2$$
$$= \mathbf{1.30 \text{ sq. units (2 dec. pl.)}}$$

Example 14

The diagram shows part of the curve $y = \frac{1}{x^2}$.

(a) Show that the area under the curve $y = \frac{1}{x^2}$, the lines $x = a$, $x = b$ $(b > a > 0)$ and the x-axis is $\frac{1}{a} - \frac{1}{b}$.

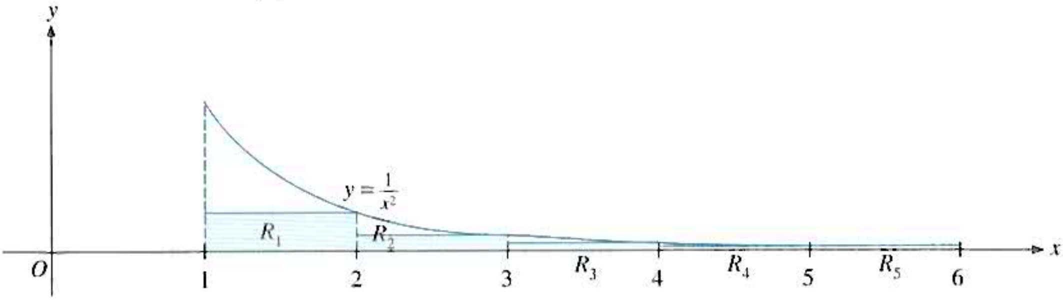
(b) Show that $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} < \frac{5}{6}$.



Solution:

(a) $$\text{Area under the curve} = \int_a^b \frac{1}{x^2} \, dx$$
$$= \left[-\frac{1}{x}\right]_a^b$$
$$= \frac{1}{a} - \frac{1}{b}.$$

(b)



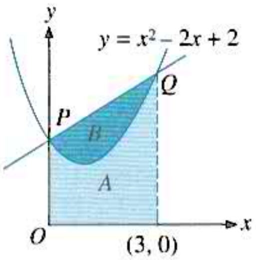
Area $(R_1 + R_2 + R_3 + R_4 + R_5) <$ Area under the curve

$y = \frac{1}{x^2}$ between $x = 1$ and $x = 6$

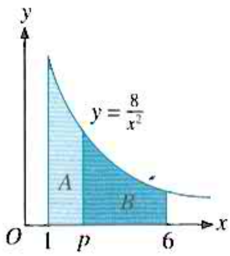
$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} < \frac{1}{1} - \frac{1}{6} = \frac{5}{6}$$

Miscellaneous Exercise 21

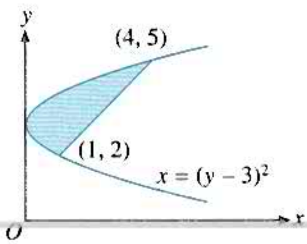
1. The diagram shows part of the curve $y = x^2 - 2x + 2$ and a chord PQ. Find
   (a) the coordinates of P and Q,
   (b) the area of the shaded region A,
   (c) the area of the shaded region B.



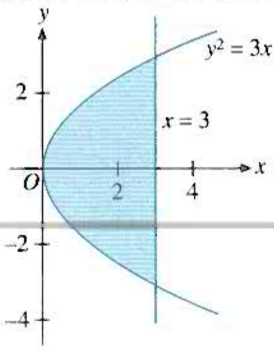
2. The diagram shows part of the curve $y = \frac{8}{x^2}$. Find
   (a) the area of region A,
   (b) the value of p for which regions A and B are equal in area.



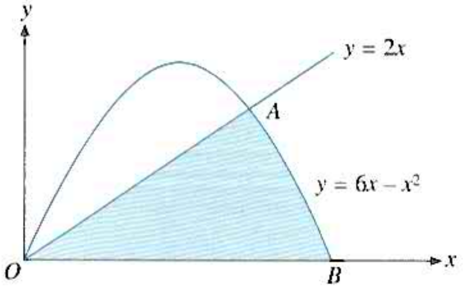
3. Calculate the area of the shaded region shown in the figure.



4. The diagram shows a shaded region bounded by part of the curve $y^2 = 3x$ and the line $x = 3$. Find, to two decimal places, the area of the shaded region.

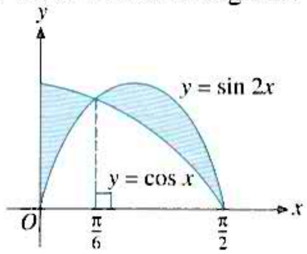


5. The figure shows part of the curve $y = 6x - x^2$ and part of the line $y = 2x$. Find
   (a) the coordinates of A and of B,
   (b) the area of the shaded region.



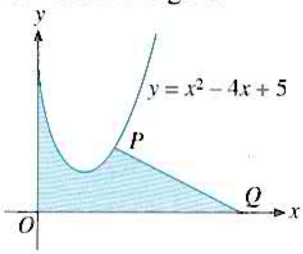
6. **(a)** Sketch the curve $y = x(5 - x)$. If a line $x = 2$ divides the region enclosed by the curve and the x-axis into two parts, show that the areas are in the ratio 44 : 81.
   (b) Find the area of the region bounded by the curve $y = x(x - 3)$ and the line $y = -2$.

7. The figure shows part of the graphs of the curves $y = \cos x$ and $y = \sin 2x$. Calculate the total area of the two shaded regions. *(C)*

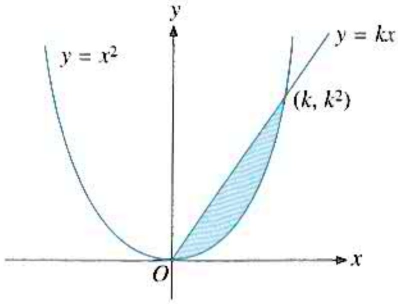


8. Sketch the graphs of $y = \cos x$ and $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$. Calculate the area enclosed
   (a) by the curves and the x-axis,
   (b) by the curves and the y-axis.

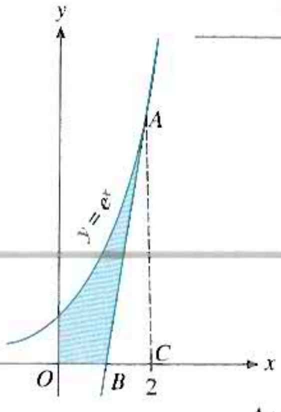
9. The line PQ is the normal to the curve $y = x^2 - 4x + 5$ at the point where $x = 3$. Show that the equation of PQ is $2y + x = 7$. Find the area of the shaded region.



10. The diagram shows part of the curve $y = x^2$ and of the line $y = kx$, where k is a positive constant. Calculate the value of k for which the area of the shaded region is 0.288 square units. *(C)*

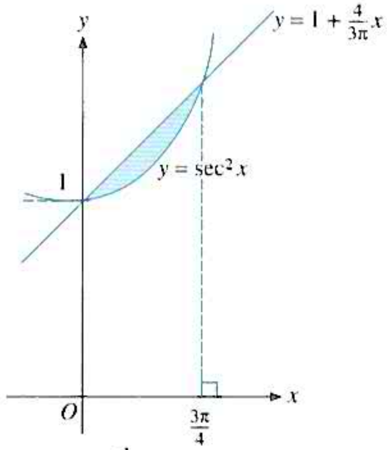


11. The diagram shows a region bounded by the curve $y = e^x$, the tangent to the curve at A where $x = 2$ and the x- and y-axes.
   (a) Find the coordinates of the point where the tangent cuts the x-axis.
   (b) Show that the area of the bounded region is $\frac{1}{2}e^2 - 1$ square units.



12. Find the maximum value of $4x - x^2 - 1$ and sketch the curve $y = 4x - x^2 - 1$ for $0 \leqslant x \leqslant 4$. Determine the equations of the tangents to the curve at the points whose x-coordinates are 1 and 3. Show that these tangents intersect at the point (2, 4). Calculate the area of the region bounded by the curve and the two tangents. *(C)*

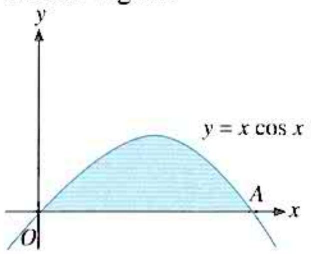
13. The diagram shows part of the line $y = 1 + \frac{4}{3\pi}x$ and of the curve $y = \sec^2 x$, intersecting at the points where $x = 0$ and $x = \frac{3\pi}{4}$. Show that the area of the shaded region is $\frac{9\pi}{8} + 1$.



14. Show that
$\frac{\mathrm{d}}{\mathrm{d}x}[x \sin x + \cos x] = x \cos x$. The diagram shows part of the curve $y = x \cos x$.

(a) Find the coordinates of the point A.

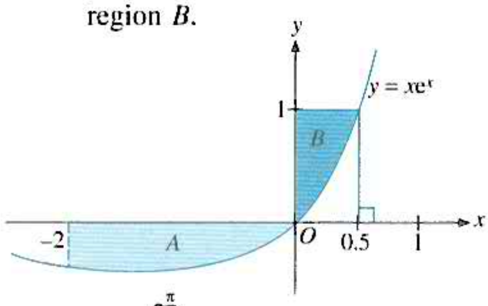
(b) Calculate, correct to two decimal places, the area of the shaded region.



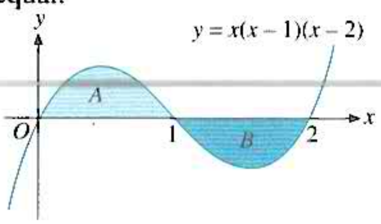
***15.** Show that $\frac{\mathrm{d}}{\mathrm{d}x}[(x - 1)\mathrm{e}^x] = x\mathrm{e}^x$.
The diagram shows part of the curve $y = x\mathrm{e}^x$.

(a) Find the area of the shaded region A.

(b) Calculate, correct to two decimal places, the area of the shaded region B.

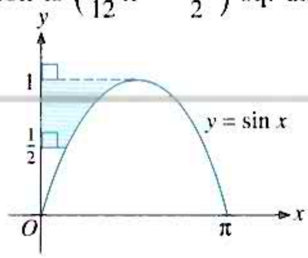


***16.** The diagram shows part of the curve $y = x(x - 1)(x - 2)$ and the two shaded regions A and B. Show that the areas of the regions A and B are equal.



***17.** Evaluate $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin x \,\mathrm{d}x$.

Hence show that the area of the shaded region is $\left(\frac{5}{12}\pi - \frac{\sqrt{3}}{2}\right)$ sq. unit.



22 Kinematics

Kinematics is the study of the motion of a body without considering the cause of its motion. In this chapter, we shall discuss the kinematics of a particle moving along a straight line using such terms as **displacement**, **velocity** and **acceleration**.

22.1 Displacement, Velocity and Acceleration

The diagram below shows a particle P moving in a straight line and its distance s m, from a fixed point O, t seconds after passing O, is given by $s = 4t - t^2$.

The position of P at time t is specified by s:

when $t = 1$, $s = 4 - 1 = 3$ and P is at A,
when $t = 2$, $s = 8 - 4 = 4$ and P is at B,
when $t = 3$, $s = 12 - 9 = 3$ and P is back at A.

Using the concept of rate of change discussed in Chapter 16, we note that the rate of change of s with respect to t is $\frac{ds}{dt} = 4 - 2t$.

When $t = 1$, $\frac{ds}{dt} = 4 - 2 = 2$.
This implies that s increases at a rate of 2 m s^{-1} and P moves away from O at a speed of 2 m s^{-1}.

When $t = 2$, $\frac{ds}{dt} = 4 - 4 = 0$.
This implies that s changes at a rate of 0 m s^{-1} and P is said to be instantaneously at rest.

When $t = 3$, $\frac{ds}{dt} = 4 - 6 = -2$.
What does this imply?

Note that $\frac{ds}{dt}$ determines both the **speed** and the **direction** of motion of P at that instant.

$\frac{ds}{dt}$ is called the **velocity** of the particle at time t and is denoted by v.

Example 1 A particle Q moves in a straight line so that its distance, s m, from a fixed point O on the line, is given by $s = 6t^2 - t^3$, where t is the time in seconds after passing O. Find

(a) the velocity and speed of Q after 5 seconds,

(b) the distance of Q from O when it is instantaneously at rest after passing through O,

(c) the total distance travelled during the first 5 seconds.

Solution:

$s = 6t^2 - t^3 \Rightarrow v = \frac{ds}{dt} = 12t - 3t^2$

(a) At $t = 5$, $v = 12 \times 5 - 3 \times 25 = -15$ m s^{-1}

$\therefore$ after 5 seconds, its velocity is **-15 m s^{-1}** and its speed is **15 m s^{-1}**.

(b) When Q is instantaneously at rest,

$$\begin{aligned} v = 0 \quad &\Rightarrow \quad 12t - 3t^2 = 0 \\ &\Rightarrow \quad 3t(4 - t) = 0 \\ &\Rightarrow \quad t = 0 \text{ or } t = 4 \end{aligned}$$

Since Q has passed O, $t = 4$ and so $s = 6(4^2) - 4^3$
$= 96 - 64 = 32$

$\therefore$ the distance of Q from O is **32 m**.

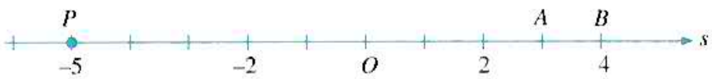
(c) When $t = 5$, $s = 6(5^2) - 5^3 = 25$

Hence the motion of Q is as shown:

$\therefore$ the total distance travelled during the first 5 seconds is

$32 + (32 - 25) =$ **39 m**

Recall the motion of the particle P at the start of our discussion. Since $s \geqslant 0$, the distance function $s = 4t - t^2$ is defined only for $0 \leqslant t \leqslant 4$ and so we cannot study the motion of the particle when $t = 5$ for which $s = 20 - 25 = -5$. However, if we treat the line as the s-axis with its origin at the fixed point O, then the s-coordinate, which can be negative, gives the position of a particle. We call this s-coordinate the **displacement** of the particle from O. Hence the position of P at $t = 5$ is as shown below:



Example 2

A particle P moves in a straight line so that its displacement, s m, from a fixed point O on the line, is given by $s = 3t^2 - 12$, where t is the time in seconds after passing a point A on the line.

(a) Find the distance OA.

(b) Find the velocity of P when it passes O.

(c) Show that P never comes to rest after passing A.

(d) Find the average speed of P during the first 3 seconds.

Solution

(a) $s = 3t^2 - 12$

At A, $t = 0 \Rightarrow s = -12$

$\therefore$ the distance OA is **12 m**.

(b) At O, $s = 0 \Rightarrow t^2 = 4$

$\Rightarrow t = 2 \quad (t \geqslant 0)$

$s = 3t^2 - 12 \Rightarrow v = \frac{ds}{dt} = 6t$

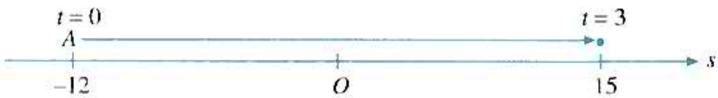
So, at O, $t = 2 \Rightarrow v = 12$.

$\therefore$ the velocity of P is **12 m s^{-1}**.

(c) After passing A, $t > 0 \Rightarrow$ $\boldsymbol{v = 6t > 0}$ and so P never comes to rest.

(d) At $t = 3$, $s = 27 - 12 = 15$

Since P never comes to rest, the motion of P during the first 3 seconds is as shown below:



$$\therefore \text{ average speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

$$= \frac{15 - (-12)}{3}$$

$$= \mathbf{9\ m\ s^{-1}}$$

Beside displacement and distance, velocity and speed, another useful quantity for describing motion is acceleration. We define the **acceleration** as the rate of change of velocity with time. In calculus notation, we have

$$\text{acceleration, } a = \frac{dv}{dt}$$

$$\Rightarrow a = \frac{d^2s}{dt^2}$$

Example 3 A particle travels in a straight line so that, t seconds after it started moving, its displacement, s metres, from a fixed point O on the line is given by $s = 6 - \frac{9}{t+3}$. Find

(a) the expressions for the velocity and acceleration of the particle in terms of t,

(b) the velocity of the particle when it is 5 metres from O,

(c) the time t when the acceleration of the particle is $-\frac{1}{12}$ m s^{-2}.

Solution:

(a)
$$s = 6 - \frac{9}{t+3}$$
$$v = \frac{ds}{dt} = \frac{9}{(t+3)^2}$$
$$a = \frac{dv}{dt} = -\frac{18}{(t+3)^3}$$

(b) When $s = 5$,
$$5 = 6 - \frac{9}{t+3}$$
$$t = 6$$
$$\Rightarrow \quad v = \frac{9}{9^2}$$
$$= \frac{1}{9} \text{ m s}^{-1}$$

(c) When $a = -\frac{1}{12}$ m s^{-2}
$$-\frac{1}{12} = -\frac{18}{(t+3)^3}$$
$$(t+3)^3 = 18 \times 12$$
$$t = 3$$

Note: Can we have $s = -5$ in (b)?

For a particle moving along the x-axis (along a straight line), its position at time t is given by its displacement x from a fixed point O. By differentiation, we obtain:

$$\text{velocity, } v = \frac{dx}{dt} \text{ and acceleration, } a = \frac{dv}{dt}$$

Conversely, if v is known, we can find x by reversing the process of differentiation, that is, by integrating v.

$$x = \int v \, dt$$

For example,
$$v = 2t + 1$$
$$\Rightarrow \quad x = \int (2t + 1) \, dt$$
$$= t^2 + t + c$$

If $x = 1$ when $t = 0$, then $c = 1$ and so
$$x = t^2 + t + 1$$

Similarly, if the acceleration a is given, then its velocity v is given by:

$$v = \int a \, dt$$

Example 4 A particle P moves in a straight line with a velocity v m s^{-1} given by $v = 4 - t^2$, where t is the time in seconds after passing through a fixed point O on the line. Find

(a) the acceleration of P after 4 seconds,

(b) the distance of P from O after 6 seconds,

(c) the displacement of P from O when P is instantaneously at rest.

Solution:

(a) The acceleration, $a = \dfrac{dv}{dt} = -2t$

$\therefore$ after 4 seconds, $t = 4$ and the acceleration is $\mathbf{-8\ m\ s^{-2}}$.

(b) The displacement x from O is given by

$$x = \int v\, dt$$

$$= \int (4 - t^2)\, dt$$

$$= 4t - \frac{1}{3}t^3 + c$$

When $t = 0$, $x = 0$ and so $c = 0$.

Hence at time t, $x = 4t - \dfrac{1}{3}t^3$

When $t = 6$, $x = 24 - 72 = -48$

$\therefore$ the distance of P from O is **48 m**.

(c) When P is instantaneously at rest,

$$\begin{aligned} v &= 0 \\ 4 - t^2 &= 0 \\ t &= 2 \quad (\text{since } t \geqslant 0) \\ \Rightarrow \quad x &= 8 - \frac{8}{3} \\ &= 5\frac{1}{3} \end{aligned}$$

$\therefore$ the displacement is $\mathbf{5\frac{1}{3}}$ **m**.

Example 5 A particle starts from rest from a point O and moves in a straight line such that its acceleration, a m s^{-2}, is given by $a = 12t^2 - 24t + 8$, where t is the time in seconds after the start of its motion. Its displacement, s m, is measured from O. Find s in terms of t.

Calculate the time taken for the particle to return to O and its acceleration at this instant.

Solution: Its velocity, v m s^{-1}, is given by

$$v = \int a\, dt$$

$$= \int (12t^2 - 24t + 8)\,dt$$
$$= 4t^3 - 12t^2 + 8t + c$$

Particle starts from rest $\Rightarrow$ when $t = 0$, $v = 0$ and so $c = 0$.
Hence at time t, $v = 4t^3 - 12t^2 + 8t$.

Now, $$s = \int v\,dt$$
$$= \int (4t^3 - 12t^2 + 8t)\,dt$$
$$= t^4 - 4t^3 + 4t^2 + d$$

Particle starts from O $\Rightarrow$ when $t = 0$, $s = 0$ and so $d = 0$.
Hence at time t, $s = t^4 - 4t^3 + 4t^2$.

When the particle returns to O, $t > 0$ and

$$\begin{aligned} s = 0 \quad \Rightarrow \quad t^4 - 4t^3 + 4t^2 &= 0 \\ t^2(t^2 - 4t + 4) &= 0 \\ t^2(t-2)^2 &= 0 \\ t &= 2 \\ \Rightarrow \quad a &= 12 \times 4 - 48 + 8 \\ &= 8 \end{aligned}$$

$\therefore$ the particle returns to O in **2 seconds** with an acceleration of **8 m s^{-2}**.

Example 6

A particle moves along the x-axis. It passes the origin O with a velocity of 8 m s^{-1} and its acceleration a m s^{-2} t seconds after passing O is given by $a = 3 - t$. Calculate
(a) the velocity of the particle when $t = 2$,
(b) the maximum velocity of the particle,
(c) the displacement of the particle from O when it is at rest,
(d) the average speed during the first 12 seconds.

Solution

(a) The velocity v is given by

$$v = \int a\,dt$$
$$= \int (3 - t)\,dt$$
$$= 3t - \frac{1}{2}t^2 + c$$

When $t = 0$, $v = 8$ and so $c = 8$.
Hence at time t, $v = 3t - \frac{1}{2}t^2 + 8$.
When $t = 2$, $v = 6 - 2 + 8 = 12$
$\therefore$ its velocity is **12 m s^{-1}** when $t = 2$.

(b) For maximum velocity, $\frac{dv}{dt} = 0$

$$\Rightarrow \quad 3 - t = 0$$
$$t = 3$$

Furthermore, $\frac{d^2v}{dt^2} = -1 < 0 \;\Rightarrow\; v$ is maximum when $t = 3$.

$\therefore$ maximum velocity $= 3 \times 3 - \frac{1}{2} \times 9 + 8 =$ **12.5 m s^{-1}**.

(c) The displacement, x m, from O is given by

$$x = \int v \, dt$$
$$= \int \left(3t - \frac{1}{2}t^2 + 8\right) dt$$
$$= \frac{3}{2}t^2 - \frac{1}{6}t^3 + 8t + d$$

When $t = 0$, $x = 0$ and so $d = 0$.
Hence at time t, $x = \frac{3}{2}t^2 - \frac{1}{6}t^3 + 8t$.
When the particle is at rest,

$$v = 0 \quad \Rightarrow \quad 3t - \frac{1}{2}t^2 + 8 = 0$$
$$t^2 - 6t - 16 = 0$$
$$(t - 8)(t + 2) = 0$$

Since $t \geqslant 0$, $t = 8 \quad \Rightarrow \quad x = \frac{3}{2} \times 8^2 - \frac{1}{6} \times 8^3 + 64$

$$= 74\frac{2}{3} \text{ m}$$

$\therefore$ the displacement of the particle from O when it is at rest is **$74\frac{2}{3}$ m.**

(d) When $t = 12$, $x = \frac{3}{2} \times 12^2 - \frac{1}{6} \times 12^3 + 96 = 24$ m

Hence the motion of the particle is as shown below:

$\therefore$ average speed during the first 12 seconds

$$= \frac{\text{distance travelled}}{\text{time taken}}$$
$$= \frac{74\frac{2}{3} + \left(74\frac{2}{3} - 24\right)}{12}$$
$$= \mathbf{10\frac{4}{9} \text{ m s}^{-1}}$$

Exercise 22.1

1. A particle P moves in a straight line so that its distance, s m, from a fixed point O on the line, is given by $s = t(t-6)^2$, where t is the time in seconds after passing O. Find
   (a) the velocity of P when $t = 1$,
   (b) the values of t when P is instantaneously at rest,
   (c) the acceleration of P when $t = 3$.

2. A particle moves in a straight line and its displacement, x m, from a fixed point O, t seconds after passing O, is given by $x = 12t - t^3$. Calculate
   (a) the acceleration of the particle when it comes instantaneously to rest,
   (b) the velocity of the particle when it is next at O,
   (c) the distance travelled by the particle during the first 3 seconds.

3. A particle P travels in a straight line so that its displacement, x metres, from a fixed point O is given by $x = 3t^2 - 4t^3 + 60$, where t is the time in seconds measured from the start of the motion. Calculate
   (a) the initial distance of P from O,
   (b) the magnitudes of the acceleration of P when it is instantaneously at rest,
   (c) the average speed of P over the first two seconds.

4. A particle P moves in a straight line so that its distance, s metres, from a fixed point O is given by $s = 18t^2 - t^4$, where t is the time in seconds after leaving O. Calculate
   (a) the non-zero value of s when P comes instantaneously to rest,
   (b) the distance travelled by P in the third second (i.e. from $t = 2$ to $t = 3$),
   (c) the velocity of P when the acceleration is -12 m s^{-2}.

5. A particle moves in a straight line so that its distance, s m, from a fixed point O on the line, is given by $s = t(t-2)^2$, where t is the time in seconds after passing O. Calculate
   (a) the velocity of the particle after 3 seconds,
   (b) the distance of the particle from O when its velocity is 7 m s^{-1},
   (c) the acceleration of the particle when it is next at O.

6. The height, h m, of a stone t seconds after it has been thrown vertically upwards from ground level is given by $h = 24t - 3t^2$. Find
   (a) its velocity after 3 seconds,
   (b) the maximum height reached,
   (c) the time of flight.

7. A particle P travels in a straight line through a fixed point O. Its distance, s metres, from O is given by $s = t^3 - 9t^2 + 15t + 40$, where t is the time in seconds after motion has begun. Calculate
   (a) the distances of P from O when its velocity is instantaneously zero,
   (b) the values of t when the acceleration has a magnitude of 9 m s^{-2},
   (c) the average speed of P during the first 2 seconds,
   (d) the total distance travelled in the first 6 seconds.

*8. A particle P moves in a straight line so that its displacement, s m, from a fixed point O, t seconds after passing through point A on the line, is given by $s = t^3 - 3t^2 - 9t + 30$. Calculate
(a) the speed of P when $t = 2$,
(b) the acceleration of P when $t = 4$,
(c) the distance of P from A when it is instantaneously at rest,
(d) the average speed over the first 4 seconds.

*9. A particle P moves in a straight line so that its displacement, s metres, from a fixed point O is given by $s = t^3 - 9t^2 + 24t + 2$ where t is the time in seconds after the start of motion. Calculate
(a) the initial velocity and acceleration,
(b) the values of t when P is instantaneously at rest,
(c) the distance travelled in the first 4 seconds,
(d) the average speed of P during the first 5 seconds.
Show that P will never return to its starting point.

10. A particle moves in a straight line with a velocity v m s^{-1} given by $v = 2t^2 - 3t - 2$. When $t = 0$, its displacement from the origin O is 3 m, find
(a) the value of t when the particle is at rest and the displacement at this instant,
(b) the displacement when $t = 3$ and the total distance travelled in the first 3 seconds.

11. A particle travels in a straight line so that its velocity v m s^{-1} is given by $v = -3t^2 + 8t + 5$, where t is the time in seconds after passing O. Find an expression in terms of t for
(a) its acceleration,
(b) its displacement from O.
Calculate
(c) the value of t when the velocity is 2 m s^{-1},
(d) the value of t at which the particle passes through O again.

12. A particle moves in a straight line so that its velocity v m s^{-1} is given by $v = 9 - t^2$, where t is the time in seconds, measured from the start of the motion. Find
(a) the value of t at which the particle is instantaneously at rest,
(b) the value of t and the speed when the particle is again at its starting point,
(c) the total distance travelled when the particle returns to its starting point.

13. A particle passes a fixed point O with a velocity of 7 m s^{-1} and moves in a straight line with an acceleration of $2(3 - t)$ m s^{-2} where t is the time in seconds after passing O. Calculate
(a) the value of t when the velocity is again 7 m s^{-1},
(b) the position of the particle at this instant.

14. A particle travelling in a straight line passes a fixed point O with a velocity of 1 m s^{-1}. Its acceleration a m s^{-2} is given by $a = 12t^2$, where t seconds is the time after passing O. Calculate
(a) the acceleration of the particle as it passes O,
(b) its distance from O when $t = 1$ and when $t = 3$,
(c) the average speed of the particle between $t = 1$ and $t = 3$.

15. A girl runs in a straight line for 25 seconds. Her speed after t seconds, where $0 \leq t \leq 25$, is v m s^{-1}, where $v = 0.75t - 0.03t^2$. Find
(a) the time at which the girl's acceleration is zero,
(b) the distance the girl runs. *(C)*

16. The velocity, v m s^{-1}, of a particle moving in a straight line is given by $v = 6 + pt + qt^3$, where t is the time in seconds after the particle passed through a fixed point O. Given that when $t = 2$, the distance of the particle from O is 16 m and its acceleration is 32 m s^{-2}, calculate
(a) the value of p and of q,
(b) the velocity of the particle at the instant when the acceleration is zero. *(C)*

17. A particle travelling in a straight line passes a fixed point O with a velocity of $1\frac{1}{2}$ m s^{-1}. It moves in such a manner that, t seconds after passing O, its acceleration, a m s^{-2}, is given by $a = p + qt$, where p and q are constants.
Given that its velocity is $3\frac{1}{2}$ m s^{-1} when $t = 2$, and that it comes instantaneously to rest when $t = 3$, calculate the value of p and of q.
Find the distance travelled by the particle between $t = 1$ and $t = 2$. *(C)*

18. A particle starts from rest at O and moves in a straight line with an acceleration a m s^{-2} given by $a = 8 - 4t$, where t is the time in seconds since leaving O. The particle comes to instantaneous rest at A. Find
(a) the time taken for the particle to reach A,
(b) the distance OA,
(c) the maximum speed of the particle during its motion from O to A.

19. A particle moving in a straight line passes a fixed point O on the line with velocity 30 m s^{-1}. The acceleration, a m s^{-2}, of the particle, t seconds after passing O, is given by $a = 13 - 6t$. Calculate
(a) the velocity of the particle 3 seconds after passing O,
(b) the time taken to reach the maximum distance from O in the direction of the initial motion,
(c) the value of this maximum distance.

20. A particle moves in a straight line so that, at time t seconds after leaving a fixed point O, its displacement, s m, is given by $s = 4 - 4e^{-t} - \frac{2}{5}t$. Calculate
(a) the initial velocity of the particle,
(b) the value of t when the particle is instantaneously at rest,
(c) the acceleration of the particle at this instant.

21. A particle moves in a straight line from a point O in the line so that, t seconds after leaving O, its velocity, v m s^{-1}, is given by $v = 3(e^{0.4t} + 2)$. Calculate
(a) the initial acceleration of the particle,
(b) the displacement of the particle from O when $t = 1$.
Will the particle return to O?

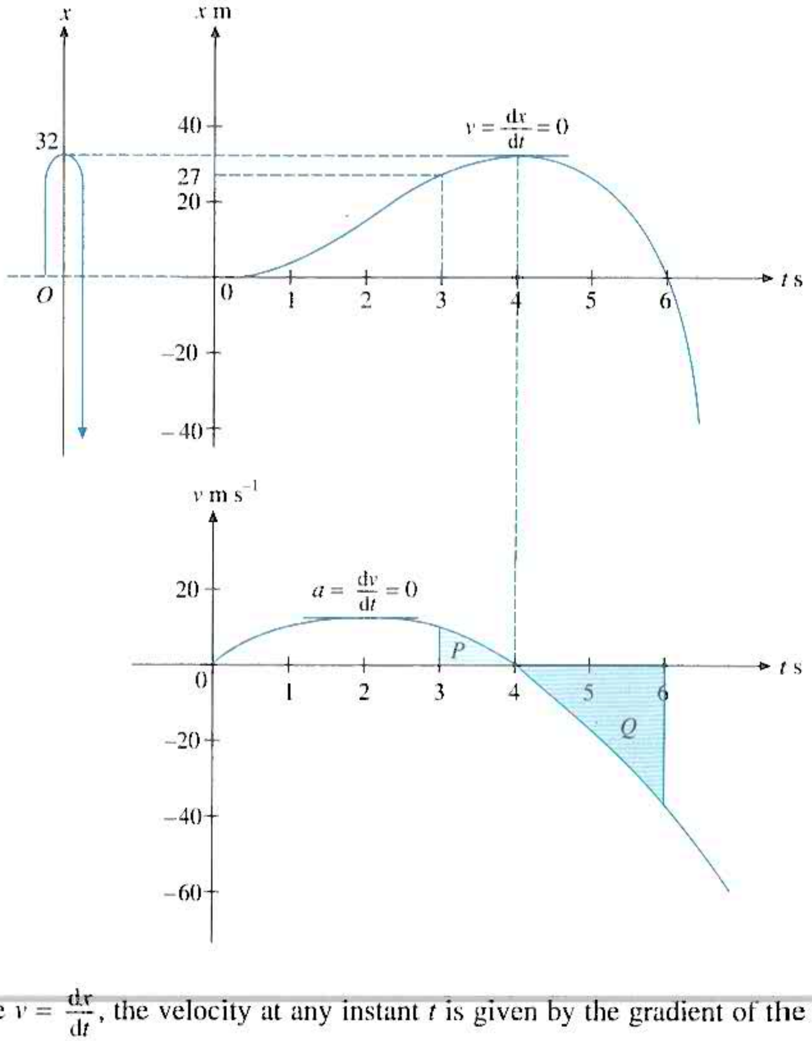
22.2 Displacement-Time and Velocity-Time Graphs

A particle moves in a vertical line such that its displacement, x m, from a point O on the line t seconds after leaving O is given by $x = 6t^2 - t^3$.

Then its velocity, $v = \frac{dx}{dt} = 12t - 3t^2$.

The diagram below shows the motion of the particle along the vertical line and its displacement-time(x-t) and velocity-time(v-t) graphs.

1. Observe how the x-t graph is related to the motion along the straight line.



2. Since $v = \frac{dx}{dt}$, the velocity at any instant t is given by the gradient of the x-t graph at that instant.

3. Similarly, since $a = \frac{dv}{dt}$, the acceleration at any instant t is given by the gradient of the v-t graph at that instant.
In the interval $0 < t < 2$, note that $a > 0$ and v is increasing; the particle is **accelerating**.

For $t > 2$, $a < 0$ and v is decreasing; the particle is **decelerating** or **retarding**. So, *negative* acceleration is also known as **deceleration** or **retardation**.

4. Displacement, $x = \int v \, dt$. Hence, we note that

area of $P = \int_3^4 v \, dt = x(4) - x(3) = 32 - 27 = 5$ and area of $Q = -\int_4^6 v \, dt = 32$.

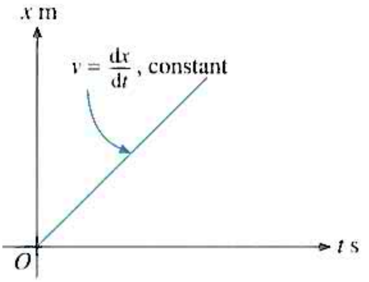
So, the change in displacement from $t = 3$ to $t = 6$ is $\int_3^6 v \, dt = \int_3^4 v \, dt + \int_4^6 v \, dt$

$= \text{area of } P + (-\text{ area of } Q) = -27$

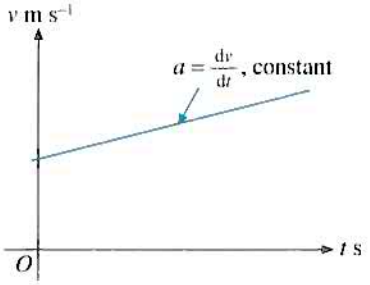
but the distance travelled during this interval $= 5 + 32$

$= \text{area of } P + \text{area of } Q = \text{shaded area.}$

5. A linear graph has a constant gradient and its rate of change is said to be **uniform**. Hence a linear x-t graph shows motion of uniform velocity while a linear v-t graph shows motion of uniform acceleration, as seen below:

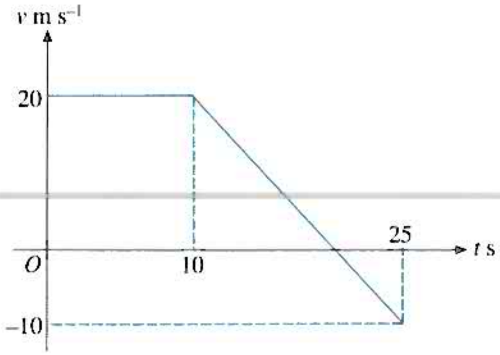


Uniform velocity motion



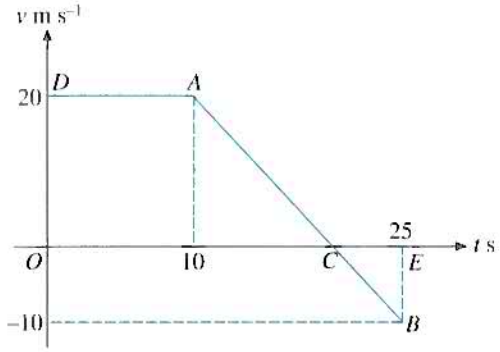
Uniform or constant acceleration motion

Example 7 The diagram shows the velocity-time graph of a particle P moving in a straight line.



(a) Describe very briefly the motion of the particle.
(b) Find the deceleration of P for the interval $t = 10$ to $t = 25$.
(c) Find the time when P is instantaneously at rest.
(d) Find the total distance travelled in the 25 s.
(e) Find the distance of P from its starting point after 25 s.

Solution:



(a) P moves with a constant velocity of 20 m s^{-1} for 10 seconds and decelerates uniformly for the next 15 seconds to a velocity of -10 m s^{-1}.

(b) The acceleration is given by the gradient of the v-t graph. From $t = 10$ to $t = 25$,

$$a = \text{gradient of } AB = \frac{(-10) - 20}{25 - 10} = -2$$

$\therefore$ the required deceleration is **2 m s^{-2}**.

(c) Let T be the time when P is at instantaneous rest. Then gradient of $AC = a$

$$\frac{0 - 20}{T - 10} = -2$$
$$T = \mathbf{20}$$

(d) Total distance travelled in the 25 s

$= \text{area of trapezium } OCAD + \text{area of triangle } CBE$

$= \frac{1}{2} \times 20(20 + 10) + \frac{1}{2} \times 5 \times 10$

$=$ **325 m**

(e) Change in displacement in the 25 s is given by area under the graph from $t = 0$ to $t = 25$.
So, change in displacement

$= \text{area of trapezium } OCAD + (-\text{ area of triangle } CBE)$

$= \frac{1}{2} \times 20(20 + 10) - \frac{1}{2} \times 5 \times 10$

$= 275$ m

$\therefore$ the distance of P from the starting point after 25 s is **275 m**.

Example 8

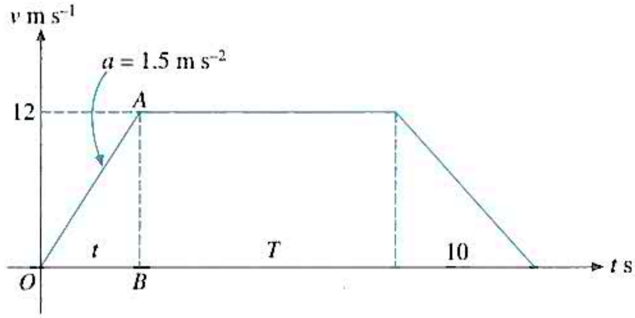
A car accelerates uniformly from rest, at a rate of 1.5 m s^{-2}, to a speed of 12 m s^{-1}. It then continues moving at 12 m s^{-1} for T seconds before decelerating uniformly to rest in a further 10 seconds.

(a) Sketch the velocity-time graph for the motion of the car.
(b) Find the time taken while accelerating.
(c) Find the distance moved while accelerating.

Given that the total distance moved by the car is 852 m, find the value of T.

Solution:

(a) The velocity-time graph for the motion is sketched as follows:



(b) Gradient of $OA = a = 1.5 \quad \Rightarrow \quad \frac{12}{t} = 1.5$

$$\Rightarrow \quad t = 8$$

$\therefore$ the time taken while accelerating is **8 s**.

(c) The distance moved while accelerating

$$= \text{area of triangle } OAB$$
$$= \frac{1}{2} \times t \times 12$$
$$= \mathbf{48\ m}$$

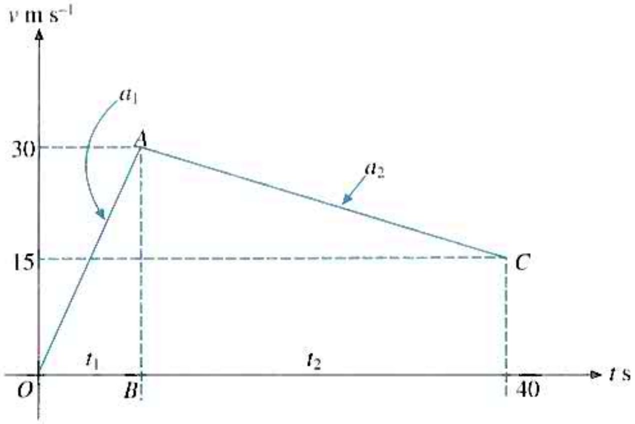
(d) Total distance moved = 852 m

$$\Rightarrow \quad \text{area of trapezium} = 852$$
$$\frac{1}{2} \times 12[(8 + T + 10) + T] = 852$$
$$2T + 18 = 142$$
$$T = \mathbf{62}$$

Example 9

A train accelerates at a constant rate from rest to a velocity of 30 m s^{-1}. On attaining this velocity, it retards uniformly to a velocity of 15 m s^{-1}. The train has been in motion for 40 seconds. Sketch the velocity-time graph for the motion of the train. Given that the magnitude of the acceleration is 6 times the magnitude of the retardation, find the total distance travelled during this time.

Solution: The velocity-time graph for the motion is sketched as follows:



$$
\begin{aligned}
|a_1| &= 6 \times |a_2| \\
\Rightarrow \quad \frac{30}{t_1} &= 6 \times \frac{15}{t_2} \\
t_2 &= 3t_1 \\
40 - t_1 &= 3t_1 \\
t_1 &= 10 \\
\Rightarrow \quad t_2 &= 40 - t_1 \\
&= 30
\end{aligned}
$$

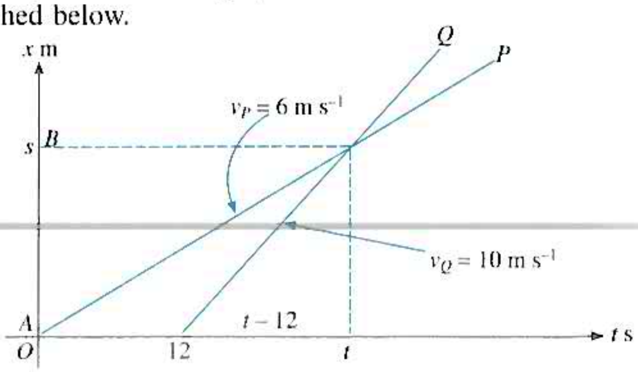
$\therefore$ total distance travelled = area under graph

$$= \frac{1}{2} \times t_1 \times 30 + \frac{1}{2} \times t_2(30 + 15)$$

$$= \mathbf{825\ m}$$

Example 10 A man P, running along a straight road, passes a point A at a constant speed of 6 m s^{-1}. Twelve seconds later a second man Q, running in the same direction as P, passes A at a constant speed of 10 m s^{-1}. Q overtakes P at a point B, where $AB = s$ metres, t seconds after P left A.
Draw, on the same diagram, the displacement-time graphs for the motions of P and Q from A to B, and evaluate s and t.

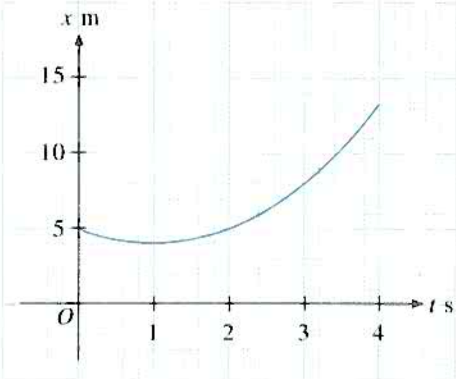
Solution: The displacement-time graphs for the motions of the two men are sketched below.



$$v_P = \frac{s}{t} = 6 \quad \Rightarrow \quad s = 6t$$
$$v_Q = \frac{s}{t-12} = 10 \quad \Rightarrow \quad s = 10(t-12)$$
$$\therefore \quad 10(t-12) = 6t \quad \Rightarrow \quad t = \mathbf{30}$$
$$\Rightarrow \quad s = 6t$$
$$= \mathbf{180}$$

Exercise 22.2

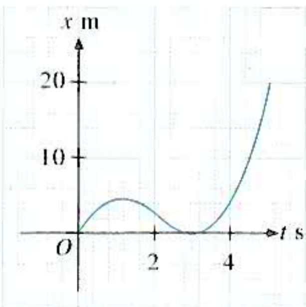
1. The diagram shows an x-t graph for a moving particle along a straight line and the equation of motion is $x = (t-1)^2 + 4$ for $0 \leqslant t \leqslant 4$. Find
   (a) the velocity at $t = 1$ and $t = 3$,
   (b) the total distance travelled during the interval $0 \leqslant t \leqslant 4$ and hence the average speed over this interval.



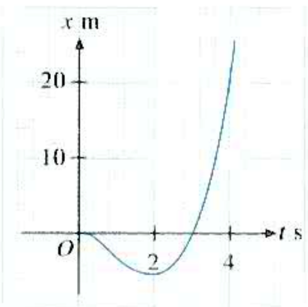
2. The displacement, x m, of a particle from a fixed point O on the line it is travelling is given by $x = 6t - t^2$.
   (a) Sketch the displacement-time graph. With the help of this graph, find the total distance travelled in the first 6 seconds.
   (b) Find v in terms of t and sketch the velocity-time graph.
   (c) Is the particle accelerating or decelerating? Is it doing so uniformly?

3. Match the x-t graphs (a), (b), and (c) with their corresponding v-t graphs (d), (e), and (f):

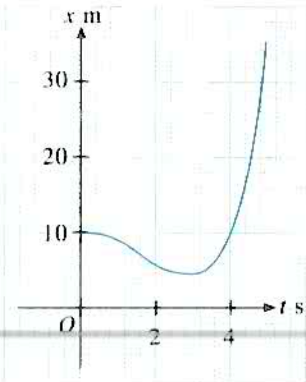
(a)



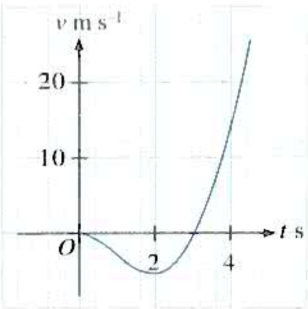
(b)



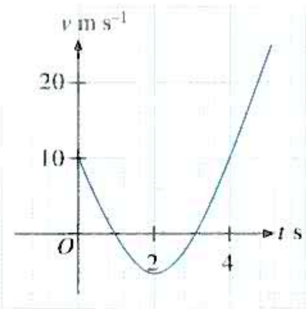
(c)



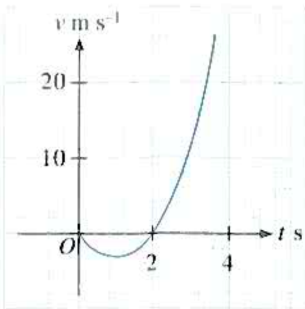
(d)



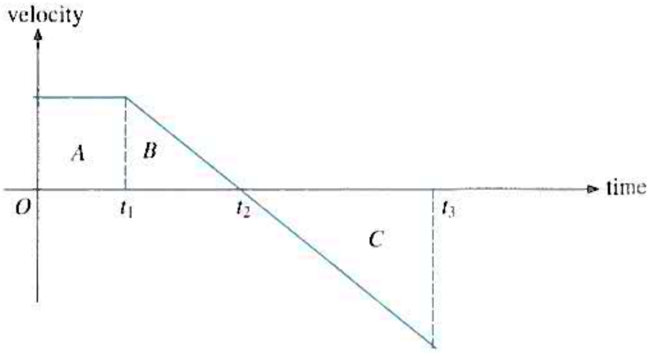
(e)



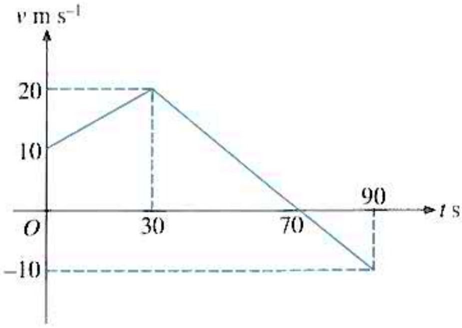
(f)



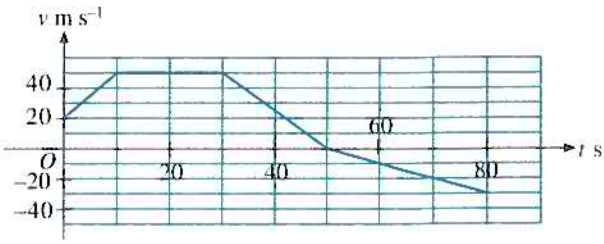
4. The diagram shows the velocity-time graph of a particle moving in a straight line.
   (a) Describe very briefly the motion of the particle.
   (b) If the particle has returned to its starting point after time t_3, what deduction can be made about the regions marked A, B and C? (C)



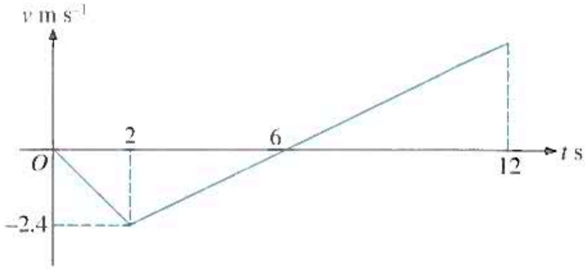
5. The velocity-time graph shows the velocities at different times of a particle travelling along a straight line. Calculate
   (a) the acceleration during the first 30 s,
   (b) the distance travelled in the first 70 s,
   (c) the total distance travelled in the 90 s,
   (d) the displacement of the particle from the starting point after 90 s.



6. The velocity-time graph shows the velocities at different times of a particle moving in a straight line. Calculate
   (a) the acceleration during the first 10 seconds,
   (b) the distance travelled in the first 50 seconds,
   (c) the average velocity during the first 50 seconds,
   (d) the total distance travelled in the 80 seconds,
   (e) the distance from the starting point after 80 seconds.



7. The figure shows the velocity-time diagram for the first 12 seconds of the motion of a particle P. This particle starts from rest at a point X and moves in a straight line. When $t = 2$ its velocity is -2.4 m s^{-1} and when $t = 6$ it is instantaneously at rest. Find



(a) the distance of P from X when $t = 6$,
(b) the acceleration of P during the time interval $t = 2$ to $t = 12$,
(c) the speed of P when $t = 12$,
(d) the distance of P from X when $t = 12$,
(e) the value of t, correct to one decimal place, when P returns to X. *(C)*

8. A boy running a 400-m race accelerates uniformly from rest for the first T seconds and reaches a velocity of 5 m s^{-1}. He maintains this velocity for the rest of the race. His time for the race is 82 s. Sketch a velocity-time graph. Calculate
(a) the value of T,
(b) the acceleration,
(c) the distance the boy runs before he reaches his maximum speed.

9. A motorist starting a car from rest accelerates uniformly to a speed of v m s^{-1} in 12 seconds. He maintains this speed for another 40 seconds and then applies the brakes and decelerates uniformly to rest. His deceleration is numerically equal to four times his previous acceleration.
(a) Sketch a velocity-time graph.
(b) Calculate the time during which deceleration takes place.
(c) Given that the total distance moved is 570 m, calculate the value of v.
(d) Calculate the final deceleration.

10. A particle P, moving in a straight line, passes a point M with a constant speed of 6 m s^{-1}. Five seconds later a second particle Q, travelling in the same direction as P, passes M with a constant speed of 8 m s^{-1}. The particle Q overtakes P at a point N, where $MN = s$ metres, t seconds after P left M.
Draw, on the same diagram, the displacement-time graphs for the motions of P and Q from M to N, and evaluate s and t.

11. A lift cage descends a mine shaft 155 m deep. It accelerates uniformly to a speed of 5 m s^{-1}, maintains this constant speed for 22 s, and then decelerates uniformly to rest. Given that the acceleration is twice the deceleration, sketch the velocity-time graph. Hence or otherwise calculate the acceleration and the total time of descent.

12. Two cyclists, A and B, start from rest at the same instant. Cyclist A accelerates at a uniform rate to reach a speed of v m s^{-1} in t seconds. Again accelerating uniformly, cyclist A's speed is doubled during the next $3t$ seconds. Sketch a velocity-time graph for the motion of cyclist A.
Cyclist B attains the same final speed as cyclist A in the same total time by accelerating from rest at 2 m s^{-2}. Find v in terms of t.
Given that the total distance travelled by cyclist A during the given $4t$ seconds is 80 m, calculate the value of v and of t.

13. A car starts from rest and accelerates uniformly at a m s^{-2} for t_1 seconds to a speed of 10 m s^{-1}. It then accelerates at $2a$ m s^{-2} for a further t_2 seconds to a speed of 20 m s^{-1}. It maintains this speed for a further 24 seconds. The total distance covered is 555 m.

(a) Show that $t_1 = 2t_2$.

(b) Sketch a velocity-time diagram.

(c) Find the total time taken.

(d) Find the value of a. *(C)*

14. A car accelerates uniformly from rest, at a rate of 1.6 m s^{-2}, to a speed of 20 m s^{-1}. It then continues moving at 20 m s^{-1} for T seconds before decelerating uniformly to a speed of 12 m s^{-1} in a further 5 seconds.

(a) Sketch the velocity-time diagram for the motion of the car.

Find

(b) the time taken while accelerating,

(c) the distance moved while accelerating,

(d) the distance moved while decelerating.

Given that the total distance moved by the car is 1245 m, find the value of T.

On another occasion the car accelerates uniformly from rest at a point A until it reaches B with a speed of 15 m s^{-1}. At B the driver observes an obstruction ahead at a point C and immediately applies the brakes, causing the car to decelerate uniformly to rest 5 m before the obstruction. Given that the car is in motion for 12 seconds, sketch the velocity-time diagram for this motion and find the distance AC. Given also that the magnitude of the deceleration is four times that of the acceleration, find the distance BC. *(C)*

15. A motorcyclist travelling along a straight road passes a fixed point O with a speed of 10 m s^{-1} and continues at this speed for t_1 seconds. Over the next 10 seconds he accelerates at a constant rate to a speed of 15 m s^{-1}. He then brings the motorcycle to rest in a further t_2 seconds by retarding at a constant rate. His acceleration and retardation are of equal magnitude.

(a) Sketch a velocity-time graph to illustrate the motion of the motorcyclist after passing O.

(b) Calculate t_2.

(c) Given that the total distance represented by the graph is 850 m, calculate t_1.

22.3 Equations of Motion with Constant Acceleration (Optional)

Example 11 A particle moves in a straight line with an initial velocity of u m s^{-1} and constant acceleration a m s^{-2}. At any time t seconds after the start of motion, its velocity is v m s^{-1} and its displacement from its starting point is s m. Use integration to prove the following equations of motion with constant acceleration

(a) $v = u + at$, **(b)** $s = ut + \frac{1}{2}at^2$

(c) $s = \frac{1}{2}(u + v)t$ **(d)** $v^2 = u^2 + 2as$

Solution:

(a) $\frac{dv}{dt} = a$, a constant $\Rightarrow \quad v = \int a \, dt$

$$= at + c$$

When $t = 0$, $v = u$ and so $c = u$.

$\therefore \quad v = u + at$

(b) So, $s = \int v \, dt$

$$= \int (u + at) \, dt$$

$$= ut + \frac{1}{2}at^2 + d$$

When $t = 0$, $s = 0$ and so $d = 0$.

So, $s = ut + \frac{1}{2}at^2$

(c) $s = ut + \frac{1}{2}at^2$

$$= \frac{1}{2}t(2u + at)$$

$$= \frac{1}{2}t(u + u + at)$$

$$s = \frac{1}{2}(u + v)t$$

(d) $v = u + at \quad \Rightarrow \quad t = \frac{v - u}{a}$

$$\therefore \quad s = \frac{1}{2}(u + v)t$$

$$\Rightarrow \quad s = \frac{1}{2}(u + v)\frac{v - u}{a}$$

$$2as = (u + v)(v - u)$$

$$= v^2 - u^2$$

$$\Rightarrow \quad v^2 = u^2 + 2as$$

Note: If the displacement of the particle is not measured from its starting point, how would this affect the above results?

For linear motion with constant acceleration:

$$v = u + at \qquad s = ut + \frac{1}{2}at^2$$

$$s = \frac{1}{2}(u + v)t \qquad v^2 = u^2 + 2as$$

Examples 12 to 16 illustrate how to apply these equations for a particle moving in a straight line with constant or uniform acceleration. Where necessary, we shall use an arrow (⟶) to indicate velocity and double arrows (⇾⇾) to indicate acceleration.

Example 12 A car moving with a constant acceleration of 3 m s^{-2} passes point A with a speed of 6 m s^{-1}. Find its speed and the distance travelled after 8 seconds.

Solution:

We have $u = 6 \text{ m s}^{-1}$, $a = 3 \text{ m s}^{-2}$ and $t = 8$.
Using $v = u + at$ from A to B,

$$v = 6 + 3 \times 8 = 30$$

$\therefore$ its speed after 8 seconds is **30 m s^{-1}**.

Using $s = ut + \frac{1}{2}at^2$ from A to B,

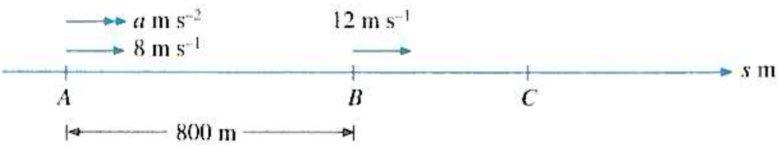
$$s = 6 \times 8 + \frac{1}{2} \times 3 \times 8^2$$
$$= 144$$

$\therefore$ its displacement from the starting point A, and hence the distance travelled, is **144 m**.

Note: Alternatively, s could be found using $s = \frac{1}{2}(u + v)t$ or $v^2 = u^2 + 2as$.

Example 13 A car travelling with constant acceleration along a straight road passes two bus stops A and B with speeds 8 m s^{-1} and 12 m s^{-1} respectively. Given that the two bus stops are 800 m apart, calculate
(a) its acceleration,
(b) the time taken to move from A to B,
(c) the distance BC if it reaches C after another 10 seconds.

Solution:



(a) Using $v^2 = u^2 + 2as$ from A to B,

$$12^2 = 8^2 + 2a(800)$$
$$a = \frac{12^2 - 8^2}{2 \times 800}$$
$$= 0.05$$

$\therefore$ its acceleration is **0.05 m s^{-2}**.

(b) Using $v = u + at$ from A to B,

$$12 = 8 + 0.05t$$
$$t = 80$$

$\therefore$ the time taken to move from A to B is **80 s**.

(c) Using $s = ut + \frac{1}{2}at^2$ from B to C,

$$s = 12 \times 10 + \frac{1}{2} \times 0.05 \times 10^2 = 122.5$$

$\therefore$ the distance BC is **122.5 m**.

Example 14

Two particles, X and Y, are moving in the same direction on parallel horizontal tracks. As X passes through a point P on its track, X overtakes Y. At this instant X is travelling at a speed of 28 m s^{-1} and retarding uniformly at 4 m s^{-2}, and Y is travelling at a speed of 4 m s^{-1} and accelerating uniformly at 2 m s^{-2}. Calculate

(a) the speed of Y when X comes to rest,

(b) the distance of X from P when Y overtakes X.

Solution:

(a) Since X is decelerating, its acceleration is negative, i.e. -4 m s^{-2}.

Using $v = u + at$, X comes to rest when

$$v = 0 \quad \Rightarrow \quad 28 + (-4)t = 0$$
$$t = 7$$

So, using $v = u + at$ for Y, we have

$$v = 4 + 2 \times 7 = 18$$

$\therefore$ the speed of Y when X comes to rest is **18 m s^{-1}**.

(b) Let s_X and s_Y be the displacements of X and Y respectively from P.

Using $s = ut + \frac{1}{2}at^2$, we have

$$s_X = 28t + \frac{1}{2}(-4)t^2$$

$$\text{and } s_Y = 4t + \frac{1}{2}(2)t^2$$

Y overtakes X at $t > 0$ and

$$s_X = s_Y \quad \Rightarrow \quad 28t + \frac{1}{2}(-4)t^2 = 4t + \frac{1}{2}(2)t^2$$
$$24t = 3t^2$$
$$t = 8$$
$$\Rightarrow \quad s_X = 28 \times 8 + \frac{1}{2}(-4)\,8^2 = 96$$

$\therefore$ the distance of X from P when Y overtakes X is **96 m**.

Vertical Motion under Gravity (Optional)

When an object falls under gravity, its downward acceleration is called the **acceleration due to gravity** and is denoted by g. To simplify calculations, we shall take the value of g to be constant at 10 m s^{-2}, although the value of g varies slightly at different points on the earth.

If an object falls vertically and we neglect the effect of air resistance, its motion is a constant acceleration motion in a straight line and the equations of motion hold.

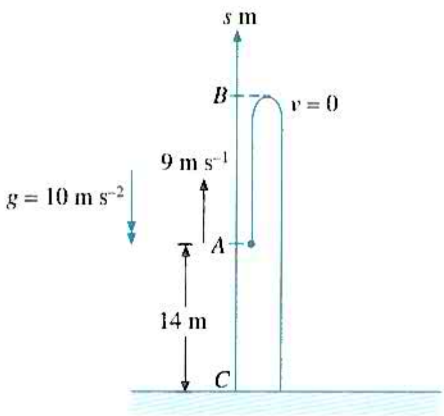
Example 15 A stone is projected vertically upwards from a height of 14 m above the ground with velocity 9 m s^{-1}. Calculate

(a) its greatest height above the ground,

(b) its speed when it hits the ground,

(c) its time of flight.

Solution:



(a) Using $v^2 = u^2 + 2as$ from A to B,

at the greatest height, $v = 0 \Rightarrow 0 = 9^2 + 2(-10)s$

$s = 4.05$

$\therefore$ its greatest height above the ground is

$14 + 4.05 =$ **18.05 m.**

(b) The stone hits the ground at C where $s = -14$.

Using $v^2 = u^2 + 2as$ from A to C (through B),

$$\begin{aligned} v^2 &= 9^2 + 2(-10)(-14) \\ &= 361 \\ v &= -19 \quad (v < 0 \text{ as the stone is moving downwards!}) \end{aligned}$$

$\therefore$ the stone hits the ground with a speed of **19 m s^{-1}**.

(c) Using $v = u + at$ from A to C (through B),

$-19 = 9 + (-10)t$

$t = 2.8$

$\therefore$ the time of flight is **2.8 s.**

Example 16

A particle falls from rest from a point h m above the ground. During the last 2 seconds before it hits the ground, the particle moves through 140 m. Calculate

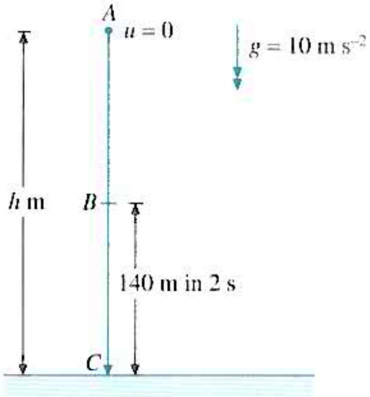
(a) its speed at the beginning of the last 2 seconds,

(b) the speed at which it hits the ground,

(c) the total time it takes to reach the ground,

(d) the value of h.

Solution:



(a) Using $s = ut + \frac{1}{2}at^2$ from B to C,

$$140 = u(2) + \frac{1}{2} \times 10 \times 2^2$$

$$u = 60$$

$\therefore$ its speed at the beginning of the last 2 seconds is **60 m s^{-1}**.

(b) Using $v = u + at$ from B to C,

$$v = 60 + 10 \times 2 = 80$$

$\therefore$ it hits the ground with a speed of **80 m s^{-1}**.

(c) Using $v = u + at$ from A to C,

$$80 = 0 + 10t$$

$$t = 8$$

$\therefore$ it takes **8 seconds** to reach the ground.

(d) Using $s = \frac{1}{2}(u + v)t$ from A to C,

$$h = \frac{1}{2}(0 + 80)8$$

$$= \mathbf{320}$$

Note: Remember that the displacement s is taken from the starting point. Hence in **(a)** for the motion B to C, B is taken as the starting point while in **(d)** for the motion A to C, A is taken as the starting point.

Exercise 22.3 (Optional)

1. A particle moving in a straight line with constant acceleration increases its speed from 6 m s^{-1} to 34 m s^{-1} in 4 seconds. Find its acceleration and the distance covered.

2. A man started from rest and attained a speed of 8 m s^{-1} after 10 seconds. If his acceleration is uniform, how far did he run? How far will he cover if he runs a further 5 seconds?

3. A car travels on a straight road with a constant acceleration of 2 m s^{-2} and its initial velocity is 10 m s^{-1}. After T seconds, its speed is 24 m s^{-1}, find the value of T and the distance travelled during these T seconds.

4. A lorry travelling in a straight line increases its speed uniformly from 40 km h^{-1} to 62 km h^{-1} over a distance of 0.935 km. Find its acceleration and its velocity when it has travelled 0.64 km.

5. A cyclist travels on a straight road with constant acceleration a m s^{-2}. He passes through points A, B and C with speeds of 3 m s^{-1}, 5 m s^{-1} and 12 m s^{-1} respectively. Given that the distance of B from A is 80 m, find
   (a) the value of a,
   (b) the distance of C from B.

6. A train is travelling at a speed of 30 m s^{-1}. On approaching a station, the brakes are applied and the train retarded uniformly and came to a stop in 20 seconds. Calculate
   (a) the retardation of the train,
   (b) the distance covered before the train came to a stop,
   (c) the speed of the train 5 seconds before stopping.

7. A stone is released from rest at a height of 45 m above the ground. Calculate the time taken for it to hit the ground. What is the speed with which it hits the ground?

8. A stone is projected vertically upwards from a height of 13 m above the ground with velocity 8 m s^{-1}. Calculate
   (a) its greatest height above the ground,
   (b) its velocity when it hits the ground.

9. A particle is projected vertically upwards from the ground with a speed of 40 m s^{-1}. Calculate
   (a) the time for which it is above a height of 75 m,
   (b) the speed which it has at this height on its way down,
   (c) the total time of flight.

10. A particle is thrown vertically downwards with a speed of 5 m s^{-1} from a point x m above the ground. During the last second before it hits the ground, the particle moves through 17 m. Calculate its speed at the beginning of the last second. Hence find the value of x.

11. A marble is thrown vertically upwards with a velocity of V m s^{-1}, where $V > 0$, from a point 30 m above the ground. Given that its velocity is 5 m s^{-1} after 2 seconds, calculate
   (a) the value of V,
   (b) its maximum height from the ground and the time taken to reach this height,
   (c) its velocity just before it hits the ground,
   (d) the time when the marble was in the air.

12. A particle is projected vertically upwards from ground level. Between 2 seconds and 4 seconds after leaving the ground it rises 40 m. Calculate
   (a) the speed of projection,
   (b) the maximum height reached,
   (c) the time interval for which the particle is above a level of 120 m.

13. A balloon is ascending vertically at a constant velocity of 4 m s^{-1}. The crew releases some gas and as a result the balloon experiences a constant downward acceleration of 0.2 m s^{-2}. How much further will the balloon ascend, and how long will it be before the balloon returns to its original height? If this height is 120 m above the ground and the balloon continues to descend with the same acceleration, how much longer will it be before the balloon strikes the ground and what will be its velocity at impact?

14. A car moves along a straight level road, accelerating from rest at a constant rate for 10 s over a distance of S_1 m until it reaches a speed of V m s^{-1}. Express S_1 in terms of V.
   It then accelerates at a constant rate of 4 m s^{-2} over a distance of S_2 m until it reaches a speed of 24 m s^{-1}. Express S_2 in terms of V.
   Given that the car has now travelled a total distance of 104 m, calculate the possible values of V. Using the smaller of these values, calculate the time taken to travel this distance.

15. **(a)** Two particles, X and Y, are moving in the same direction on parallel horizontal tracks. As X passes through a point P on its track, X overtakes Y. At this instant X is travelling with a speed of 22 m s^{-1} and retarding uniformly at 3 m s^{-2}, and Y is travelling with a speed of 7 m s^{-1} and accelerating uniformly at 2 m s^{-2}. Calculate
   (i) the speed of Y when X comes to rest,
   (ii) the distance of X from P when Y overtakes X.
   (b) A car, moving with a constant acceleration of 2 m s^{-2} passes three points A, B and C, on a straight horizontal road, where $AB = BC$. Given that the car passes A and B with speeds of 7 m s^{-1} and 17 m s^{-1} respectively, calculate its speed when it passes C.
   The point D also lies between A and C and the car took the same time to travel from A to D as it did from D to C. Calculate the speed of the car when it passed D and the distance of D from A. *(C)*

Important Notes

1. For a particle moving in a straight line with displacement x, velocity v and acceleration a:

$$v = \frac{dx}{dt} \qquad \text{and} \qquad a = \frac{dv}{dt}$$

Conversely,

$$x = \int v\,dt \qquad \text{and} \qquad v = \int a\,dt$$

That is,

$$\begin{tikzcd} x \arrow[r, bend left, "\text{differentiate}"] & v \arrow[l, bend left, "\text{integrate}"] \arrow[r, bend left, "\text{differentiate}"] & a \arrow[l, bend left, "\text{integrate}"] \end{tikzcd}$$

When the particle is at instantaneous rest, $v = 0$.

2. Average speed $= \dfrac{\text{distance travelled}}{\text{time taken}}$

3. In a displacement-time graph, the velocity at any instant is given by the gradient of the graph at that instant.

4. In a velocity-time graph,
   (a) a straight line is obtained when the acceleration is uniform (constant),
   (b) the acceleration at any instant is given by the gradient of the graph at that instant,
   (c) both the change in displacement and the distance travelled may be found by considering the appropriate areas under the graph.

5. (Optional) For a particle moving in a straight line with constant acceleration, including vertical motion under gravity, the equations of motion are

$$v = u + at,$$
$$s = ut + \frac{1}{2}at^2,$$
$$s = \frac{1}{2}(u + v)\,t,$$
$$v^2 = u^2 + 2as,$$

where s is the displacement from the starting point,
u is the initial velocity,
v is the final velocity,
a is the acceleration, and
t is the time.

Miscellaneous Examples

Example 17

A particle moves in a straight line so that at time t seconds after leaving a fixed point O, its velocity v m s^{-1}, is given by $v = 200e^{-2t}$.

(a) Find its initial acceleration.

(b) Calculate, to 2 decimal places, its displacement from O when $t = 3$.

(c) Sketch the velocity-time and displacement-time graphs for the particle.

Solution:

(a) $v = 200e^{-2t} \quad \Rightarrow \quad a = \dfrac{dv}{dt} = 200(-2e^{-2t})$

$$= -400e^{-2t}$$

When $t = 0$, $a = -400 \times 1 = -400$

$\therefore$ its initial acceleration is **-400 m s^{-2}**.

(b) Displacement from O, $x = \displaystyle\int v \, dt = \int 200e^{-2t} \, dt$

$$= 200 \times \frac{1}{-2}e^{-2t} + c$$

$$= -100e^{-2t} + c$$

When $t = 0$, $x = 0$ and so

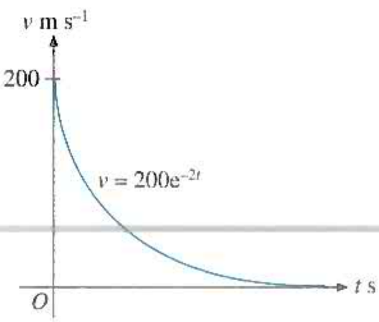
$$0 = -100 \times 1 + c$$

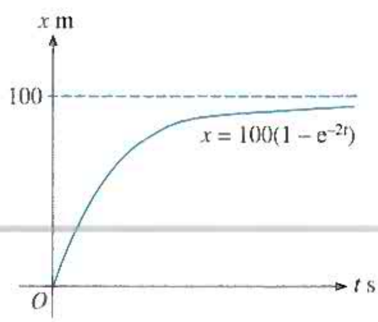
$$c = 100$$

$\therefore \; x = 100(1 - e^{-2t})$

When $t = 3$, $x = 100(1 - e^{-6}) =$ **99.75**

(c) Its velocity-time graph and displacement-time graph are as follows:





Example 18 During a certain stage of its journey, a car decelerates uniformly from a speed of 30 m s^{-1} to a speed of 20 m s^{-1} which it maintains for a time before accelerating uniformly to its former speed of 30 m s^{-1}. Sketch a velocity-time graph to illustrate this stage. Given that, for this stage of the journey, the total distance travelled is 7200 m, the total time taken is 320 seconds, and the magnitude of the acceleration is three times that of the deceleration, calculate

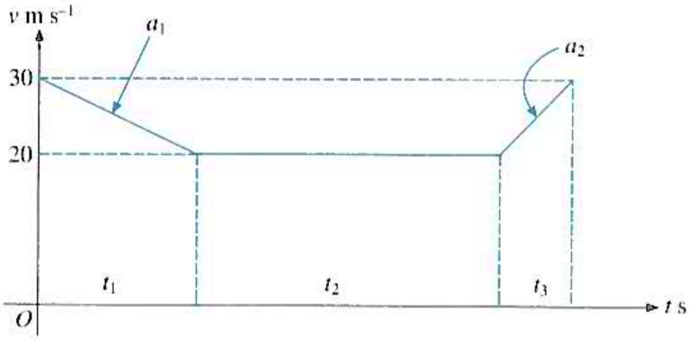
(a) the time during which the car is accelerating,

(b) the speed of the car 160 seconds after the start of this stage,

†**(c)** the speed of the car 290 seconds after the start of this stage.

†Optional

Solution: Its velocity-time graph is sketched below:



(a) Total distance travelled = 7200

$$\Rightarrow \quad \frac{1}{2}t_1(30 + 20) + 20t_2 + \frac{1}{2}t_3(30 + 20) = 7200$$

$$\Rightarrow \quad 5t_1 + 4t_2 + 5t_3 = 1440 \quad \text{.............................. (1)}$$

Total time taken = 320

$$\Rightarrow \quad t_1 + t_2 + t_3 = 320 \quad \text{.................................. (2)}$$

Magnitude of acceleration = 3 × Magnitude of deceleration

$$|a_2| = 3 \times |a_1|$$

$$\frac{10}{t_3} = 3 \times \frac{10}{t_1}$$

$$t_1 = 3t_3 \quad \text{...................... (3)}$$

Substitute (3) into (2):

$$3t_3 + t_2 + t_3 = 320$$

$$t_2 = 320 - 4t_3 \quad \text{............ (4)}$$

Substitute (4) and (3) into (1):

$$5 \times 3t_3 + 4(320 - 4t_3) + 5t_3 = 1440$$

$$4t_3 = 1440 - 4(320)$$

$$t_3 = 40$$

$\therefore$ the time taken for accelerating is **40 s.**

(b) $t_1 = 3t_3 = 120 \quad \Rightarrow \quad$ when $t = 160$, $v = 20$

$\therefore$ after 160 seconds, its speed is **20 m s^{-1}**.

(c) $t_2 = 320 - 4t_3 = 160$

$$t_1 + t_2 = 120 + 160 = 280 \quad \Rightarrow \quad \text{at } t = 290,$$

the car is accelerating with $a_2 = \dfrac{10}{t_3} = 0.25$

It is into $(290 - 280) = 10$ s of this acceleration.

Using $v = u + at$, we get

$$\begin{aligned} v &= 20 + 0.25 \times 10 \\ &= 22.5 \end{aligned}$$

$\therefore$ after 290 seconds, its speed is **22.5 m s^{-1}**.

Example 19

(Optional)

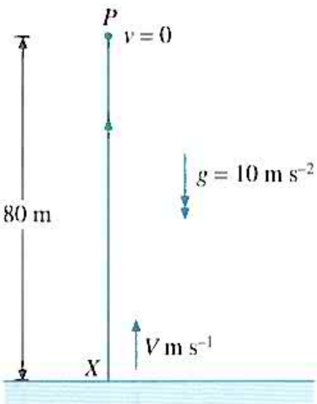
A particle P is projected vertically upwards from a point X with a speed of V m s^{-1}. Given that it reaches a maximum height of 80 m, find the value of V. When P reaches its maximum height, a second particle Q is projected vertically upwards, also from X. Given that 3 s after Q is projected, P and Q collide at a point Y, find

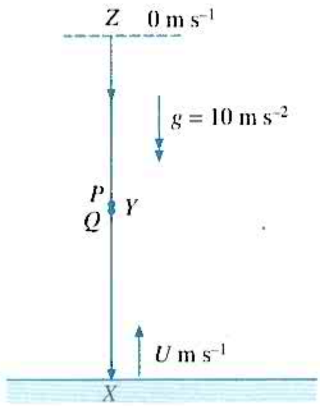
(a) the height of Y above the ground,

(b) the speed of the projection of Q,

(c) the speed of Q at Y and state whether it is moving upwards or downwards.

Solution:





At the maximum height, $v = 0$. So, using $v^2 = u^2 + 2as$,

$$\begin{aligned} 0 &= V^2 + 2(-10)80 \\ V &= \mathbf{40} \end{aligned}$$

(a) Using $s = ut + \frac{1}{2}at^2$ for P from Z to Y,

$$\begin{aligned} s &= 0 + \frac{1}{2}(10)3^2 \\ &= 45 \end{aligned}$$

$\therefore$ the height of Y above the ground is $80 - 45 =$ **35 m**.

(b) Using $s = ut + \frac{1}{2}at^2$ for Q from X to Y,

$$35 = U \times 3 + \frac{1}{2}(-10)3^2$$

$$U = 26\frac{2}{3}$$

$\therefore$ the speed of projection of Q is **$26\frac{2}{3}$ m s^{-1}**.

(c) Using $v = u + at$ for Q from X to Y,

$$v = 26\frac{2}{3} + (-10)3$$

$$= -3\frac{1}{3}$$

$\therefore$ at Y, Q is moving downwards with a speed of **$3\frac{1}{3}$ m s^{-1}**.

Example 20

(Optional)

A balloon is ascending at a uniform speed of 5 m s^{-1}. A small stone is released from the balloon at a height of 60 m above the ground and moves freely under gravity. Find

(a) the greatest height above the ground attained by the stone,

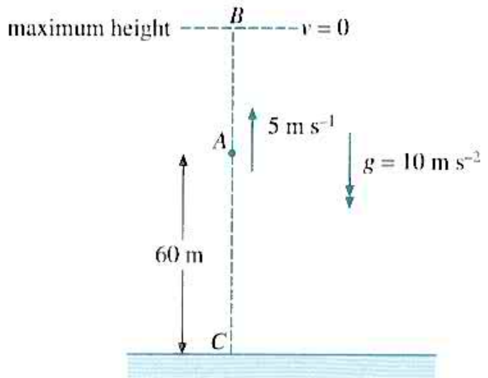
(b) the speed with which the stone hits the ground,

(c) the time taken by the stone to reach the ground from the moment of release.

After the release of the stone, the balloon accelerates uniformly upwards at 0.6 m s^{-2}. Find the height of the balloon above the ground when the stone hits the ground.

Solution:

For the stone,



(a) At the maximum height, $v = 0$.

Using $v^2 = u^2 + 2as$ for the stone from A to B,

$$0 = 5^2 + 2(-10)\,s$$

$$s = 1.25$$

$\therefore$ the greatest height of the stone above the ground is $(60 + 1.25) =$ **61.25 m**.

(b) Using $v^2 = u^2 + 2as$ for the stone from A to C,

$$v^2 = 5^2 + 2(-10)(-60)$$
$$= 1225$$
$$v = -35 \quad (v < 0 \text{ as the stone is moving downwards!})$$

$\therefore$ the stone hits the ground with a speed of **35 m s^{-1}**.

(c) Using $v = u + at$ for the stone from A to C,

$$-35 = 5 - 10t$$
$$t = 4$$

$\therefore$ the time taken is **4 s**.

Using $s = ut + \frac{1}{2}at^2$ for the balloon with $t = 4$,

$$s = 5 \times 4 + \frac{1}{2}(0.6)4^2$$
$$= 24.8$$

$\therefore$ when the stone hits the ground, the height of the balloon above the ground is $(60 + 24.8) =$ **84.8 m**.

Note: Can you think of an alternative solution for **(c)**?

Miscellaneous Exercise 22

1. A particle moves in a straight line and its displacement, s cm, from a fixed point O, t seconds after passing O, is given by $s = 27t - t^3$. Calculate
(a) the acceleration of the particle when it comes instantaneously to rest,
(b) the velocity of the particle when it is next at O,
(c) the distance travelled in the fourth second,
(d) the distance travelled in the first 4 seconds.

2. A particle passes a fixed point O with a velocity of 6 m s^{-1} and moves in a straight line with an acceleration of $(2 - t)$ m s^{-2}, where t is the time in seconds after passing O. Calculate
(a) its velocity when $t = 5$,
(b) the distance travelled during the first 6 seconds after passing O.

3. A particle moves in a straight line so that, at time t seconds after leaving a fixed point O, its velocity v m s^{-1} is given by $v = \dfrac{144}{(2t + 3)^2} - 4k$, where k is a constant. Its initial velocity is 12 m s^{-1}. Find
(a) the value of k,
(b) the value of t for which the particle is at instantaneous rest,
(c) the acceleration and the displacement of the particle from O when $t = \frac{1}{2}$.

4. A particle starts from rest and travels in a straight line so that t seconds after passing a fixed point O, its acceleration a m s^{-2} is given by $a = 2(3 - e^{-t})$. Calculate
(a) the velocity of the particle when $t = 2$,
(b) the displacement of the particle from O when $t = 1$.

5. A particle moves in a straight line so that its distance, s m, from a fixed point A on the line is given by $s = 2t^2 - 4t + 9$, for $t \leqslant 3$, where t is the time in seconds after passing through a point B on the line. Find
   (a) the distance AB,
   (b) the distance from A of the particle when it is instantaneously at rest,
   (c) the total distance travelled by the particle in the period $t = 0$ to $t = 3$,
   (d) the velocity of the particle when $t = 3$.
   At $t = 3$ the acceleration of the particle is changed to $(t - 8)$ m s^{-2}, the instantaneous velocity remaining unchanged. Find the next value of t at which the particle comes to instantaneous rest. *(C)*

*6. A particle moves in a straight line so that, t seconds after passing through O, its velocity, v m s^{-1}, is given by $v = t^2 - 8t + 7$. The particle comes to instantaneous rest, firstly at A and then at B. Find
   (a) an expression for the displacement, x m, of the particle from O at time t,
   (b) the distance AB,
   (c) the total distance travelled in the first 9 seconds after passing through O.
   Given that C is the point at which the particle has zero acceleration, determine, with full working whether C is nearer to O or to B.

7. A particle X moves along a horizontal straight line so that its displacement, s m, from a fixed point O, t seconds after motion has begun, is given by $s = 28 + 4t - 5t^2 - t^3$. Obtain expressions, in terms of t, for the velocity and acceleration of X, and state the initial velocity and the initial acceleration of X.
   A second particle Y moves along the same horizontal straight line as X and starts from O at the same instant that X begins to move. The initial velocity of Y is 2 m s^{-1} and its acceleration, a m s^{-2}, t seconds after motion has begun, is given by $a = 2 - 6t$. Find the value of t at the instant when X and Y collide and determine whether or not X and Y are travelling in the same direction at this instant. *(C)*

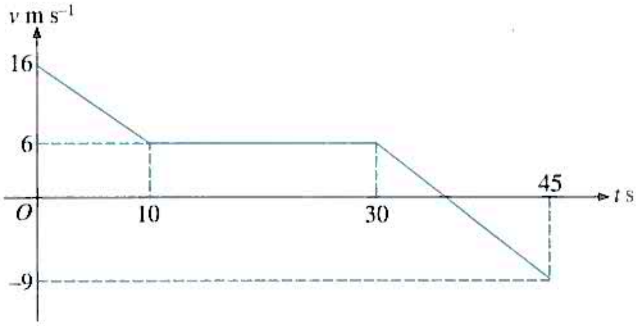
8. A particle moves in a straight line passing a fixed point O with a velocity of 3 m s^{-1}. It moves in such a manner that t seconds after passing O, its velocity is given by $v = at^2 + b$. If the particle is again at O after 3 seconds, find its speed at that instant. Find the total distance travelled between $t = 0$ and $t = 3$.

9. A particle moves in a straight line so that at time t seconds after leaving a fixed point O, its velocity, v m s^{-1}, is given by $v = 12 \sin \frac{1}{2}t$. Find
   (a) the time at which the particle first has a speed of 8 m s^{-1} and its acceleration at this instant,
   (b) an expression for the displacement of the particle from O in terms of t.

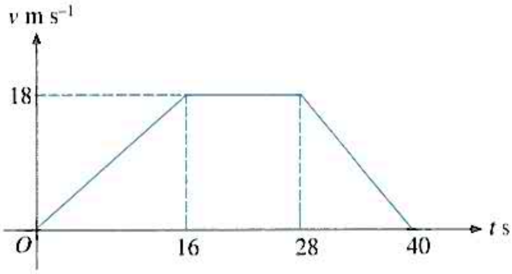
10. A particle P moves in a straight line so that, after t seconds, its displacement, x metres, from a fixed point O is given by the equation $x = 12\sin(2t) - 6$. Determine
   (a) expressions for the velocity and acceleration of P in terms of t,
   (b) the time at which P first returns to O,
   (c) the maximum distance of P from O during the motion,
   (d) the speed of P when $x = 3$.

11. A particle moves in a straight line so that at time t seconds after leaving a fixed point O, its velocity v m s^{-1}, is given by $v = 8e^{-\frac{1}{2}t}$.
(a) Sketch the velocity-time graph for the particle.
(b) Calculate, to 1 decimal place, the acceleration of the particle when $t = 3$.
(c) Find an expression for the displacement of the particle from O and sketch the displacement-time graph.

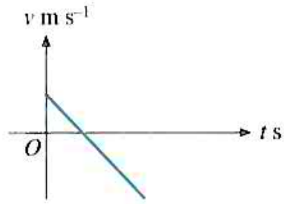
12. The diagram shows the velocity-time graph of a particle P moving in a straight line.
(a) Describe very briefly the motion of the particle.
(b) Find the deceleration of P at $t = 5$.
(c) Find the time when P is instantaneously at rest.
(d) Find the total distance travelled in the 45 s.
(e) Find the distance of P from its starting point after 45 s.



13. The figure shows the velocity-time diagram for the motion of a delivery van travelling on a horizontal road from shop A to shop B. Find
(a) the distance AB,
(b) the average speed of the journey,
(c) the deceleration during the last 12 seconds,
(d) the speed of the car when it has travelled 441 m from A,
(e) the two values of t at which the speed of the van is 4.5 m s^{-1}.



14. Tom was asked to sketch a velocity-time graph to illustrate the motion of a stone which is projected vertically upwards from the ground. His sketch is shown in the given diagram. What is wrong with his v-t graph?



15. Two bus stops are 570 m apart. A bus accelerates uniformly from rest at the first stop to a speed of V m s^{-1}. It maintains this speed for T s and then, with a uniform retardation, comes to rest at the second stop. Sketch the velocity-time diagram for the journey.
(a) Given that $T = 6$ and that the bus accelerates for 15 s and slows down for 30 s, find the value of V.
(b) Given that $V = 10$, and that the magnitude of the acceleration and of the retardation are unchanged, find the value of T.
A car starts from rest at the first bus stop and accelerates uniformly to 30 m s^{-1} in 18 s. On attaining this speed it retards uniformly, coming to rest at the second bus stop. Find the retardation. *(C)*

16. **(a)** A particle P, moving in a straight line, passes a point A with a constant speed of 8 m s^{-1}. Three seconds later a second particle Q, travelling in the same direction as P, passes A with a constant speed of 10 m s^{-1}. The particle Q overtakes P at a point B, where $AB = s$ metres, t seconds after P left A.
Draw, on the same diagram, the distance-time graphs for the motions of P and Q from A to B, and evaluate s and t.

(b) A car passes a service depot on a motorway with a speed of 20 m s^{-1} and a constant retardation of 0.2 m s^{-2}. At the same instant a motor-cyclist leaves the service depot, starting from rest, and with a constant acceleration of 0.6 m s^{-2}. The motor-cyclist passes the car at time T seconds after leaving the service depot at a point whose distance from the service depot is S metres. Find

(i) the value of T and of S.

(ii) the speed of each vehicle at time T seconds.

(iii) Sketch, on the same diagram, the velocity-time graphs of the motor-cyclist and the car over the time interval of T seconds.

(iv) Find the time, after leaving the service depot, when the speeds of the vehicles are the same. *(C)*

[Questions 17 to 26 are optional.]

17. A stone is projected vertically upwards from a height of 55 m above the ground with velocity 60 m s^{-1}. Calculate

(a) its greatest height above the ground and the time taken for it to reach this height,

(b) the total distance the stone covered in the first 8 seconds,

(c) its velocity when it hits the ground.

18. A missile is fired vertically upwards from a point on the ground, level with the foot of a tower 51 m high. The missile is level with the top of the tower 0.6 s after being fired. Calculate

(a) the initial velocity of the missile,

(b) the time taken to reach its greatest height,

(c) this greatest height,

(d) the length of time for which the missile is higher than the top of the tower.

19. Two particles, P and Q, are 40 m apart on a smooth horizontal surface. Particle P is moving directly towards Q with a speed of 3 m s^{-1} and an acceleration of 0.2 m s^{-2}. Particle Q is moving in the same direction as P with a speed of 7 m s^{-1} and a deceleration of 0.4 m s^{-2}. Calculate

(a) the time taken before the particles collide,

(b) the velocity of Q just before the collision and state whether it is still moving in its original direction.

20. Two particles, P and Q are moving in the same direction along parallel horizontal lines. The particles pass a point A simultaneously, P moving with a speed of 3 m s^{-1} and a constant acceleration of 0.5 m s^{-2}, and Q moving with a speed of 1.8 m s^{-1} and a constant acceleration of 0.8 m s^{-2}.
Given that, t seconds after passing A, the speeds of P and Q are equal, calculate

(a) the value of t,

(b) the distance separating the particles at this time.

Given also that, x seconds after passing A, the particles pass a second point B simultaneously, calculate

(c) the value of x,

(d) the distance AB,

(e) the speed of each particle as it passes B.

21. A particle A moves in a straight line with uniform acceleration from a point O towards a point P. After 3 seconds it is 33 m from O and after 6 seconds it is 84 m from O. Calculate

(a) the initial speed and the acceleration,

(b) the speed when A is 48 m from O.

At the instant that A moves from O, a particle B leaves P and travels towards O at a constant speed of 19 m s^{-1}. Given that $OP = 940$ m, calculate

(c) the time it will take for A and B to collide,

(d) the speed of A at the moment of collision.

22. (a) A car starts from rest and accelerates uniformly for t seconds to a speed of V m s^{-1}. It maintains this speed for 8 seconds and then comes to rest with a uniform retardation of 3 m s^{-2}. Sketch a velocity-time diagram to represent this motion. Given that the total time of motion is 13 seconds and that the total distance moved is 63 m, find the value of

(i) V, (ii) t.

(b) Two particles, initially at rest at points A and B, move towards each other with constant acceleration a m s^{-2} and b m s^{-2} respectively. Given that $AB = 375$ m, and that after 12 seconds the particles are 240 m apart, find

(i) the value of $a + b$,

(ii) the time taken from the start of motion to the point C where the particles meet.

Given that $AC = 150$ m, find the speed of each particle when they meet. *(C)*

23. (a) A car accelerates uniformly from rest to a speed of 20 m s^{-1} in T s. It maintains this speed for 40 s and then decelerates uniformly to rest in a further $2T$ s. Sketch a velocity-time diagram for this motion. Given that the total distance travelled is 1160 m, find

(i) the value of T,

(ii) the magnitude of the deceleration of the car in the final stage of the motion.

(b) A particle starts from rest at a point X and moves in a straight line with an acceleration of 0.75 m s^{-2}. Five seconds later a second particle is also projected from X in the same direction at a constant speed of 8 m s^{-1}. Find

(i) the distance between the two particles 6 seconds after the first started moving,

(ii) the distance from X of the point at which the second particle collides with the first. *(C)*

24. From the foot of a vertical cliff 45 m high a stone was projected vertically upwards so as just to reach the top. Find its velocity of projection.

One second after the first stone was projected, another stone was allowed to fall from rest from the top of the cliff. The stones passed one another after a further t seconds at a height h m above the ground. Calculate the value of t and of h.

25. **(a)** A particle moves with constant acceleration a m s^{-2}. Its velocity changes from u m s^{-1} to v m s^{-1} in t seconds. Sketch a velocity-time graph to illustrate its motion. With the help of this graph, derive the equations of motion

(i) $v = u + at$, **(ii)** $s = \frac{1}{2}(u + v)t$.

From the above two equations, show that $s = ut + \frac{1}{2}at^2$. Hence, **deduce** that $s = vt - \frac{1}{2}at^2$.

(b) A car travels in a straight lane with a constant acceleration of 2 m s^{-2}. It passes points X and Y with velocities 20 m s^{-1} and 36 m s^{-1} respectively. Calculate the distance XY and the distance it covers during the 3 seconds before reaching Y.

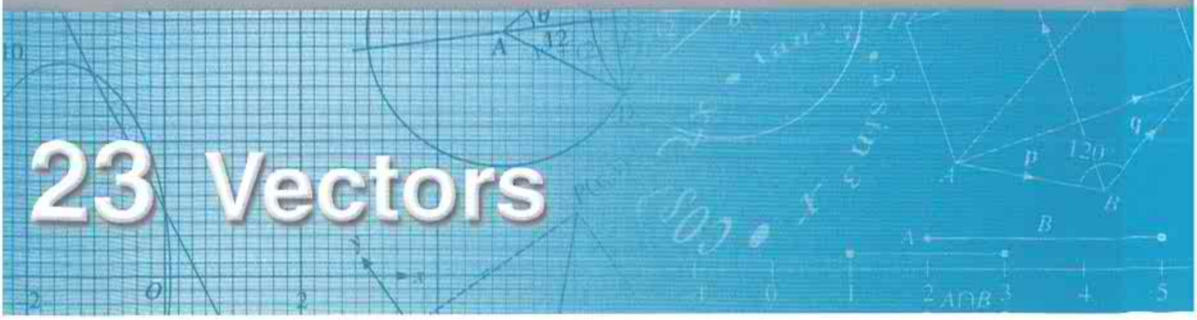
26. Two cars P and Q, travelling in the horizontal straight road, pass a fixed point A at the same instant. Each car subsequently passes the fixed points B, C and D. Car P travels at a constant speed of 20 m s^{-1}. Car Q accelerates uniformly from a speed of u m s^{-1} at A until it reaches a speed of 22 m s^{-1} at the point B. Given that car Q overtakes car P at B, find the value of u.

Car Q then travels from B to the point C at a constant speed of 22 m s^{-1}, taking 15 s. From C car Q decelerates uniformly until it reaches the point D, with a speed of 13 m s^{-1}. Given that AB = 250 m and BD = 750 m,

(a) show that car P reaches D before car Q,

(b) draw on graph paper, the velocity-time graph for car Q's journey from A to D,

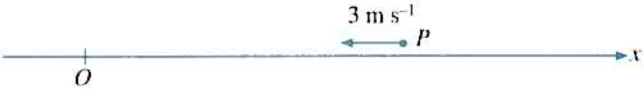
(c) calculate the distance of car Q from D at the instant when car P reaches D, giving your answer to the nearest metre. *(C)*



23 Vectors

23.1 Basic Concepts

Scalar and Vector Quantities



In the earlier chapters, we calculate distances between points and areas of regions enclosed by curves. These quantities involving only magnitude are called **scalar** quantities. However, when studying the motion of a particle moving along the x-axis, we use positive and negative signs to indicate direction of displacement, velocity and acceleration. For example, when a particle P is moving with a velocity $v = -3$ m s^{-1} at a certain instant, P is actually moving with a speed of 3 m s^{-1} in the direction opposite to the positive direction of the x-axis as shown.

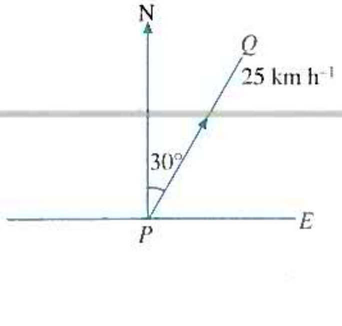
Displacement, velocity and acceleration, which involve not only magnitude but also direction, are known as **vector** quantities.

In this chapter, we shall explore the basic concepts on vectors and study the general properties of vectors and its applications in simple geometry. These same properties for all vector quantities will be applied to solve relative velocity problems in Chapter 24.

Representation of Vectors by Directed Line Segments

A simple way to study vectors is to represent them by directed line segments and define operations on them by using the intuitive properties of directed line segments.

For example, a displacement of 10 m on the bearing 032° is represented by the directed line segment $\overrightarrow{AB}$ and a velocity of 25 km h^{-1} in the direction N 30°E is represented by the directed line segment $\overrightarrow{PQ}$ as shown below:

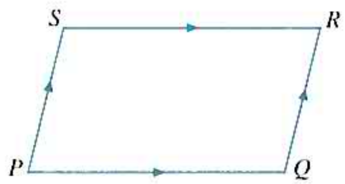


For the displacement, its magnitude is represented by the length AB and its direction is indicated by the arrow from A to B. Similarly, the magnitude of the velocity is represented by the length PQ and its direction is from P to Q.

The above displacement represented by $\overrightarrow{AB}$ can be written as $\mathbf{AB}$ or denoted by a single letter $\underset{\sim}{d}$ or $\mathbf{d}$. The magnitude of the directed line segment $\overrightarrow{AB}$ is denoted by $|\overrightarrow{AB}|$, $|\mathbf{AB}|$, $|\underset{\sim}{d}|$, $|\mathbf{d}|$ or simply d.

Equal Vectors

The diagram shows a parallellogram $PQRS$, the directed line segments $\overrightarrow{PQ}$ and $\overrightarrow{SR}$ have the same direction and are of equal length. The vectors $\overrightarrow{PQ}$ and $\overrightarrow{SR}$ are said to be equal and we write $\overrightarrow{PQ} = \overrightarrow{SR}$. Similarly, $\overrightarrow{PS} = \overrightarrow{QR}$.



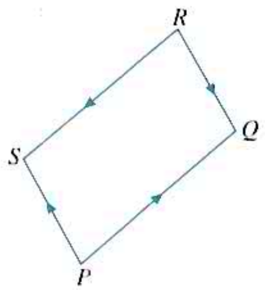
$\mathbf{a} = \mathbf{b} \Leftrightarrow$ $\mathbf{a}$ and $\mathbf{b}$ have the same direction and $|\mathbf{a}| = |\mathbf{b}|$.

Negative Vectors

Refer to the parallelogram $PQRS$ shown below. The directed line segments $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are of equal length but in opposite directions. The vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are negative vectors of each other and we write:

$$\overrightarrow{PQ} = -\overrightarrow{RS} \text{ or } \overrightarrow{RS} = -\overrightarrow{PQ}$$

Similarly, $$\overrightarrow{PS} = -\overrightarrow{RQ} \text{ or } \overrightarrow{RQ} = -\overrightarrow{PS}$$



$\mathbf{a} = -\mathbf{b}$ $\Leftrightarrow$ $\mathbf{a}$ and $\mathbf{b}$ are in opposite directions and $|\mathbf{a}| = |\mathbf{b}|$.

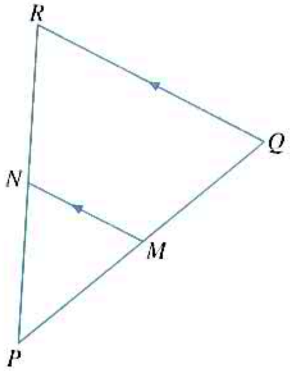
Zero Vector

The directed line segment $\overrightarrow{AA}$ has zero magnitude and it represents a zero vector denoted by $\underset{\sim}{0}$ or $\mathbf{0}$.

$\mathbf{a}$ is a zero vector $\Leftrightarrow$ $|\mathbf{a}| = 0$.

Scalar Multiplication of a Vector

In the diagram, the points M and N are midpoints of the sides PQ and PR of the triangle PQR respectively.



Since $\overrightarrow{QR}$ and $\overrightarrow{MN}$ have the same direction,

$$QR = 2MN \quad \Rightarrow \quad \overrightarrow{QR} = 2\overrightarrow{MN},$$

and

$$MN = \frac{1}{2}QR \quad \Rightarrow \quad \overrightarrow{MN} = \frac{1}{2}\overrightarrow{QR}.$$

Also,

$$\overrightarrow{QR} = 2\overrightarrow{MN} \quad \Rightarrow \quad \overrightarrow{RQ} = -2\overrightarrow{MN},$$

$$\overrightarrow{MN} = \frac{1}{2}\overrightarrow{QR} \quad \Rightarrow \quad \overrightarrow{MN} = -\frac{1}{2}\overrightarrow{RQ}.$$

Note that the negative sign indicates that the related vectors are opposite in direction.

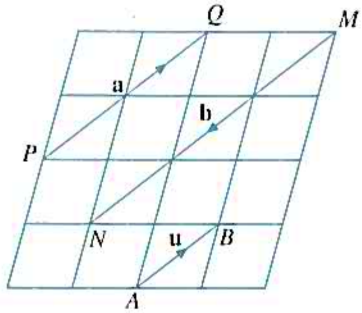
> $\mathbf{b} = k\mathbf{a}$ and $k > 0 \iff$ $\mathbf{a}$ and $\mathbf{b}$ are in the same direction and $|\mathbf{b}| = |k||\mathbf{a}|$,
> $\mathbf{b} = k\mathbf{a}$ and $k < 0 \iff$ $\mathbf{a}$ and $\mathbf{b}$ are opposite in direction and $|\mathbf{b}| = |k||\mathbf{a}|$.

Unit Vectors

If $|\mathbf{u}| = 1$, $\mathbf{u}$ is known as a unit vector. The unit vector in the direction of a given vector $\mathbf{a}$ is $\frac{\mathbf{a}}{|\mathbf{a}|}$ and we write: $\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$

Example 1 The diagram shows a grid with parallel lines, $\overrightarrow{AB} = \mathbf{u}$, $\overrightarrow{PQ} = \mathbf{a}$ and $\overrightarrow{MN} = \mathbf{b}$.

(a) Express $\mathbf{a}$ and $\mathbf{b}$ in terms of $\mathbf{u}$.
(b) Express $\mathbf{b}$ in terms of $\mathbf{a}$.
(c) If $\mathbf{u}$ is a unit vector, find the magnitude of $\mathbf{a}$ and of $\mathbf{b}$.



Solution:

(a) $\overrightarrow{PQ}$ and $\overrightarrow{AB}$ are in the same direction and $PQ = 2AB$.

$$\overrightarrow{PQ} = 2\overrightarrow{AB}$$
$$\mathbf{a} = 2\mathbf{u}$$

$\overrightarrow{MN}$ and $\overrightarrow{AB}$ are in opposite directions and $MN = 3AB$.

$$\overrightarrow{MN} = -3\overrightarrow{AB}$$
$$\mathbf{b} = -3\mathbf{u}$$

(b) $\mathbf{b} = -3\mathbf{u} = -3\left(\frac{1}{2}\mathbf{a}\right) = -\frac{3}{2}\,\mathbf{a}.$

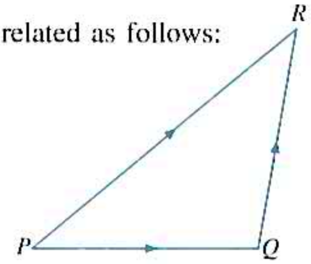
(c) $|\mathbf{a}| = 2|\mathbf{u}| =$ **2 units** $(\because |\mathbf{u}| = 1)$

$|\mathbf{b}| = |-3||\mathbf{u}| =$ **3 units**

Addition of Vectors

1. **Triangle Law of addition**

In the triangle PQR, the vectors $\overrightarrow{PQ}$, $\overrightarrow{QR}$ and $\overrightarrow{PR}$ are related as follows:



Let $\overrightarrow{PQ}$ be the displacement from P to Q,
$\overrightarrow{QR}$ be the displacement from Q to R, and
$\overrightarrow{PR}$ be the displacement from P to R.

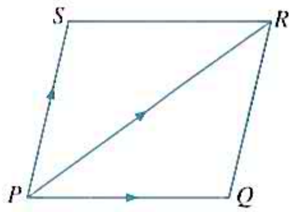
In physical situations, the displacement $\overrightarrow{PQ}$ followed by the displacement $\overrightarrow{QR}$ is the displacement $\overrightarrow{PR}$ and we write:

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$

This process of adding the two vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ using the triangle PQR is known as the **triangle law of addition**.

2. **Parallelogram Law of addition**

Refer to the parallelogram $PQRS$ shown on the right.



$$\overrightarrow{PS} = \overrightarrow{QR}$$

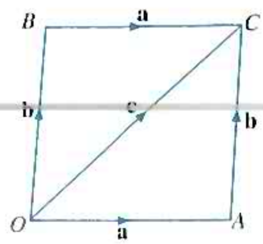
$$\overrightarrow{PQ} + \overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$

Hence: $\overrightarrow{PQ} + \overrightarrow{PS} = \overrightarrow{PR}$

This process of adding $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ using the parallelogram $PQRS$ is known as the **parallelogram law of addition**.

3. **Commutative Law**

In the diagram, $OACB$ is a parallelogram. We have $\overrightarrow{OA} = \overrightarrow{BC} = \mathbf{a}$, $\overrightarrow{OB} = \overrightarrow{AC} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.



By the triangle law, $\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$

and $\overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OC}$

i.e. $\mathbf{a} + \mathbf{b} = \mathbf{c}$

and $\mathbf{b} + \mathbf{a} = \mathbf{c}$

Hence $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

That is, vector addition **is commutative**

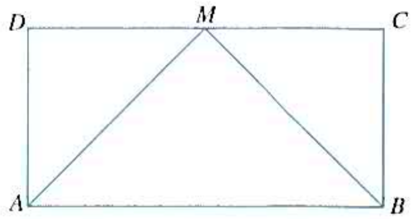
By the triangle law, $\overrightarrow{OA} + \overrightarrow{AA} = \overrightarrow{OA}$.

i.e. $\mathbf{a} + \mathbf{0} = \mathbf{a}$

$\mathbf{0} + \mathbf{a} = \mathbf{a}$

Example 2

The diagram shows a rectangle $ABCD$ in which $AB = 2AD$. Given that M is the midpoint of CD, express each of the following sums as a single vector.



(a) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{MD}$ **(b)** $\overrightarrow{AM} + \overrightarrow{MD} + \overrightarrow{DB}$

Solution:

(a) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{MD} = (\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{MD}$

$= \overrightarrow{AC} + \overrightarrow{MD}$ *(triangle law)*

$= \overrightarrow{AC} + \overrightarrow{CM}$ $(\because \overrightarrow{MD} = \overrightarrow{CM})$

$= \overrightarrow{AM}$

(b) $\overrightarrow{AM} + \overrightarrow{MD} + \overrightarrow{DB} = \overrightarrow{AD} + \overrightarrow{DB}$ *(triangle law)*

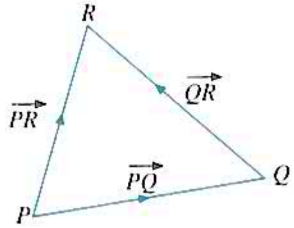
$= \overrightarrow{AB}$

Subtraction of Vectors

Refer to the triangle PQR in the diagram.

$$\begin{aligned}\overrightarrow{QR} &= \overrightarrow{QP} + \overrightarrow{PR} \\ &= \overrightarrow{PR} + \overrightarrow{QP} \\ &= \overrightarrow{PR} + \left(-\overrightarrow{PQ}\right) = \overrightarrow{PR} - \overrightarrow{PQ}\end{aligned}$$

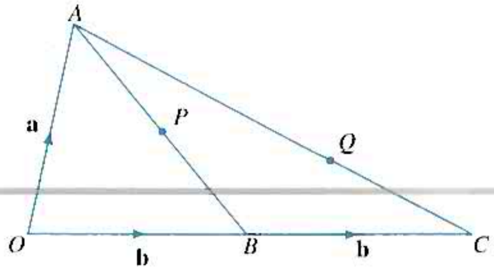
Hence $\overrightarrow{QR} = \overrightarrow{PR} - \overrightarrow{PQ}$.



Example 3

In the diagram $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ represent vectors $\mathbf{a}$, $\mathbf{b}$ and $2\mathbf{b}$ respectively. P and Q are points such that $AP = \frac{1}{2}AB$ and $AQ = \frac{2}{3}AC$.

Express $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.



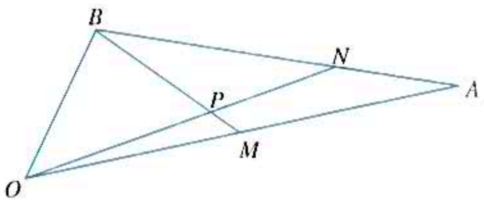
Solution:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} \quad \textit{(subtraction of vectors)}$$
$$= \mathbf{b} - \mathbf{a}$$
$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} \quad \textit{(triangle law)}$$
$$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$$
$$= \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a})$$
$$= \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} \quad \textit{(subtraction of vectors)}$$
$$= 2\mathbf{b} - \mathbf{a}$$
$$\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{AQ} \quad \textit{(triangle law)}$$
$$= \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AC}$$
$$= \mathbf{a} + \frac{2}{3}(2\mathbf{b} - \mathbf{a})$$
$$= \frac{1}{3}\mathbf{a} + \frac{4}{3}\mathbf{b}$$

Example 4

In the diagram, OAB is a triangle with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. M is the midpoint of OA, N is a point on AB such that $AN = \frac{1}{3}AB$ and P is the point of intersection of ON and BM.



Given that $\overrightarrow{OP} = \lambda\overrightarrow{ON}$ and $\overrightarrow{BP} = \mu\overrightarrow{BM}$, express $\overrightarrow{OP}$
(a) in terms of λ, $\mathbf{a}$ and $\mathbf{b}$,
(b) in terms of μ, $\mathbf{a}$ and $\mathbf{b}$.

Solution:

(a) In $\triangle OAN$, $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN}$
$$= \overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB}$$
$$= \mathbf{a} + \frac{1}{3}(\mathbf{b} - \mathbf{a})$$
$$= \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$$
$$\overrightarrow{OP} = \lambda\overrightarrow{ON}$$
$$= \frac{2\lambda}{3}\mathbf{a} + \frac{\lambda}{3}\mathbf{b}$$

(b) In $\triangle OBP$, $\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP}$

$$= \overrightarrow{OB} + \mu\,\overrightarrow{BM}$$

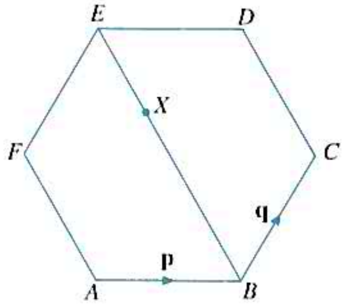
$$= \mathbf{b} + \mu\left(\frac{1}{2}\mathbf{a} - \mathbf{b}\right)$$

$$= \frac{\mu}{2}\mathbf{a} + (1-\mu)\mathbf{b}$$

In Example 4, we found different expressions for the vector $\overrightarrow{OP}$ by considering different triangles.

Example 5

In the diagram, $ABCDEF$ is a regular hexagon and X is a point on the line BE. It is given that $\overrightarrow{AB} = \mathbf{p}$, $\overrightarrow{BC} = \mathbf{q}$ and $\overrightarrow{BX} = k\overrightarrow{BE}$ where k is a scalar.



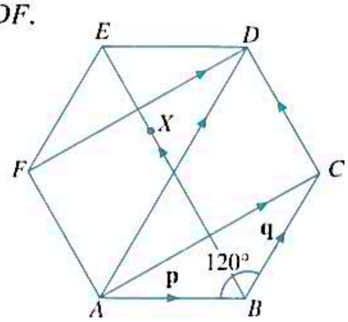
(a) Express $\overrightarrow{FD}$, $\overrightarrow{CD}$ and $\overrightarrow{BE}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

(b) Given that $\mathbf{p}$ is a unit vector, evaluate $|\overrightarrow{AC}|$.

(c) Express $\overrightarrow{AX}$ in terms of $\mathbf{p}$, $\mathbf{q}$ and k.

Solution:

(a) Consider the parallelogram $ACDF$.



$$\overrightarrow{FD} = \overrightarrow{AC}$$

$$= \overrightarrow{AB} + \overrightarrow{BC}$$

$$= \mathbf{p} + \mathbf{q}$$

Consider the $\triangle ACD$.

$$\overrightarrow{CD} = \overrightarrow{AD} - \overrightarrow{AC}$$

$$= 2\mathbf{q} - (\mathbf{p} + \mathbf{q})$$

$$= \mathbf{q} - \mathbf{p}$$

Consider the trapezium $BCDE$.

$$\overrightarrow{BE} = 2\overrightarrow{CD}$$

$$= 2(\mathbf{q} - \mathbf{p})$$

(b) $|\mathbf{p}| = 1 \quad \Rightarrow \quad |\mathbf{q}| = 1$

i.e. $AB = 1$ and $BC = 1$

By the cosine rule,

$$AC^2 = AB^2 + BC^2 - 2AB.\,BC\cos 120°$$

$$= 1 + 1 - 2 \times \left(-\frac{1}{2}\right)$$

$$= 3$$

$$AC = \sqrt{3}$$

$$|\overrightarrow{AC}| = \sqrt{3}\text{ units}$$

(c) $\overrightarrow{AX} = \overrightarrow{AB} + \overrightarrow{BX}$

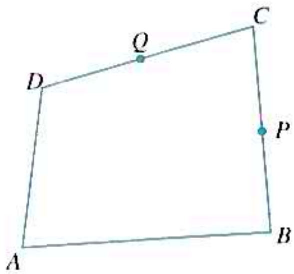
$= \overrightarrow{AB} + k\overrightarrow{BE}$

$= \mathbf{p} + 2k(\mathbf{q} - \mathbf{p})$

$= (1 - 2k)\mathbf{p} + 2k\mathbf{q}$

Exercise 23.1

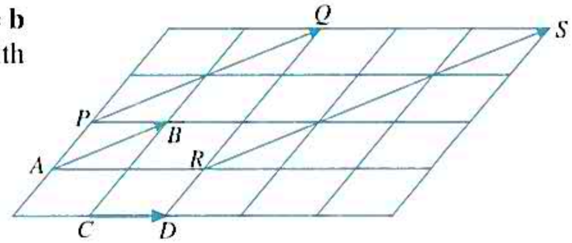
1. In the figure, $ABCD$ is a quadrilateral. P and Q are the midpoints of BC and CD respectively. Express each of the following sums as a single vector.



   (a) $\overrightarrow{AB} + \overrightarrow{BQ} + \overrightarrow{QP}$ (b) $\overrightarrow{AC} + \overrightarrow{CD} + \overrightarrow{DB}$

   (c) $\overrightarrow{AQ} + \overrightarrow{QB} + \overrightarrow{BD}$

2. In the diagram, $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{CD} = \mathbf{b}$ are two vectors shown in a grid with parallel lines. Express

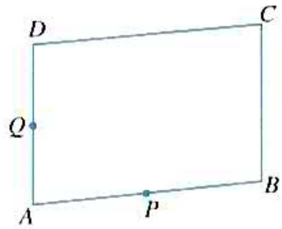


   (a) $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ in terms of $\mathbf{a}$,

   (b) $\overrightarrow{QS}$ in terms of $\mathbf{b}$,

   (c) $\overrightarrow{PS}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

3. $ABCD$ is a parallelogram. P and Q are the midpoints of AB and AD respectively. Show that



   (a) $\overrightarrow{AP} + \overrightarrow{AQ} = \frac{1}{2}\overrightarrow{AC}$, (b) $\overrightarrow{PC} + \overrightarrow{QC} = \frac{3}{2}\overrightarrow{AC}$,

   (c) $\overrightarrow{PD} + \overrightarrow{QB} = \frac{1}{2}\overrightarrow{AC}$.

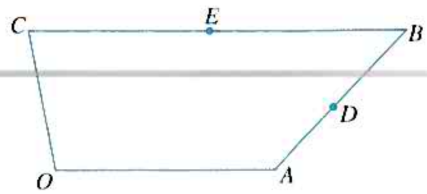
4. OAB is a triangle. Given that M is the midpoint of AB and N is a point on OB such that $\overrightarrow{ON} = m\overrightarrow{OB}$, express

   (a) $\overrightarrow{AM}$ and $\overrightarrow{OM}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$,

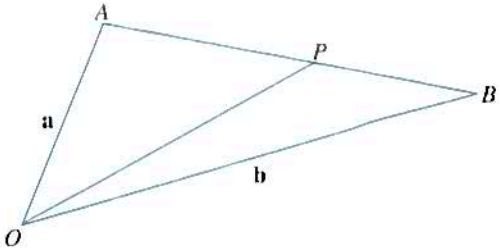
   (b) $\overrightarrow{MN}$ in terms of $\overrightarrow{OA}$, $\overrightarrow{OB}$ and m.

5. OAB is a triangle with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. M and N are points between A and B such that $AM = \frac{1}{3}AB$ and $AN = \frac{3}{5}AB$. Express $\overrightarrow{OM}$ and $\overrightarrow{ON}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

6. $OABC$ is a trapezium with $\overrightarrow{CB} = k\overrightarrow{OA}$. The points D and E are midpoints of AB and BC respectively. Given that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, express $\overrightarrow{OE}$ and $\overrightarrow{DE}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and k.



7. In the diagram, A and B are two points on a line and OAB is a triangle with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. P is a point on the line such that $AP = kAB$ where k is a scalar. Show that $\overrightarrow{OP} = (1 - k)\mathbf{a} + k\mathbf{b}$. Describe briefly the position of P for each of the following cases.



(a) $k = \frac{1}{2}$ (b) $k = 1$ (c) $k = 2$

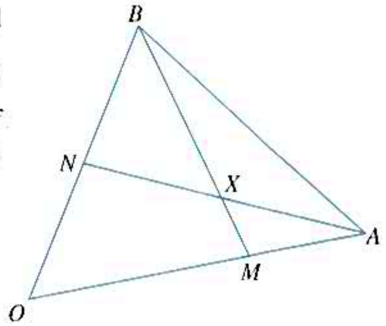
8. $ABCD$ is a rectangle. Given that $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{BC} = \mathbf{q}$, express in terms of $\mathbf{p}$ and $\mathbf{q}$,

(a) $\overrightarrow{AC}$, (b) $\overrightarrow{DB}$, (c) $\overrightarrow{BX}$, where X is the midpoint of CD.

Given that $|\mathbf{p}| = 2|\mathbf{q}|$ and $\mathbf{q}$ is a unit vector, evaluate $|\overrightarrow{BX}|$.

9. In the diagram, OAB is a triangle with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. M is a point on OA such that $OM = \frac{2}{3}OA$ and N is the midpoint of OB. X is the point of intersection of AN and BM. Given that $\overrightarrow{AX} = \lambda\overrightarrow{AN}$ and $\overrightarrow{BX} = \mu\overrightarrow{BM}$, express $\overrightarrow{OX}$

(a) in terms of λ, $\mathbf{a}$ and $\mathbf{b}$,
(b) in terms of μ, $\mathbf{a}$ and $\mathbf{b}$.



10. $OABCDE$ is a regular hexagon with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) Given that P is on BC such that $\overrightarrow{BP} = \lambda\overrightarrow{BC}$, express $\overrightarrow{OP}$ in terms of λ, $\mathbf{a}$ and $\mathbf{b}$.

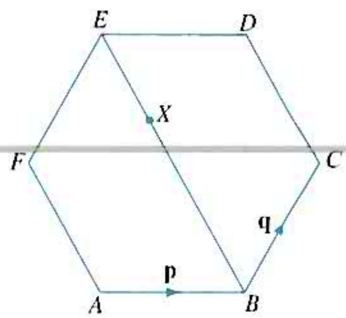
(b) Given that Q is on EC such that $\overrightarrow{FQ} = \mu\overrightarrow{FD}$, express $\overrightarrow{OQ}$ in terms of μ, $\mathbf{a}$ and $\mathbf{b}$.

*11. $ABCDEFGH$ is a regular octagon and $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{BC} = \mathbf{q}$. Express $\overrightarrow{AH}$ in terms of $\mathbf{p}$ and $\mathbf{q}$ and show that $\overrightarrow{AE} + \overrightarrow{BH} + \overrightarrow{CG} + \overrightarrow{DF} = 2(2 + \sqrt{2})(\mathbf{q} - \sqrt{2}\,\mathbf{p})$.

23.2 Vectors Expressed in Terms of Two Non-Parallel Vectors

In Example 5, $\mathbf{p}$ and $\mathbf{q}$ are non-parallel vectors and

$\overrightarrow{FD} = \mathbf{p} + \mathbf{q}$,
$\overrightarrow{CD} = \mathbf{q} - \mathbf{p} = -\mathbf{p} + \mathbf{q}$,
$\overrightarrow{BE} = 2(\mathbf{q} - \mathbf{p}) = -2\mathbf{p} + 2\mathbf{q}$,
$\overrightarrow{AX} = (1 - 2k)\mathbf{p} + 2k\mathbf{q}$.

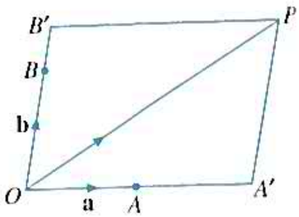


Each of the above vectors is expressed in terms of the non-parallel vectors **p** and **q** and is of the form $m\mathbf{p} + n\mathbf{q}$, where m and n are scalars.

If **a** and **b** are two non-zero and non-parallel vectors, any vector $\overrightarrow{OP}$ in the plane containing **a** and **b** can be expressed in terms of **a** and **b**. That is,

$$\overrightarrow{OP} = m\mathbf{a} + n\mathbf{b}, \text{ where } m \text{ and } n \text{ are constants.}$$

The above result may be explained as follows:



For any vector $\overrightarrow{OP}$, there is a parallelogram $OA'PB'$ such that

$\overrightarrow{OA'} = m\mathbf{a}$, $\overrightarrow{OB'} = n\mathbf{b}$

and $\overrightarrow{OP} = \overrightarrow{OA'} + \overrightarrow{OB'}$ (*parallelogram rule*)

$= m\mathbf{a} + n\mathbf{b}$

An important nature of vectors is that two non-parallel vectors cannot be equal. This enables us to have the following results:

If non-zero vectors **a** and **b** are non-parallel, then:

(a) $\lambda\mathbf{a} = \mu\mathbf{b} \Rightarrow \lambda\mathbf{a} = \mathbf{0}$ and $\mu\mathbf{b} = \mathbf{0}$

$\Rightarrow \lambda = 0$ and $\mu = 0$

(b) $p\mathbf{a} + q\mathbf{b} = r\mathbf{a} + s\mathbf{b} \Rightarrow (p - r)\mathbf{a} = (s - q)\mathbf{b}$

$\Rightarrow (p - r) = 0$ and $(s - q) = 0$

which gives: $p\mathbf{a} + q\mathbf{b} = r\mathbf{a} + s\mathbf{b} \Rightarrow p = r$ and $q = s$

Example 6

Given that $\mathbf{p} = \mathbf{a} + t(\mathbf{a} + 2\mathbf{b})$ and $\mathbf{q} = 2\mathbf{a} + \mathbf{b} + s\mathbf{a}$ where **a** and **b** are non-parallel and non-zero, find

(a) the values of t and s if $\mathbf{p} = \mathbf{q}$,

(b) the relationship between t and s if **p** and **q** are parallel.

Solution:

(a) $\mathbf{p} = \mathbf{q} \Rightarrow (1 + t)\mathbf{a} + 2t\mathbf{b} = (2 + s)\mathbf{a} + \mathbf{b}$

$\Rightarrow \quad 1 + t = 2 + s$

and $\quad 2t = 1$

$t = \frac{1}{2}$ and $s = -\frac{1}{2}$

(b) **p** and **q** are parallel $\Rightarrow \mathbf{p} = k\mathbf{q}$ where k is a scalar

$\Rightarrow (1 + t)\mathbf{a} + 2t\mathbf{b} = k[(2 + s)\mathbf{a} + \mathbf{b}]$

$\Rightarrow 1 + t = k(2 + s)$ and $2t = k$

$\Rightarrow 1 + t = 2t(2 + s)$

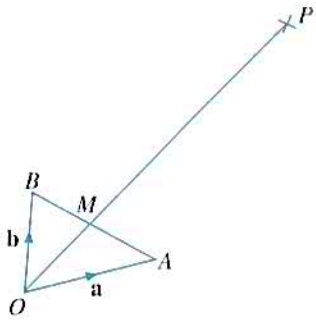
$\Rightarrow 1 = t(3 + 2s)$

Example 7 In the diagram, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OP} = 2\mathbf{a} + 3\mathbf{b}$ and M is the point of intersection of the lines OP and AB. Given that $\overrightarrow{OM} = \lambda\overrightarrow{OP}$ and $\overrightarrow{AM} = \mu\overrightarrow{AB}$, express $\overrightarrow{OM}$

(a) in terms of λ, $\mathbf{a}$ and $\mathbf{b}$,

(b) in terms of μ, $\mathbf{a}$ and $\mathbf{b}$.

Hence find the values of λ and μ and express $\overrightarrow{OM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.



Solution:

(a) $\overrightarrow{OM} = \lambda\overrightarrow{OP}$

$= \lambda(2\mathbf{a} + 3\mathbf{b})$ .... (1)

(b) $\overrightarrow{AM} = \mu\overrightarrow{AB}$

$= \mu(\overrightarrow{OB} - \overrightarrow{OA})$

$= \mu(\mathbf{b} - \mathbf{a})$

By the triangle law,

$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$

$= \mathbf{a} + \mu(\mathbf{b} - \mathbf{a})$

$= (1 - \mu)\mathbf{a} + \mu\mathbf{b}$ .... (2)

From (1) and (2):

$2\lambda\mathbf{a} + 3\lambda\mathbf{b} = (1 - \mu)\mathbf{a} + \mu\mathbf{b}$

$\Rightarrow \quad 2\lambda = 1 - \mu$ and $3\lambda = \mu$ ($\because$ **a** *and* **b** *are non-parallel vectors*)

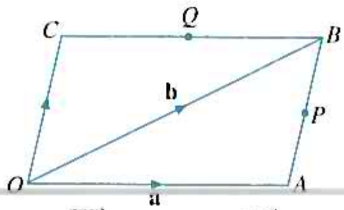
$\Rightarrow \quad \lambda = \frac{1}{5}, \mu = \frac{3}{5}$

$\overrightarrow{OM} = \frac{1}{5}(2\mathbf{a} + 3\mathbf{b})$

$= \frac{2}{5}\mathbf{a} + \frac{3}{5}\mathbf{b}$

Exercise 23.2

1. $OABC$ is a parallelogram with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. P and Q are the midpoints of AB and BC respectively. Find $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and $\overrightarrow{AQ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.



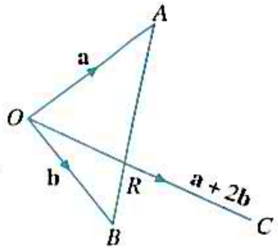
2. $ABCDEF$ is a regular hexagon with centre O. Suppose $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{AF} = \mathbf{q}$, express $\overrightarrow{AO}$, $\overrightarrow{AC}$, $\overrightarrow{AE}$ and $\overrightarrow{CE}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

3. Given that the non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are non-parallel and that
$$3\mathbf{a} + t(2\mathbf{a} - 3\mathbf{b}) = \mathbf{a} + \mathbf{b} + s(\mathbf{a} + 2\mathbf{b}),$$
find the value of t and of s.

4. Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are non-parallel. If $2\mathbf{a} + t(\mathbf{a} - \mathbf{b})$ and $2\mathbf{b} + \mathbf{a} + t\mathbf{b}$ are parallel, find the value of t.

5. $OABC$ is a square with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. M is the midpoint of BC and AM produced intersects OC produced at P. Given that $\overrightarrow{AP} = k\overrightarrow{AM}$ and $\overrightarrow{OP} = n\overrightarrow{OC}$, express $\overrightarrow{OP}$
   (a) in terms of k, $\mathbf{a}$ and $\mathbf{c}$,
   (b) in terms of n, $\mathbf{a}$ and $\mathbf{c}$.
   Hence find the value of n and of k.

6. In the figure, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{a} + 2\mathbf{b}$. AB meets OC at R so that $\overrightarrow{AR} = k\overrightarrow{AB}$ and $\overrightarrow{OR} = n\overrightarrow{OC}$. Express $\overrightarrow{OR}$
   (a) in terms of k, $\mathbf{a}$ and $\mathbf{b}$, **(b)** in terms of n, $\mathbf{a}$ and $\mathbf{b}$.
   Hence find the value of n and of k.



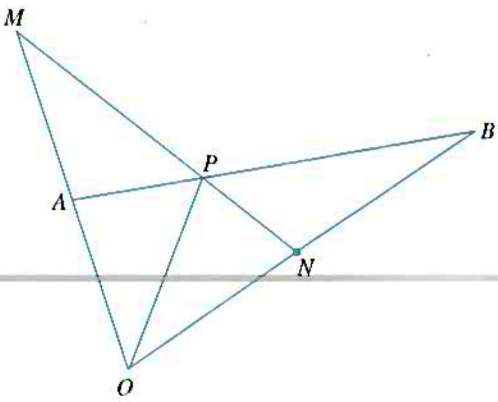
7. Given that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OP} = \frac{2}{3}\overrightarrow{OA}$ and $\overrightarrow{OQ} = 2\mathbf{b}$, express $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
   PQ meets AB at R so that $\overrightarrow{PR} = n\overrightarrow{PQ}$ and $\overrightarrow{AR} = k\overrightarrow{AB}$. Express $\overrightarrow{OR}$
   (a) in terms of n, $\mathbf{a}$ and $\mathbf{b}$, **(b)** in terms of k, $\mathbf{a}$ and $\mathbf{b}$.
   Hence find the value of n and of k.

23.3 Position Vectors

In the diagram, the position of a point P with respect to an origin O is indicated by the directed line segment $\overrightarrow{OP}$. The vector $\overrightarrow{OP}$ is called the **position vector** of P relative to O.

Example 8

Relative to an origin O, the position vectors of the points A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively. P is the point on AB such that $AP = mAB$ where $0 < m < 1$. Find, in terms of $\mathbf{a}$, $\mathbf{b}$ and m, the position vector of P.



Given that the position vectors of the points M and N are $2\mathbf{a}$ and $\frac{1}{2}\mathbf{b}$ respectively, and P lies on the line MN, express $\overrightarrow{MP}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and m. Hence find the value of m.

Solution:

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA} + \overrightarrow{AP}\\ &= \overrightarrow{OA} + m\overrightarrow{AB}\\ &= \mathbf{a} + m(\mathbf{b} - \mathbf{a})\\ &= (1 - m)\mathbf{a} + m\mathbf{b}\end{aligned}$$

$$\begin{aligned}\overrightarrow{MN} &= \overrightarrow{ON} - \overrightarrow{OM}\\ &= \frac{1}{2}\mathbf{b} - 2\mathbf{a}\end{aligned}$$

$$\begin{aligned}\overrightarrow{MP} &= \overrightarrow{OP} - \overrightarrow{OM}\\ &= (1 - m)\mathbf{a} + m\mathbf{b} - 2\mathbf{a}\\ &= (-1 - m)\mathbf{a} + m\mathbf{b}\end{aligned}$$

Since P lies on MN, $\overrightarrow{MP} = k\overrightarrow{MN}$

$(-1 - m)\mathbf{a} + m\mathbf{b} = k(-2\mathbf{a} + \frac{1}{2}\mathbf{b})$

$$-1 - m = -2k \quad\text{.................................. (1)}$$

$$m = \frac{1}{2}k \quad\text{.................................. (2)}$$

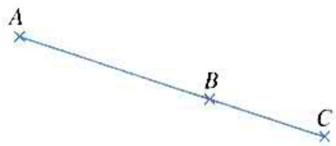
Solving (1) and (2):

$$m = \frac{1}{3}$$

In the above example, we apply the basic concept on parallel vectors for a point lying on a line segment. In the following section we shall further apply the method for collinear points in geometry.

Collinear Points

The diagram shows three distinct points A, B and C. If A, B and C lie on a straight line, the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel. The converse is also true. Hence we have the following result:



Three distinct points A, B and C are collinear

$\Leftrightarrow$ $\overrightarrow{AB} = k\overrightarrow{BC}$, where k is a scalar.

Example 9 Relative to an origin O, the position vectors of the points A, B and C are $2\mathbf{p} - 2\mathbf{q}$, $3\mathbf{p} + \lambda\mathbf{q}$ and $(2 + \lambda)\mathbf{p} + 6\mathbf{q}$ where $\mathbf{p}$ and $\mathbf{q}$ are non-parallel vectors.

(a) Express $\overrightarrow{AB}$ and $\overrightarrow{BC}$ in terms of λ, $\mathbf{p}$ and $\mathbf{q}$.

(b) Given that A, B and C are collinear, find the possible values of λ.

Solution:

(a)
$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA}\\ &= (3\mathbf{p} + \lambda\mathbf{q}) - (2\mathbf{p} - 2\mathbf{q})\\ &= \mathbf{p} + (\lambda + 2)\mathbf{q}\end{aligned}$$

$$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{OC} - \overrightarrow{OB}\\ &= (2 + \lambda)\mathbf{p} + 6\mathbf{q} - (3\mathbf{p} + \lambda\mathbf{q})\\ &= (\lambda - 1)\mathbf{p} + (6 - \lambda)\mathbf{q}\end{aligned}$$

(b) Since A, B and C are collinear,

$$\begin{aligned}\overrightarrow{BC} &= k\overrightarrow{AB}\\ (\lambda - 1)\mathbf{p} + (6 - \lambda)\mathbf{q} &= k(\mathbf{p} + (\lambda + 2)\mathbf{q})\\ \Rightarrow \quad \lambda - 1 &= k \text{ and } 6 - \lambda = k(\lambda + 2)\\ \Rightarrow \quad 6 - \lambda &= (\lambda - 1)(\lambda + 2)\\ \Rightarrow \quad \lambda^2 + 2\lambda - 8 &= 0\\ \Rightarrow \quad (\lambda - 2)(\lambda + 4) &= 0\\ \lambda &= 2 \text{ or } -4\end{aligned}$$

The possible values of λ are **2** and **−4**.

Example 10 Relative to an origin O, the position vectors of the points A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively. The position vectors of the points C, D and E are $\overrightarrow{OC} = \frac{2}{3}\overrightarrow{OA}$, $\overrightarrow{OD} = 2\overrightarrow{OB}$ and $\overrightarrow{AE} = \frac{1}{2}\overrightarrow{AB}$ respectively.

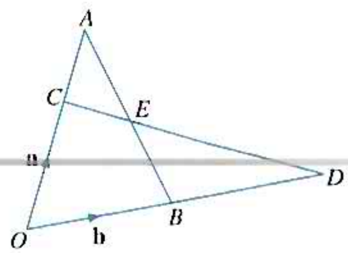
(a) Express $\overrightarrow{CE}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) Express $\overrightarrow{ED}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(c) Show that C, D and E are collinear points and find the ratio $CE : ED$.

Solution:

(a)
$$\begin{aligned}\overrightarrow{CE} &= \overrightarrow{CA} + \overrightarrow{AE}\\ &= \frac{1}{3}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}\\ &= \frac{1}{3}\mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a})\\ &= \frac{1}{6}(-\mathbf{a} + 3\mathbf{b})\end{aligned}$$



(b) $\overrightarrow{ED} = \overrightarrow{EB} + \overrightarrow{BD}$

$= \frac{1}{2}\overrightarrow{AB} + \overrightarrow{OB}$

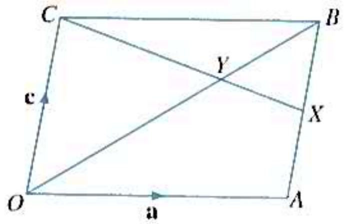
$= \frac{1}{2}(\mathbf{b} - \mathbf{a}) + \mathbf{b}$

$= \frac{1}{2}(-\mathbf{a} + 3\mathbf{b})$

(c) From (a) and (b),

$\overrightarrow{ED} = 3\overrightarrow{CE} \quad \Rightarrow \quad C, D$ and E are collinear and $CE : ED = 1 : 3$.

Example 11 $OABC$ is a parallelogram with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. The midpoint of AB is X and CX meets OB at Y.
Given that $\overrightarrow{OY} = \lambda\overrightarrow{OB}$ and $\overrightarrow{CY} = \mu\overrightarrow{CX}$,
(a) express $\overrightarrow{OY}$ in terms of λ, $\mathbf{a}$ and $\mathbf{c}$,
(b) express $\overrightarrow{OY}$ in terms of $\mathbf{a}$ and $\mathbf{c}$.



Solution:

(a) $\overrightarrow{OY} = \lambda\overrightarrow{OB} = \lambda(\mathbf{a} + \mathbf{c})$ .... (1)

(b) $\overrightarrow{OY} = \overrightarrow{OC} + \overrightarrow{CY}$

$= \overrightarrow{OC} + \mu\overrightarrow{CX}$

$= \mathbf{c} + \mu\left(\mathbf{a} - \frac{1}{2}\mathbf{c}\right) \quad (\overrightarrow{CX} = \overrightarrow{CB} + \overrightarrow{BX})$

$= \mu\mathbf{a} + \left(1 - \frac{\mu}{2}\right)\mathbf{c}$ .... (2)

(c) From (1) and (2):

$\lambda\mathbf{a} + \lambda\mathbf{c} = \mu\mathbf{a} + \left(1 - \frac{\mu}{2}\right)\mathbf{c}$

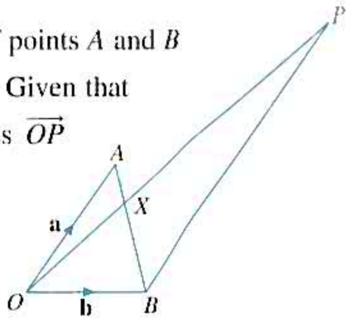
$\Rightarrow \lambda = \mu$ and $\lambda = 1 - \frac{\mu}{2}$

$\Rightarrow \lambda = \mu = \frac{2}{3}$

From (1), $\overrightarrow{OY} = \lambda(\mathbf{a} + \mathbf{c})$

$= \frac{2}{3}(\mathbf{a} + \mathbf{c})$

Example 12 In the diagram, the position vectors of points A and B relative to O are $\mathbf{a}$ and $\mathbf{b}$ respectively. Given that $\overrightarrow{BP} = m\mathbf{a}$, where m is a scalar, express $\overrightarrow{OP}$ in terms of m, $\mathbf{a}$ and $\mathbf{b}$.
The lines AB and OP intersect at X.
Given that $\overrightarrow{AX} = \frac{1}{4}\overrightarrow{AB}$ and $\overrightarrow{OP} = n\overrightarrow{OX}$, express $\overrightarrow{OP}$ in terms of n, $\mathbf{a}$ and $\mathbf{b}$ and hence evaluate m and n.



Solution:

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OB} + \overrightarrow{BP}\\ &= \mathbf{b} + m\mathbf{a}\\ &= m\mathbf{a} + \mathbf{b} \text{ .... (1)}\end{aligned}$$

Next,

$$\begin{aligned}\overrightarrow{OX} &= \overrightarrow{OA} + \overrightarrow{AX}\\ &= \overrightarrow{OA} + \frac{1}{4}\overrightarrow{AB}\\ &= \mathbf{a} + \frac{1}{4}(\mathbf{b} - \mathbf{a})\\ &= \frac{3}{4}\mathbf{a} + \frac{1}{4}\mathbf{b}\end{aligned}$$

$$\begin{aligned}\overrightarrow{OP} &= n\overrightarrow{OX}\\ &= n\left(\frac{3}{4}\mathbf{a} + \frac{1}{4}\mathbf{b}\right) \text{ .... (2)}\end{aligned}$$

From (1) and (2):

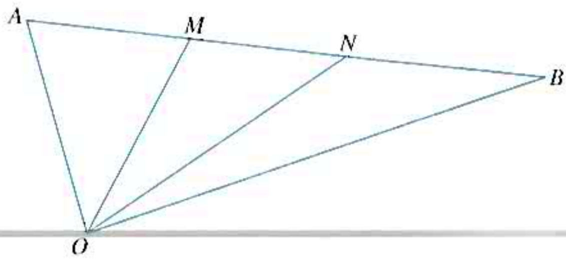
$$m\mathbf{a} + \mathbf{b} = \frac{3}{4}n\mathbf{a} + \frac{1}{4}n\mathbf{b}$$

$$\Rightarrow \quad m = \frac{3}{4}n \quad \text{and} \quad 1 = \frac{1}{4}n$$

$$\Rightarrow \quad m = 3 \quad \text{and} \quad n = 4$$

Exercise 23.3

1. The position vectors of the points A and B relative to an origin O are $\mathbf{a}$ and $\mathbf{b}$ respectively. The points M and N lie between A and B such that $AM = MN = NB$ and the point P is such that $\overrightarrow{AP} = k\overrightarrow{AB}$. Express



   (a) the position vectors of M and N in terms of $\mathbf{a}$ and $\mathbf{b}$,

   (b) the position vector $\overrightarrow{OP}$ in terms of k, $\mathbf{a}$ and $\mathbf{b}$.

2. $OABC$ is a trapezium and $\overrightarrow{CB} = m\overrightarrow{OA}$. Relative to O, the position vectors of A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively. Express the position vector of C in terms of m, $\mathbf{a}$ and $\mathbf{b}$.

3. The points P, Q and R have position vectors $\mathbf{a} + \mathbf{b}$, $3\mathbf{a} - \mathbf{b}$ and $6\mathbf{a} - 4\mathbf{b}$ respectively. Find $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and hence show that P, Q and R are collinear.

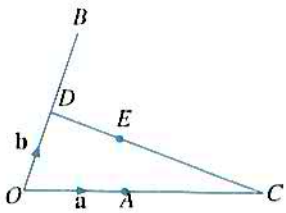
4. Relative to an origin O, the points A, B and C have position vectors $\mathbf{p} + \mathbf{q}$, $3\mathbf{p} - 2\mathbf{q}$ and $6\mathbf{p} + k\mathbf{q}$ respectively where $\mathbf{p}$ and $\mathbf{q}$ are non-parallel vectors.
   (a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$. (b) If $\overrightarrow{AB} = \lambda\overrightarrow{AC}$, find the value of k and of λ.

5. The points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $m\mathbf{a} + n\mathbf{b}$ respectively. If A, B and C are collinear, show that $m + n = 1$.

6. A, B and C are points with position vectors $\mathbf{p} - \mathbf{q}$, $\lambda(\mathbf{p} + \mathbf{q})$, and $\mathbf{p} + \lambda\mathbf{q}$ respectively, relative to an origin O. Obtain expressions for $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Given that A, B and C are collinear, and $\mathbf{p}$ and $\mathbf{q}$ are not parallel, find the value of λ.

7. $ABCD$ is a parallelogram whose diagonals meet at E. M is the midpoint of DC. Given that $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{AD} = \mathbf{q}$, express in terms of $\mathbf{p}$ and $\mathbf{q}$,
   (a) $\overrightarrow{AE}$, (b) $\overrightarrow{BD}$, (c) $\overrightarrow{MB}$.
   AD is produced to N where $AD = DN$. Prove, by a vector method, that N, M and B are collinear. *(C)*

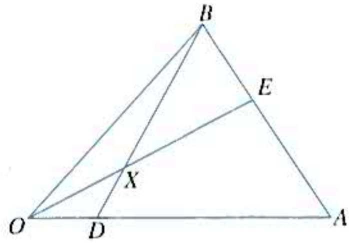
8. In the diagram, the position vectors of A and B, relative to an origin O, are $\mathbf{a}$ and $\mathbf{b}$ respectively. OA is produced to C so that $OC = kOA$, and C is joined to the midpoint, D, of OB. E is the point on CD such that $CE = nCD$.
   (a) Express the position vector of E in terms of $\mathbf{a}$, $\mathbf{b}$, k and n.
   (b) If A, E and B are collinear, find k in terms of n.



9. Relative to an origin O, P, Q and R are points with position vectors $\mathbf{p}$, $\mathbf{q}$ and $3\mathbf{q} - 9\mathbf{p}$ respectively, where $\mathbf{p}$ and $\mathbf{q}$ are non-parallel vectors. S is the point on PQ produced such that $\overrightarrow{QS} = m\overrightarrow{PQ}$ and $\overrightarrow{RS} = n\overrightarrow{OQ}$ where k and m are constants. Find the position vector of S in terms of
   (a) $\mathbf{p}$, $\mathbf{q}$ and m, (b) $\mathbf{p}$, $\mathbf{q}$ and n.
   Hence evaluate m and n and find the position vector of S.

10. Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively relative to an origin O. P is the point on OA produced such that $\overrightarrow{OP} = 3\overrightarrow{OA}$ and Q is on OB produced such that $\overrightarrow{OQ} = 2\overrightarrow{OB}$. The lines AQ and BP meet at R. Express $\overrightarrow{AQ}$ and $\overrightarrow{BP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. Given that $\overrightarrow{AR} = \lambda\overrightarrow{AQ}$ and $\overrightarrow{BR} = \mu\overrightarrow{BP}$, express $\overrightarrow{OR}$ in terms of
    (a) λ, $\mathbf{a}$ and $\mathbf{b}$, (b) μ, $\mathbf{a}$ and $\mathbf{b}$.
    Hence determine the values of λ and μ and find the position vector of R.

11. Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively relative to an origin O. The point D is such that $\overrightarrow{OD} = k\overrightarrow{OA}$ and the point E is such that $\overrightarrow{AE} = l\overrightarrow{AB}$. The line segments BD and OE intersect at X. If $\overrightarrow{OX} = \frac{2}{5}\overrightarrow{OE}$ and $\overrightarrow{XB} = \frac{4}{5}\overrightarrow{DB}$, express $\overrightarrow{OX}$ and $\overrightarrow{XB}$ in terms of $\mathbf{a}$, $\mathbf{b}$, k, l and hence evaluate k and l. *(C)*



12. $ABCDEF$ is a regular hexagon, centre G. The position vectors of A, B and D relative to an origin O are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{d}$ respectively. Find the position vectors of G, C, E and F in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{d}$.

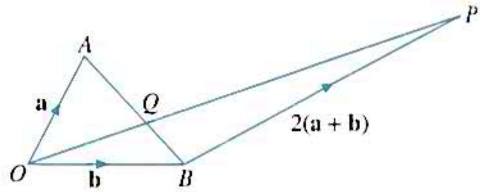
13. The position vectors of points A, B and C relative to an origin O are $\mathbf{p}$, $2\mathbf{p} + \mathbf{q}$ and $4\mathbf{p} - 5\mathbf{q}$ respectively. Given that $ABCD$ is a parallelogram, find the position vector of D in terms of $\mathbf{p}$ and $\mathbf{q}$. Given that T is a point whose position vector relative to O is $6\mathbf{p} + \lambda\mathbf{q}$ and $OBTC$ is a parallelogram, find the value of λ.

14. $ABCD$ is a parallelogram whose diagonals meet at M and the position vectors of A, B and C relative to an origin O are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. Find the position vectors of D and M in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

15. In the diagram, the position vectors of points A and B relative to O are $\mathbf{a}$ and $\mathbf{b}$ respectively. The lines AB and OP intersect at Q. Given that $\overrightarrow{BP} = 2(\mathbf{a} + \mathbf{b})$, $\overrightarrow{AQ} = \lambda\overrightarrow{AB}$ and $\overrightarrow{OQ} = \mu\overrightarrow{OP}$, express $\overrightarrow{OQ}$ in terms of

(a) λ, $\mathbf{a}$ and $\mathbf{b}$, **(b)** μ, $\mathbf{a}$ and $\mathbf{b}$,

and hence evaluate λ and μ.



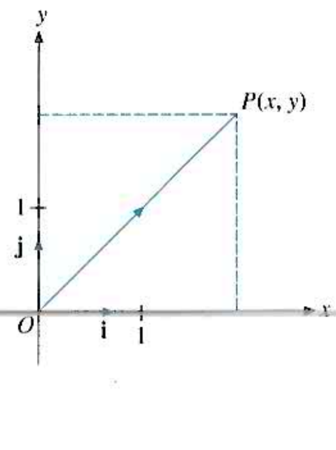
23.4 Vectors in the Cartesian Plane

Let $\mathbf{i}$ and $\mathbf{j}$ be the unit vectors in the positive directions of the x-axis and the y-axis respectively. Then the position vector of the point $P(x, y)$ is

$$\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j}$$

and its magnitude is

$$|\overrightarrow{OP}| = \sqrt{x^2 + y^2}.$$



The unit vector in the direction of $\overrightarrow{OP}$ is

$$\frac{\overrightarrow{OP}}{|\overrightarrow{OP}|} = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}}.$$

For the vector $\mathbf{a} = x\mathbf{i} + y\mathbf{j}$, $|x\mathbf{i} + y\mathbf{j}| = \sqrt{x^2 + y^2}$ and the unit vector in the direction of $\mathbf{a}$ is $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}}$.

Example 13 The position vector of the point $A(3, 4)$ is $\mathbf{a}$. Find
(a) the magnitude of $\mathbf{a}$,
(b) the unit vector in the direction of $\mathbf{a}$,
(c) the vector $\mathbf{b}$ which has a magnitude of 20 units in the direction of $\mathbf{a}$.

Solution:

(a) $\mathbf{a} = \overrightarrow{OA} = 3\mathbf{i} + 4\mathbf{j}$

$|\mathbf{a}| = \sqrt{3^2 + 4^2} = 5$

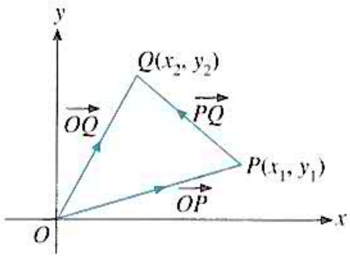
(b) The unit vector in the direction of $\mathbf{a}$ is $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{1}{5}(3\mathbf{i} + 4\mathbf{j})$

(c) $\mathbf{b} = 20\hat{\mathbf{a}} = 20 \times \dfrac{1}{5}(3\mathbf{i} + 4\mathbf{j}) = 4(3\mathbf{i} + 4\mathbf{j})$

Distance between Two Points

In the diagram, if $\overrightarrow{OP} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\overrightarrow{OQ} = x_2\mathbf{i} + y_2\mathbf{j}$ are the position vectors of the points P and Q, then $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}$ and the distance between P and Q is

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$



Example 14 Relative to an origin O, the position vectors of two fixed points A and B are $2\mathbf{i} + \mathbf{j}$ and $-\mathbf{i} + 2\mathbf{j}$ respectively. P is a point on the line passing through A and B such that $\overrightarrow{AP} = m\overrightarrow{AB}$. Find
(a) the distance between A and B,
(b) the position vector of P in terms of m,
(c) the value of m if P lies on the x-axis,
(d) the value of m if P lies on the line $y = x$.

Solution:

(a) $\overrightarrow{AB} = (-\mathbf{i} + 2\mathbf{j}) - (2\mathbf{i} + \mathbf{j}) = -3\mathbf{i} + \mathbf{j}$

The distance between A and B is $|\overrightarrow{AB}| = \sqrt{(-3)^2 + (1^2)}$

$= \sqrt{10}$ **units**

(b) $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$
$= \overrightarrow{OA} + m\overrightarrow{AB}$
$= 2\mathbf{i} + \mathbf{j} + m(-3\mathbf{i} + \mathbf{j})$
$= (2 - 3m)\mathbf{i} + (1 + m)\mathbf{j}$

(c) If P lies on the x-axis, then its y-coordinate is 0.

$$1 + m = 0$$
$$m = -1$$

(d) If P lies on the line $y = x$, then $1 + m = 2 - 3m$ and so $m = \frac{1}{4}$.

Column Vectors

The vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ may be written in the column form

$$\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}$$

which is called a **column** vector.

If $\mathbf{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, then:

(a) $\mathbf{a} = \mathbf{b} \Leftrightarrow x_1 = x_2$ and $y_1 = y_2$

(b) $k\mathbf{a} = k\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} kx_1 \\ ky_1 \end{pmatrix}$, where k is a scalar

(c) $\mathbf{a} + \mathbf{b} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$

(d) $\mathbf{a} - \mathbf{b} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}$

$$m\mathbf{a} + n\mathbf{b} = m\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + n\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} mx_1 + nx_2 \\ my_1 + ny_2 \end{pmatrix}$$

Example 15 Relative to an origin O, the position vectors of the points A, B and C are $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$ respectively. Find the value of m and of n such that $m\mathbf{a} + n\mathbf{b} = \mathbf{c}$.
Show that $\overrightarrow{AC} = 2(\overrightarrow{OA} + \overrightarrow{OB})$.

Solution: $m\mathbf{a} + n\mathbf{b} = \mathbf{c}$

$$m\begin{pmatrix}2\\1\end{pmatrix} + n\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}8\\9\end{pmatrix} \Rightarrow \quad 2m + n = 8 \quad \text{.......... (1)}$$

$$m + 3n = 9 \quad \text{.......... (2)}$$

Solving these equations, we have $m = \mathbf{3}$ and $n = \mathbf{2}$.

Then, $3\mathbf{a} + 2\mathbf{b} = \mathbf{c}$

$\Rightarrow \quad \mathbf{c} - \mathbf{a} = 2(\mathbf{a} + \mathbf{b})$

Hence $\quad \overrightarrow{AC} = 2\left(\overrightarrow{OA} + \overrightarrow{OB}\right)$.

Example 16 Relative to an origin O, the position vectors of two points A and B are $\begin{pmatrix}1\\4\end{pmatrix}$ and $\begin{pmatrix}7\\1\end{pmatrix}$ respectively. Given that the point $P(t, t+1)$ is on AB, find

(a) $\overrightarrow{AP}$ and $\overrightarrow{BP}$ in terms of t,
(b) the value of t and the ratio $AP : PB$.

Solution: **(a)** $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$$= \begin{pmatrix}t\\t+1\end{pmatrix} - \begin{pmatrix}1\\4\end{pmatrix}$$

$$= \begin{pmatrix}\mathbf{t-1}\\\mathbf{t-3}\end{pmatrix}$$

$\overrightarrow{BP} = \overrightarrow{OP} - \overrightarrow{OB}$

$$= \begin{pmatrix}t\\t+1\end{pmatrix} - \begin{pmatrix}7\\1\end{pmatrix}$$

$$= \begin{pmatrix}\mathbf{t-7}\\\mathbf{t}\end{pmatrix}$$

(b) Since $\overrightarrow{AP}$ and $\overrightarrow{BP}$ are parallel vectors,

$\overrightarrow{AP} = k\overrightarrow{BP}$ for some scalar k.

i.e. $\begin{pmatrix}t-1\\t-3\end{pmatrix} = k\begin{pmatrix}t-7\\t\end{pmatrix}$

which gives

$$t - 1 = k(t-7) \quad \text{.......... (1)}$$

$$t - 3 = kt \quad \text{.......... (2)}$$

$$\frac{t-1}{t-7} = \frac{t-3}{t} \quad (= k)$$

$$t^2 - t = t^2 - 10t + 21$$

$$t = \mathbf{\frac{7}{3}}$$

From (2):

$$k = \frac{t-3}{t} = -\frac{2}{7}$$

$$\overrightarrow{AP} = -\frac{2}{7}\overrightarrow{BP} \Rightarrow AP : PB = 2 : 7$$

Exercise 23.4

1. Given that $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{c} = 4\mathbf{i} + \mathbf{j}$, find
   (a) $\mathbf{a} + \mathbf{b}$, (b) $3\mathbf{a} + 2\mathbf{b}$, (c) $2\mathbf{b} - \mathbf{c}$, (d) $2\mathbf{a} + \mathbf{b} - 2\mathbf{c}$.

2. Given that $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, find
   (a) $2\mathbf{a} + \mathbf{b}$, (b) $\mathbf{a} + \mathbf{b} + \mathbf{c}$, (c) $3\mathbf{a} + 2\mathbf{c}$, (d) $2\mathbf{a} + 3\mathbf{b} - 2\mathbf{c}$.

3. Find the magnitude of each of the following vectors. Write down the unit vector and the vector of magnitude 65 which are in the direction of the given vector.
   (a) $3\mathbf{i} + 4\mathbf{j}$ (b) $5\mathbf{i} - 12\mathbf{j}$

4. Given that $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find the value of m and of n such that
   (a) $m\mathbf{a} + n\mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, (b) $m\mathbf{a} + n\mathbf{b} = \begin{pmatrix} 4 \\ -7 \end{pmatrix}$.

5. The position vectors of the points A and B are $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ respectively relative to an origin O. Given that
   (a) $\overrightarrow{AE} = 3\overrightarrow{AB}$, find the position vector of E,
   (b) $\overrightarrow{AP} = \lambda\overrightarrow{AB}$, express the position vector of P in terms of λ. Find the value of λ such that OP is parallel to $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

6. Relative to an origin O, the position vectors of the points P, Q and R are $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 5 \end{pmatrix}$ respectively. Find
   (a) the distance between Q and R,
   (b) the unit vector in the direction of $\overrightarrow{QR}$,
   (c) the value of m and n such that $\overrightarrow{OP} + m\overrightarrow{OQ} = n\overrightarrow{OR}$ and hence deduce that $\overrightarrow{PR} = 3\overrightarrow{OQ}$.

7. The vector $\begin{pmatrix} p \\ q \end{pmatrix}$ is parallel to $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Find the relationship between p and q. Given that $p > 0$ and $\begin{pmatrix} p \\ q \end{pmatrix}$ has a magnitude of 5, find the value of p and of q.

8. Find the relationship between p and q if
   (a) the vector $\begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ is parallel to $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$,
   (b) the vector $\begin{pmatrix} p-1 \\ q \end{pmatrix}$ has a magnitude of 2.

9. The position vectors of the points A and B, relative to an origin O, are $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ respectively. Find the position vector of C if
   (a) $\overrightarrow{AC} = 2\overrightarrow{AB}$,
   (b) $\overrightarrow{AC} = 3\overrightarrow{BA}$.

10. The position vectors of three points O, A and B are $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 6 \end{pmatrix}$ respectively. Given that $\overrightarrow{OC} = 3\overrightarrow{OA}$ and $\overrightarrow{OD} = \frac{1}{3}\overrightarrow{OB}$, write down the position vectors of C and D. Given also that AB and CD intersect at E, and that $\overrightarrow{DE} = p\overrightarrow{DC}$ and $\overrightarrow{AE} = q\overrightarrow{AB}$, find the position vector of E in terms of
    (a) p,
    (b) q.
    Hence find the value of p and of q.

11. The position vectors, relative to an origin O, of the points A and B are $\mathbf{a} = \begin{pmatrix} 15 \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Given that the position vector of C is $\mathbf{c} = \lambda\hat{\mathbf{a}} + 3\lambda\hat{\mathbf{b}}$ and ACB is a straight line, find the value of λ.

12. Relative to an origin O, the position vectors of the points A and B are $\lambda\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ respectively. Find $\overrightarrow{AB}$ and the values of λ if the distance between A and B is 5.

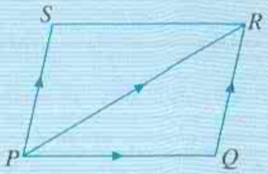
Important Notes

1. **Definitions**
   (a) $\mathbf{a} = \mathbf{b} \Leftrightarrow$ $\mathbf{a}$ and $\mathbf{b}$ have the same direction and $|\mathbf{a}| = |\mathbf{b}|$
   (b) $\mathbf{a}$ is a zero vector $\Leftrightarrow$ $|\mathbf{a}| = 0$
   (c) unit vector in the direction of $\mathbf{a}$ is $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$

2. Parallel vectors

$\mathbf{b} = k\mathbf{a}$ and $k > 0 \Leftrightarrow$ $\mathbf{a}$ and $\mathbf{b}$ are in the same direction
$\mathbf{b} = k\mathbf{a}$ and $k < 0 \Leftrightarrow$ $\mathbf{a}$ and $\mathbf{b}$ are opposite in direction
$\mathbf{b} = k\mathbf{a} \Rightarrow |\mathbf{b}| = |k||\mathbf{a}|$

3. Addition and subtraction of vectors



(a) $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$ *(triangle law of addition)*

$\overrightarrow{PQ} + \overrightarrow{PS} = \overrightarrow{PR}$ *(parallelogram law of addition)*

$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ *(commutative law)*

(b) $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} \Leftrightarrow \overrightarrow{QR} = \overrightarrow{PR} - \overrightarrow{PQ}$ *(subtraction of vectors)*

4. Vectors expressed in terms of two non-parallel vectors a and b

$$p\mathbf{a} + q\mathbf{b} = r\mathbf{a} + s\mathbf{b} \Leftrightarrow p = r \text{ and } q = s$$

5. Position vectors

(a) Relative to an origin O, the position vector of a point P is $\overrightarrow{OP}$.

(b) A, B and C are collinear points $\Leftrightarrow \overrightarrow{AB} = k\overrightarrow{BC}$

(c) $ABCD$ is a parallelogram $\Leftrightarrow \overrightarrow{AB} = \overrightarrow{DC}$

6. Vectors in the cartesian plane

(a) If P has coordinates (x, y) in a cartesian plane, then the position vector of P is

$$\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the positive direction along the x-axis and the y-axis respectively.

(b) In column form, we write $\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $|\overrightarrow{OP}| = \sqrt{x^2 + y^2}$.

(c) Unit vector in the direction of $\overrightarrow{OP}$ is

$$\frac{1}{\sqrt{x^2 + y^2}}(x\mathbf{i} + y\mathbf{j}) \text{ or } \frac{1}{\sqrt{x^2 + y^2}}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Miscellaneous Exercise 23

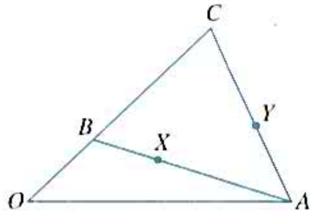
1. The position vectors of three collinear points P, Q and R relative to an origin O are $\mathbf{b} - 2\mathbf{a}$, $3\mathbf{a} - 2\mathbf{b}$ and $k\mathbf{a} + 6\mathbf{b}$ respectively.

(a) Find the value of k and state the ratio $PQ : QR$.

(b) Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, the position vector of the point S such that $\overrightarrow{PS} = 2\overrightarrow{SQ}$.

2. The position vectors of three points A, B and C relative to an origin O are $2\mathbf{p} - \mathbf{q}$, $4\mathbf{q}$ and $14\mathbf{q} - 4\mathbf{p}$ respectively. Show that the points A, B and C lie on the same straight line and state the ratio $AB : BC$.
Given that $OABD$ is a parallelogram, find the position vector of D. P is the point on DB such that $\overrightarrow{DP} = k\overrightarrow{DB}$, find the position vector of P in terms of $\mathbf{p}$, $\mathbf{q}$ and k. If O, P and C are collinear, find the value of k.

3. Given that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = 3\overrightarrow{OB}$, express $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
Given further that $\overrightarrow{AX} = 2\overrightarrow{XB}$ and $\overrightarrow{AY} = k\overrightarrow{YC}$, express $\overrightarrow{OX}$ and $\overrightarrow{OY}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and k.
If the points O, X and Y lie in a straight line, find k.



4. The position vectors of three points A, B and C relative to an origin O are $\mathbf{a}$, $\mathbf{b}$ and $5\mathbf{a}$ respectively. The point P lies on AB and is such that $AP = kAB$. The point Q lies on OP produced and is such that $OQ = \frac{1}{2k}OP$. Find the position vector of Q in terms of $\mathbf{a}$, $\mathbf{b}$ and k. If Q lies on BC, find the value of k and the ratio $BQ : QC$.

5. The position vectors of four points A, B, C and D relative to an origin O are $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 10 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$ respectively. Two points P and Q are points on AB and CD respectively, $AP = \lambda AB$ and $CQ = \mu CD$. Show that the position vectors of P and Q are $\begin{pmatrix} 2 - \lambda \\ 1 + \lambda \end{pmatrix}$ and $\begin{pmatrix} 10 - 5\mu \\ 5 - \mu \end{pmatrix}$ respectively. Hence find the position vector of M, the point of intersection of AB and CD.

6. The position vectors of points A and B, relative to an origin O, are $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 9 \\ -3 \end{pmatrix}$ respectively. P, Q and R are points such that $\overrightarrow{OP} = \frac{1}{2}\overrightarrow{OA}$, $\overrightarrow{AR} = k\overrightarrow{AB}$ and $\overrightarrow{OQ} = \frac{2}{3}\overrightarrow{OB}$. Write down the position vectors of P and Q. Express the position vector of R in terms of k. Given that P, Q and R are collinear, find the value of k.

7. Relative to an origin O, the position vectors of three points P, Q and R are $\mathbf{a} + \mathbf{b}$, $\mathbf{a}$ and $\mathbf{a} - 2\mathbf{b}$ respectively. The point L lies on OP and is such that $OL = \frac{3}{4}OP$. The point M lies on OR produced and is such that $OM = \lambda OR$. Find
   (a) in terms of $\mathbf{a}$, $\mathbf{b}$ and λ, the position vector of M,
   (b) the value of λ for which L, Q and M are collinear. Hence find the value of $\frac{QM}{LQ}$.

8. **(a)** Given that A, B and C are collinear points with position vectors $\mathbf{p}$, $\mathbf{p} + 2\mathbf{q}$, $\lambda(\mathbf{p} - 2\mathbf{q})$ respectively, find the value of

(i) λ, (ii) $\frac{AB}{BC}$.

(b) A, B and C are points with position vectors $2\mathbf{p} - 3\mathbf{q}$, $3\mathbf{p} - 2\mathbf{q}$ and $\mathbf{p} + 3\mathbf{q}$ respectively, relative to an origin O. The point X is such that $\overrightarrow{OX} = k\overrightarrow{OA}$ and $\overrightarrow{CX} = m\overrightarrow{CB}$, express $\overrightarrow{OX}$ in terms of

(i) k, $\mathbf{p}$ and $\mathbf{q}$, (ii) m, $\mathbf{p}$ and $\mathbf{q}$.

Hence evaluate k and m and write down the position vector of X in terms of $\mathbf{p}$ and $\mathbf{q}$.

9. **(a)** Find the unit vector $\mathbf{n}$ in the direction of the vector $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$. Express the vector $\begin{pmatrix} 6 \\ -8 \end{pmatrix}$ in terms of $\mathbf{n}$.

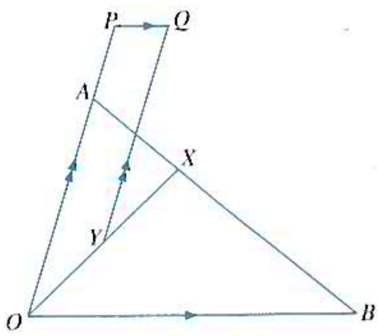
(b) Find the vector $\mathbf{p}$ which has a magnitude of 39 units and is in the same direction as $\begin{pmatrix} 5 \\ 12 \end{pmatrix}$.

(c) The position vectors of A and B relative to an origin O are $\begin{pmatrix} 8 \\ 8 \end{pmatrix}$ and $\begin{pmatrix} 12 \\ 5 \end{pmatrix}$ respectively. Find $\overrightarrow{AB}$ and the unit vector in the direction of $\overrightarrow{AB}$.

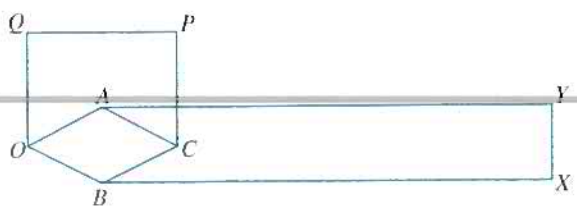
10. In the figure the points X and Y are such that $AX = \frac{1}{2}XB$ and $OY = YX$, while the point P is such that $OA = 3AP$. The lines YQ and PQ are parallel to OA and OB respectively.

Given that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, express $\overrightarrow{OP}$ and $\overrightarrow{OY}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

Given that $\overrightarrow{YQ} = m\mathbf{a}$ and $\overrightarrow{PQ} = n\mathbf{b}$, find the values of m and n. Hence show that $OAQY$ is a parallelogram. *(C)*



11. $OACB$ is a parallelogram in which $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. $ABXY$ is a parallelogram in which $\overrightarrow{AY} = 3\overrightarrow{OC}$. $OCPQ$ is a parallelogram in which $\overrightarrow{CP} = 2\overrightarrow{BA}$. Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, the position vectors P, Q, X and Y relative to O.



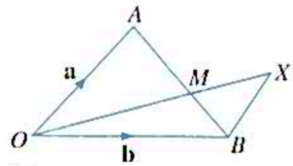
12. The position vectors of three points A, B and C, relative to an origin O, are $\mathbf{a}$, $\mathbf{b}$ and $k\mathbf{a}$ respectively. The point P lies on AB and is such that $AP = 2PB$. The point Q lies on BC and is such that $CQ = 6QB$. Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, the position vectors of P and Q. Given that OPQ is a straight line, find

(a) the value of k, **(b)** the ratio $\frac{OP}{PQ}$.

The position vector of a point R is $\frac{7}{3}\mathbf{a}$. Show that PR is parallel to BC. *(C)*

13. (a) Given that $\mathbf{a}$ is a vector of magnitude 10 units in the direction of the vector $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ and that $\mathbf{b}$ is a vector of magnitude 15 units in the direction of the vector $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$, find the vector $\mathbf{a} + \mathbf{b}$.

(b) In the diagram, the position vectors of A and B relative to an origin O are $\mathbf{a}$ and $\mathbf{b}$ respectively. Given that $\overrightarrow{BX} = p\overrightarrow{OA}$, express $\overrightarrow{OX}$ in terms of p, $\mathbf{a}$ and $\mathbf{b}$. The point M lies on AB such that $AM : MB = 2 : 1$. Given also that $\overrightarrow{OX} = q\overrightarrow{OM}$, express $\overrightarrow{OX}$ in terms of q, $\mathbf{a}$ and $\mathbf{b}$. Hence find the value of q and of p. *(C)*



14. The position vectors of the points P, Q and R are $\mathbf{p} = \begin{pmatrix} a \\ 8 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 8 \\ b \end{pmatrix}$ respectively. Find

(a) the value of a and of n such that $\mathbf{p} + n\mathbf{q} = \begin{pmatrix} 2 \\ 16 \end{pmatrix}$,

(b) the value of b such that the points P, Q and R are collinear.

15. Relative to an origin O, the position vectors of two points A and B are $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 1 \end{pmatrix}$ respectively. The point C lies on AB, between A and B. Given that the position vector of C is $\begin{pmatrix} t^2 \\ t \end{pmatrix}$, find the value of t and the ratio $AC : CB$.

16. (a) $ABCDEF$ is a regular hexagon with centre G. The position vectors of A, B and C relative to an origin O are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. Express **(i)** $\overrightarrow{OG}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, **(ii)** $\overrightarrow{CD}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

(b) The position vectors of points P, Q and R relative to an origin O are $\mathbf{p} + \mathbf{q}$, $2\mathbf{p} + 3\mathbf{q}$ and $4\mathbf{p} - \mathbf{q}$ respectively.
Given that $PQRS$ is a parallelogram find, in terms of $\mathbf{p}$ and $\mathbf{q}$, the position vector of S. Given that T is a point whose position vector relative to O is $5\mathbf{p}$, show that $OPTR$ is a parallelogram. *(C)*

24 Relative Velocity

24.1 Relative Motion in a Straight Line

True Velocity

In the diagram, A (an athlete) jogs along a straight road at a speed of 2 m s^{-1} due east. The velocity of A, denoted by $\mathbf{v}_A$, is written as

$$\mathbf{v}_A = \xrightarrow{2 \text{ m s}^{-1}}$$

which is the velocity **relative** to the earth and is known as a **true (or actual) velocity**. The true speed v_A is also known as the **ground speed**.

Similarly, $\mathbf{v}_C$ is the true velocity of C (a cyclist) who travels at a speed of 20 km h^{-1} due west and we write

$$\mathbf{v}_C = \xleftarrow{20 \text{ km h}^{-1}}.$$

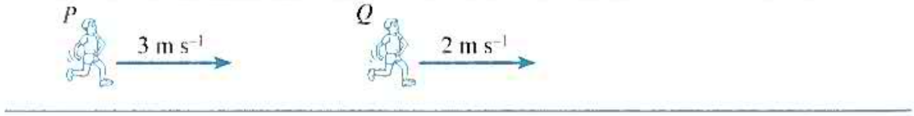
The velocity of a moving object P relative to the earth is known as a true (or actual) velocity and is denoted by $\mathbf{v}_P$.

Suppose that P and Q are two fixed points on the earth and A jogs from P to Q in 30 minutes, then the distance (true distance) from P to Q is

$$PQ = v_A t = 2(30 \times 60) \text{ m} = 3600 \text{ m (3.6 km)}$$

Relative Velocity

Let us begin with a simple example:



The diagram shows two men P and Q running at speeds of 3 m s^{-1} and 2 m s^{-1} respectively. Intuitively, we can observe the following:

To the man Q, the man P appears to move at a speed of 1 m s^{-1} towards Q. Relative to Q(observer), P is moving at a speed of 1 m s^{-1} due east as shown

and we write

$$\mathbf{v}_{P/Q} = \xrightarrow{1 \text{ m s}^{-1}}$$

which represents the velocity of P relative to Q. The velocity $\mathbf{v}_{P/Q}$ is known as a **relative velocity** and it covers the distance between P and Q. If the distance between P and Q is 20 m, then P will overtake Q after 20 seconds.

Similarly, relative to P(observer), Q appears to move at a speed of 1 m s^{-1} due west towards P and we write

$$\mathbf{v}_{Q/P} = \xleftarrow{1 \text{ m s}^{-1}}$$

which represents the velocity of Q relative to P.

(a) The velocity of a moving object P relative to another moving object Q is known as an apparent velocity and is denoted by $\mathbf{v}_{P/Q}$.

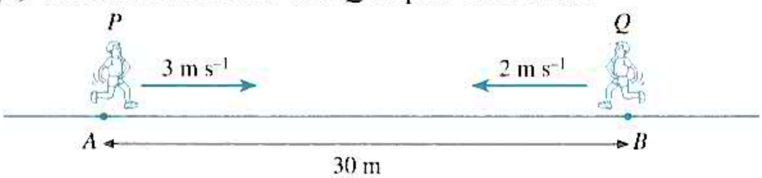
(b) $\mathbf{v}_{P/Q} = -\mathbf{v}_{Q/P}$

Example 1

In the diagram, A and B are two fixed points on a straight path and the distance apart is 30 m. A man P jogs from A at a speed of 3 m s^{-1} due east. Another man Q jogs from B at a speed of 2 m s^{-1} in the opposite direction. Find

(a) the velocity of P relative to Q,

(b) the time taken for P and Q to pass each other.



Solution:

(a) $\mathbf{v}_P = \xrightarrow{3 \text{ m s}^{-1}}$ $\qquad \mathbf{v}_Q = \xleftarrow{2 \text{ m s}^{-1}}$

To Q, P appears to move at a speed of 5 m s^{-1} towards him. In mathematical terms, we say that relative to Q(observer), P is moving at a speed of 5 m s^{-1} due east.

$$\mathbf{v}_{P/Q} = \xrightarrow{5 \text{ m s}^{-1}}$$

(b) The time taken for P and Q to pass each other,

$$t = \frac{AB}{v_{P/Q}} = \frac{30}{5} \text{ s} = 6 \text{ s}.$$

Note: AB = initial distance between P and Q.

In Example 1, we study intuitively the relative motion between two moving objects. Next, we shall apply basic laws of vectors to study relative motion and develop systematic approaches to solve more complicated problems.

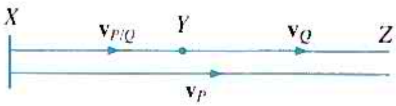
Relative Velocity Equation

For two moving objects P and Q, the relationship among the true velocities $\mathbf{v}_P$, $\mathbf{v}_Q$ and the relative velocity $\mathbf{v}_{P/Q}$ is given by the vector equation

$$\mathbf{v}_P = \mathbf{v}_{P/Q} + \mathbf{v}_Q \qquad \text{(Addition of Vectors)}$$

which is known as the **relative velocity equation**.

To solve the vector equation, we apply the Triangle Law of Addition on a straight line to draw a **velocity diagram** as shown:



(1) Draw $\mathbf{v}_{P/Q}$ represented by $\overrightarrow{XY}$ followed by $\mathbf{v}_Q$ represented by $\overrightarrow{YZ}$.

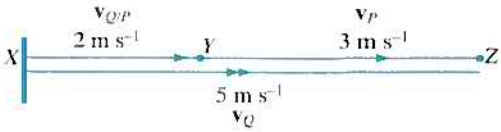
(2) Draw $\mathbf{v}_P$ represented by $\overrightarrow{XZ}$.

We shall begin by illustrating the application of the above equation as follows:

(a) A particle P is moving at a speed of 3 m s^{-1} due east. To P, another particle Q appears to move at a speed of 2 m s^{-1} due east.

$\mathbf{v}_P = \xrightarrow{3 \text{ m s}^{-1}}$ $\qquad\qquad$ $\mathbf{v}_{Q/P} = \xrightarrow{2 \text{ m s}^{-1}}$

Applying $\mathbf{v}_Q = \mathbf{v}_{Q/P} + \mathbf{v}_P$, we have the velocity diagram as follows:



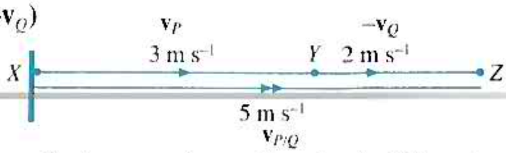
So $\mathbf{v}_Q = \xrightarrow{5 \text{ m s}^{-1}}$

(b) In Example 1, we have

$\mathbf{v}_P = \xrightarrow{3 \text{ m s}^{-1}}$ $\qquad\qquad$ $\mathbf{v}_Q = \xleftarrow{2 \text{ m s}^{-1}}$

Applying $\mathbf{v}_P = \mathbf{v}_{P/Q} + \mathbf{v}_Q$, $\quad \mathbf{v}_{P/Q} = \mathbf{v}_P - \mathbf{v}_Q$

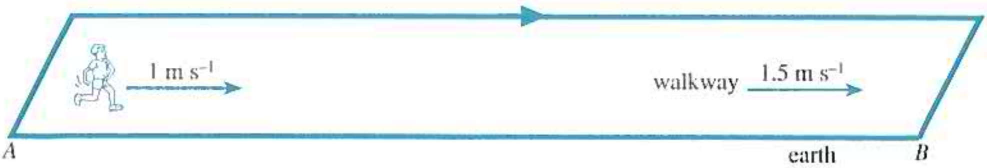
$$= \mathbf{v}_P + (-\mathbf{v}_Q)$$



So $\mathbf{v}_{P/Q} = \xrightarrow{5 \text{ m s}^{-1}}$

Note that in (b) we quote the same relative velocity equation and apply the Triangle Law of Addition.

Example 2



A straight horizontal moving walkway travels at 1.5 m s^{-1} in a direction from the fixed point A towards the fixed point B as shown in the diagram. A man walks from A to B on the moving walkway, at a speed of 1 m s^{-1}.

Write down
(a) the velocity of the man relative to the walkway,
(b) the true velocity of the man.
What is the true speed of the man if the walkway increases its speed to 1.8 m s^{-1} and the speed of the man relative to the walkway remains unchanged?

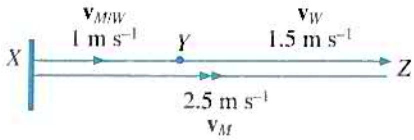
Solution:

Let W denote the walkway and M the man.
(a) The velocity of the man relative to the walkway is:

$$\mathbf{v}_{M/W} = \xrightarrow{1\text{ m s}^{-1}}$$

(b) $\mathbf{v}_W = \xrightarrow{1.5\text{ m s}^{-1}}$

Apply $\mathbf{v}_M = \mathbf{v}_{M/W} + \mathbf{v}_W$ to draw a velocity diagram.



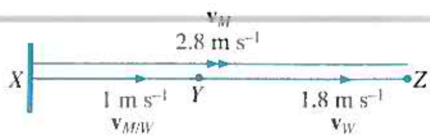
The true velocity of the man is:

$$\mathbf{v}_M = \xrightarrow{2.5\text{ m s}^{-1}}$$

When the walkway increases its speed to 1.8 m s^{-1}, the speed of the man relative to the walkway remains unchanged.

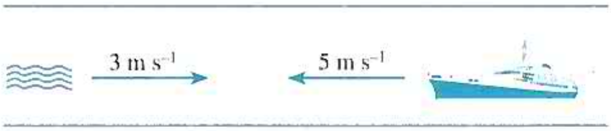
$$\mathbf{v}_M = \mathbf{v}_{M/W} + \mathbf{v}_W$$

$$= \xrightarrow{2.8\text{ m s}^{-1}}$$



From Example 2, we assume that $v_{M/W}$ (= 1 m s^{-1}) remains the same when the walkway is still or moving with different constant speeds.

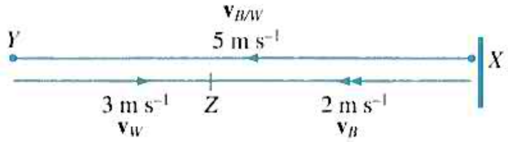
Example 3 In the diagram, a river is flowing at a speed of 3 m s^{-1} due east. A boat, whose speed in still water is 5 m s^{-1}, is moving upstream. Find the true speed and the actual direction of motion of the boat.



Solution: Let B be the boat and W the water.

$\mathbf{v}_W = \xrightarrow{3\text{ m s}^{-1}}$ and $\mathbf{v}_{B/W} = \xleftarrow{5\text{ m s}^{-1}}$

Apply $\mathbf{v}_B = \mathbf{v}_{B/W} + \mathbf{v}_W$ to draw a velocity diagram.



Hence, $\mathbf{v}_B = \xleftarrow{2\text{ m s}^{-1}}$

Again, we assume that the boat travels at a speed of 5 m s^{-1} relative to (still or moving) water.

For an object P moving in the water, we always assume that the speed of P relative to the water, $v_{P/W}$, remains unchanged when the water is still or moving at different constant speeds.

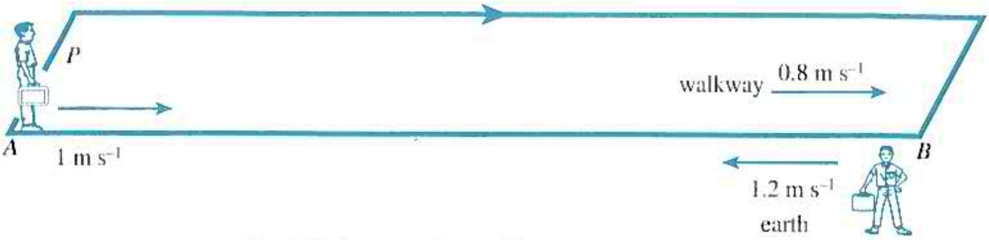
Example 4 In an international airport, a straight horizontal moving walkway travels at 0.8 m s^{-1} in a direction from the fixed point A towards the fixed point B. A passenger P walks from A to B on the moving walkway, at a speed of 1 m s^{-1} relative to the walkway. At the same instance, another passenger Q walks from B to A, on a fixed horizontal ground alongside the walkway, at a speed of 1.2 m s^{-1}. Calculate

(a) the velocity of P,

(b) the speed of P relative to Q.

The distance AB is 120 m. Suppose P and Q pass each other after t seconds, find the value of t.

Solution: (a)



Let W denote the walkway.

$$\mathbf{v}_W = \xrightarrow{0.8\ \text{m s}^{-1}} \qquad \mathbf{v}_{P/W} = \xrightarrow{1\ \text{m s}^{-1}}$$

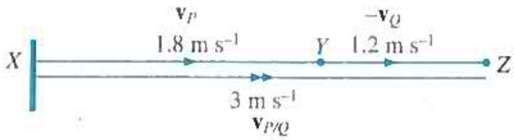
The velocity of P is $\mathbf{v}_P = \mathbf{v}_{P/W} + \mathbf{v}_W = \xrightarrow{\mathbf{1.8\ m\ s^{-1}}}$

(b) True velocities of P and Q are as follows:

$$\mathbf{v}_P = \xrightarrow{1.8\ \text{m s}^{-1}} \qquad \mathbf{v}_Q = \xleftarrow{1.2\ \text{m s}^{-1}}$$

Applying $\mathbf{v}_P = \mathbf{v}_{P/Q} + \mathbf{v}_Q$, the velocity of P relative to Q is given by:

$$\mathbf{v}_{P/Q} = \mathbf{v}_P + (-\mathbf{v}_Q)$$



From the velocity diagram

$$\mathbf{v}_{P/Q} = \xrightarrow{3\ \text{m s}^{-1}}$$

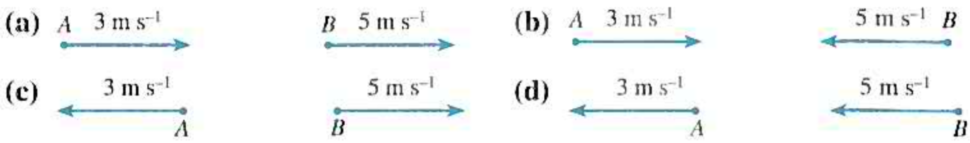
The speed of P relative to Q is $v_{P/Q} = \mathbf{3\ m\ s^{-1}}$.

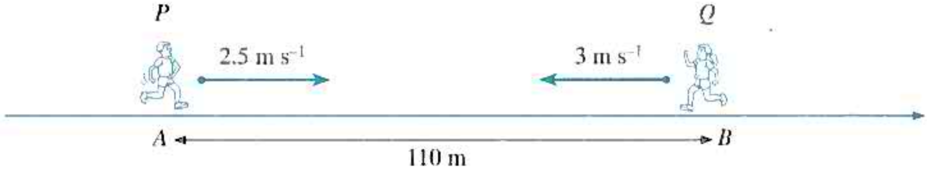
Suppose P and Q pass each other after t seconds.

Time taken, $t = \dfrac{AB}{v_{P/Q}} = \dfrac{120}{3} = \mathbf{40\ seconds}$

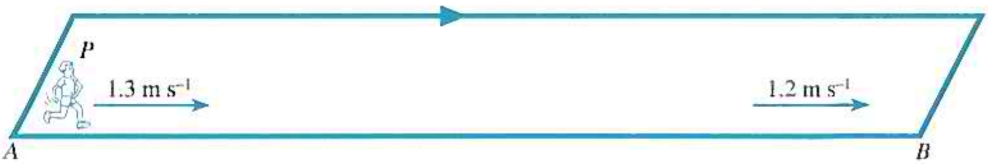
Exercise 24.1

1. In each of the following, find the velocity of particle B relative to particle A.



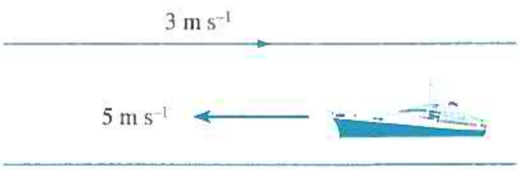
2. 

In the diagram, A and B are two fixed points on a straight path and the distance apart is 110 m. A boy P runs from A at a speed of 2.5 m s^{-1} due east. A girl Q runs from B at a speed of 3 m s^{-1} in the opposite direction. Find
(a) the velocity of P relative to Q,
(b) the time taken for P to meet Q and the distance travelled by P.

3. 

A straight horizontal moving walkway travels at 1.2 m s^{-1} in a direction from the fixed point A towards the fixed point B as shown in the diagram. A man P walks from A to B on the moving walkway at a speed of 1.3 m s^{-1}. Write down
(a) the velocity of the man relative to the walkway,
(b) the true velocity of the man.

4. In the diagram, a river is flowing at a speed of 3 m s^{-1} due east. A boat, whose speed in still water is 5 m s^{-1}, is moving upstream. Find the true speed and direction of motion of the boat. At a certain instant, the river is flowing at a slower speed of 2 m s^{-1}. Find the true speed of the boat at that instant.



5. In an international airport, a straight horizontal moving walkway is designed to travel at 0.8 m s^{-1} in a direction from the fixed point A towards the fixed point B. A passenger P walks from A to B on the moving walkway, at a speed of 1.2 m s^{-1} relative to the walkway. At the same instant, another passenger Q walks from A to B, on a fixed horizontal ground alongside the walkway at a speed of 1.5 m s^{-1}. Calculate
(a) the speed of P, **(b)** the speed of P relative to Q.
The distance AB is 120 m. Find the distance between P and Q at the instant when P reaches B.

6. A straight horizontal moving walkway travels at 2 m s^{-1} in a direction from the fixed point A towards the fixed point B. A man P walks from A to B on the moving walkway, at a speed of 1.5 m s^{-1} relative to the walkway. Another man Q walks from B to A, on a fixed horizontal ground alongside the walkway, at a speed of 1.6 m s^{-1}. Calculate the speed of P relative to Q.
The distance AB is 200 m and P and Q pass each other at a point halfway between A and B. Find the time between Q's departure from B and P's departure from A.
(C)

7. The diagram shows a car A and a van B moving in the same direction on a straight road; car A is travelling at 12 m s^{-1} and van B is travelling at 8 m s^{-1}.

Write down the velocity of B relative to A. Hence find the time taken from the instant when A and B are at a distance 200 m apart to the instant when A catches up with B.

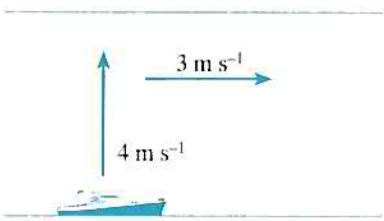
24.2 Relative Motion in a Current

By now, we have some ideas about true velocity, relative velocity (apparent velocity) and the vector equation relating true and apparent velocities for motion along a straight line. In the following, we shall study motion in a current (of water or air) and continue to apply the relative velocity equation $\mathbf{v}_P = \mathbf{v}_{P/Q} + \mathbf{v}_Q$.

Motion in the Water: River Crossing Problems

First, we begin by studying some examples on *River Crossing Problems.*

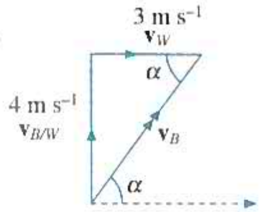
Example 5 In the diagram, a river is flowing at 3 m s^{-1} due east. A boat, of speed 4 m s^{-1} in still water, is steered in the direction due north. Find the true speed and direction of motion of the boat.



Solution: Let B denote the boat and W the water.

$$\mathbf{v}_W = \xrightarrow{3 \text{ m s}^{-1}} \quad \text{and} \quad \mathbf{v}_{B/W} = 4 \text{ m s}^{-1} \uparrow$$

Using $\mathbf{v}_B = \mathbf{v}_{B/W} + \mathbf{v}_W$, we obtain the velocity diagram as follows:



$$v_B = \sqrt{3^2 + 4^2} = 5 \text{ m s}^{-1}$$

$$\tan \alpha = \frac{4}{3} \Rightarrow \alpha = 53.1°$$

The true speed of the boat is 5 m s^{-1} and it travels downstream making an angle 53.1° with the bank.

In example 5, we use the fact that the course taken by the boat is the direction of $\mathbf{v}_{B/W}$, the velocity of the boat relative to the water.

For an object P moving in the water, the course taken by P is the direction of $\mathbf{v}_{P/W}$, the velocity of P relative to W the water.

Again, in Example 5, the velocity $\mathbf{v}_B$ is the vector sum of the velocities $\mathbf{v}_{B/W}$ and $\mathbf{v}_W$. $\mathbf{v}_B$ is also known as the **resultant velocity**.

Perpendicular Components of a Velocity

In Example 5, the velocity $\mathbf{v}_B = 5$ m s^{-1} ↗ is the composition of two perpendicular components as shown in Fig. (a).

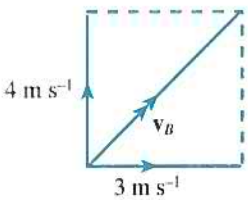


Fig. (a) Velocity diagram

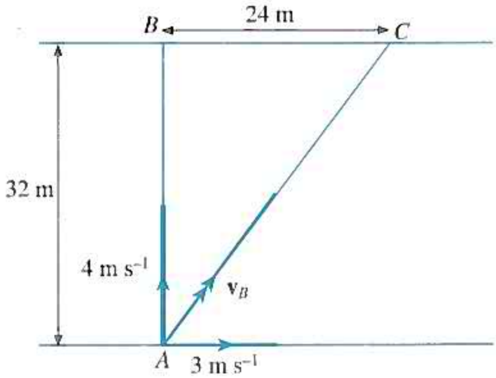


Fig. (b) Position diagram

In Fig. (b), the component of 4 m s^{-1} along the direction (↑) enables the boat to cross the river. If the width of the river is 32 m, then the time taken to cross the river is $\frac{32}{4} = 8$ seconds.

The component of 3 m s^{-1} along the bank (⟶) enables the boat to move downstream from the point of crossing. When the boat reaches the other bank of the river, it will reach a point C which is $3 \times 8 = 24$ m downstream from B as shown above.

Given that $AC = \sqrt{(32)^2 + (24)^2} = 40$ m, the time taken to cross the river from A to C is $\frac{AC}{v_B} = \frac{40}{5} = 8$ seconds.

Example 6

A boy who swims at 1 m s^{-1} in still water wishes to cross a river 20 m wide. The river is flowing between straight parallel banks at 1.5 m s^{-1}. He heads upstream in a direction making an angle of 60° with the bank. Find

(a) the speed at which he travels,

(b) the angle which his resultant velocity makes with the bank,

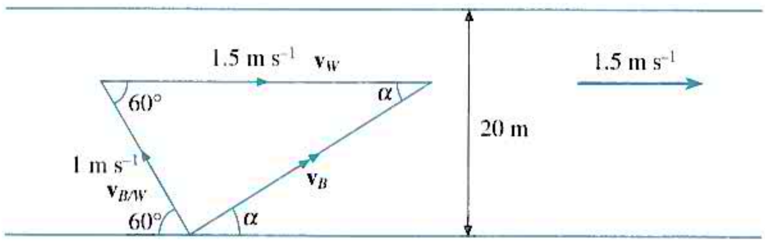
(c) the time taken for the crossing, to the nearest second.

Solution:

Let B denote the boy and W the water.

$$\mathbf{v}_W = \xrightarrow{1.5 \text{ m s}^{-1}} \qquad \mathbf{v}_{B/W} = 1 \text{ m s}^{-1} \nwarrow$$

Using $\mathbf{v}_B = \mathbf{v}_{B/W} + \mathbf{v}_W$, we draw a velocity diagram as shown.



(a) Using cosine rule:

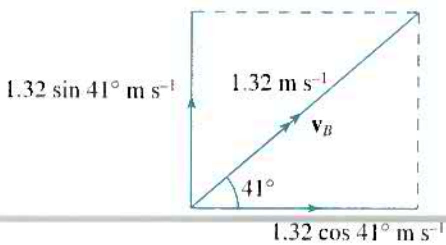
$$v_B = \sqrt{1^2 + 1.5^2 - 2 \times 1 \times 1.5 \times \cos 60^\circ} = \mathbf{1.32 \text{ m s}^{-1}}$$

(b) Using sine rule:

$$\frac{\sin \alpha}{1} = \frac{\sin 60^\circ}{1.32} \quad \Rightarrow \quad \alpha = 41^\circ$$

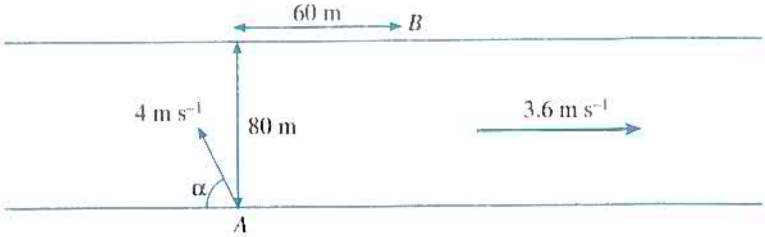
So the angle which his resultant velocity makes with the bank is **41°**.

(c) Resolve the resultant velocity into two components as shown.



Time taken for the crossing, $t = \dfrac{20}{1.32 \sin 41^\circ} = \mathbf{23 \text{ seconds}}$

Example 7



The diagram shows a river, 80 m wide, flowing at a speed of 3.6 m s^{-1}, between straight parallel banks. A boat crosses the river from a point A on one bank to a point B on the opposite bank, 60 m downstream. The speed of the boat in still water is 4 m s^{-1}. In order to travel directly from point A to B, the boat is steered in a direction making an angle α to the bank as shown. Find

(a) the value of α,
(b) the resultant speed of the boat,
(c) the time taken for the crossing, to the nearest second.

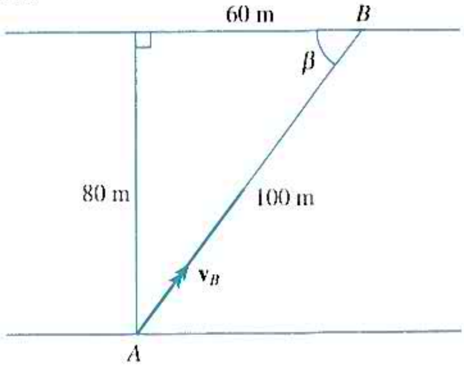
Solution:

(a) Refer to the position diagram: $\sin \beta = \dfrac{80}{100} = 0.8$ and the direction of $\mathbf{v}_B$ is known.

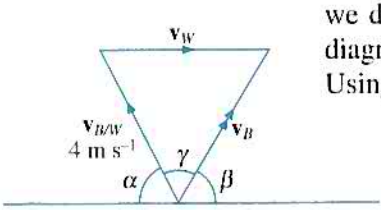
$\mathbf{v}_W = \xrightarrow{3.6 \text{ m s}^{-1}}$

$\mathbf{v}_B = v_B \nearrow$

$\mathbf{v}_{B/W} = 4 \text{ m s}^{-1} \nwarrow$



Using $\mathbf{v}_B = \mathbf{v}_{B/W} + \mathbf{v}_W$
$= \mathbf{v}_W + \mathbf{v}_{B/W}$,
we draw a velocity diagram as shown.



Velocity diagram

Using sine rule on the velocity diagram,

$$\frac{\sin \gamma}{3.6} = \frac{\sin \beta}{4}$$

$$\sin \gamma = 3.6 \times \frac{0.8}{4} = 0.72$$

$\sin \beta = 0.8$ and $\sin \gamma = 0.72 \Rightarrow \beta = 53.1°$ and $\gamma = 46.1°$
Hence $\alpha = 180° - \beta - \gamma =$ **80.8°**.

(b) Using sine rule on the velocity diagram,

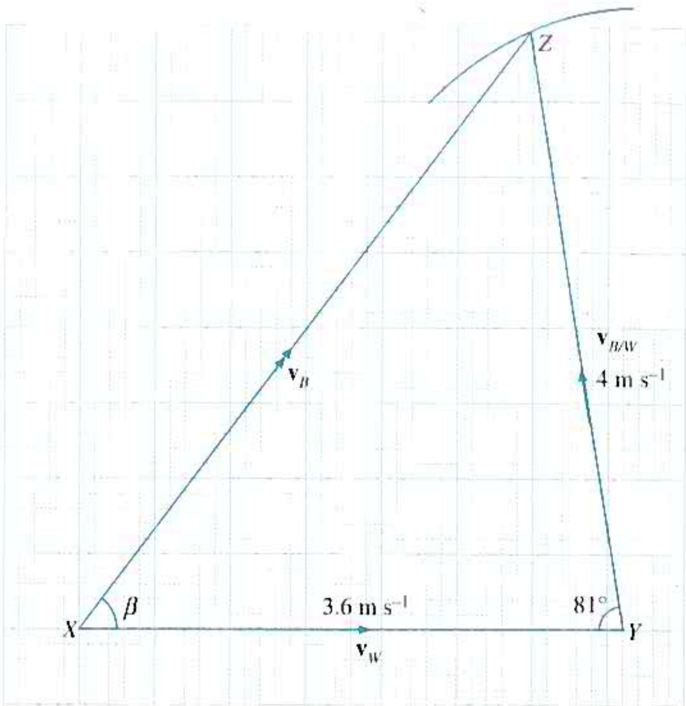
$$\frac{v_B}{\sin 80.8°} = \frac{4}{\sin \beta}$$

$$v_B = \frac{4}{0.8} \times 0.987 = 4.94 \text{ m s}^{-1}$$

(c) Time taken for the crossing $= \dfrac{AB}{v_B} = \dfrac{100}{4.94} =$ **20 seconds (to the nearest second)**

Graphical Method

In Examples 5 and 6, we use the calculation method to find the unknown speeds and directions with the aid of a rough sketch for each velocity diagram. However, we can construct a velocity diagram to scale on graph paper and measure the unknown speeds and directions. This method is most convenient for problems involving more computations.



To illustrate the graphical method, we construct the velocity diagram in Example 7 as shown.

Rewrite the relative velocity equation $\mathbf{v}_B = \mathbf{v}_{B/W} + \mathbf{v}_W$ as $\mathbf{v}_B = \mathbf{v}_W + \mathbf{v}_{B/W}$. It is because $\mathbf{v}_W$ is the only known vector (in magnitude and direction) and it is the first vector to be drawn under the construction of the velocity diagram as shown below.

1. With reference to a rough sketch, we choose the **scale : 2 cm : 1 m s^{-1}**
2. Draw a directed line segment XY of length 7.2 cm to represent $\mathbf{v}_W$.
3. Draw an arc of a circle with centre Y and of radius 8 cm for $\mathbf{v}_{B/W}$ which has unknown direction.
4. Draw a line segment from X in the direction (6 cm $\rightarrow$, 8 cm $\uparrow$) so that the line meets the arc at Z as shown.

In the velocity diagram, $\boldsymbol{YZ}$ represents $\mathbf{v}_{B/W}$ and $\boldsymbol{XZ}$ represent $\mathbf{v}_B$. By measurement, we have **(a)** $\alpha = \angle ZYX \approx 81°$

(b) $XZ \approx 10$ cm and so the speed of B is $v_B \approx \dfrac{10}{2} = 5$ m s^{-1}.

Motion of Aircraft in the Air

In this section, we shall continue our study on the motion of aircraft in a current of air (wind) using the same basic concepts and methods for problem solving. Again, we shall learn through the solutions of the following examples.

Example 8 The speed of a commercial aircraft in still air is 300 km h^{-1}. The wind velocity is 80 km h^{-1} from the west. The aircraft is steered on the course in the direction 060°. Find the true velocity of the aircraft.

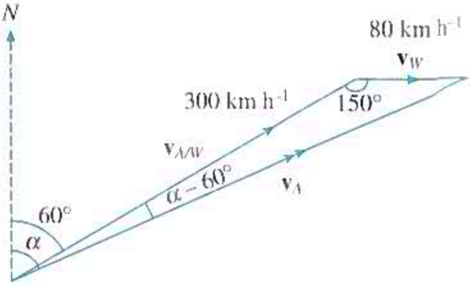
Solution:

Let A denote the aircraft and W the wind (air).

$\mathbf{v}_{A/W} = 300 \text{ km h}^{-1}$ ↗

$\mathbf{v}_W = 80 \text{ km h}^{-1}$ →

Apply $\mathbf{v}_A = \mathbf{v}_{A/W} + \mathbf{v}_W$ to draw a velocity diagram as shown.



Using the cosine rule, the true speed is

$$v_A = \sqrt{300^2 + 80^2 - 2(300)(80)\cos 150^\circ}$$
$$= 371 \text{ m s}^{-1}$$

Using the sine rule,

$$\frac{\sin(\alpha - 60^\circ)}{80} = \frac{\sin 150}{371}$$
$$\alpha - 60^\circ = 6.2^\circ$$
$$\alpha = 66.2^\circ$$

The true velocity of the aircraft is **371 m s^{-1}** on the bearing **066.2°**.

Note: **(1)** The speed of the aircraft is 300 km h^{-1} relative to the (still or moving) air.

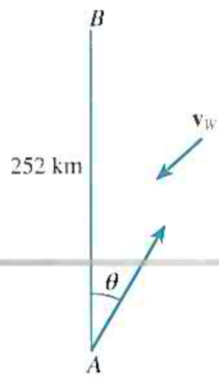
(2) The course taken by the pilot is the direction of $\mathbf{v}_{A/W}$, the direction of the plane relative to the wind.

For any object P moving in the air,
(a) the speed of P in still air is $v_{P/W}$,
(b) the course taken by P is the direction of $\mathbf{v}_{P/W}$ where $\mathbf{v}_{P/W}$ is the velocity of P relative to the wind (air).

Example 9

An aircraft flies due north from A to B where $AB = 252$ km. The wind is blowing from the direction 040° at 85 km h^{-1}. The speed of the aircraft in still air is 350 km h^{-1} and the pilot sets the course on the bearing θ, where θ is acute. Find

(a) θ,
(b) the time taken, in minutes, for the journey from A to B.



Solution:

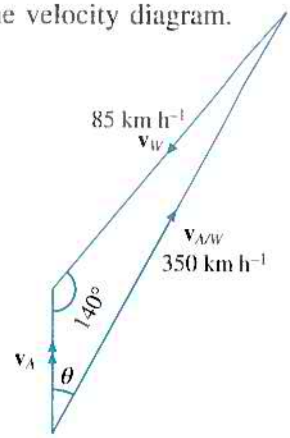
(a) Let W denote the wind and A the aircraft.

$\mathbf{v}_W = 85 \text{ km h}^{-1}$ ↙

$\mathbf{v}_{A/W} = 350 \text{ km h}^{-1}$ ↗

$\mathbf{v}_A = v_A$ ↑ (aircraft flies due north with unknown speed v_A)

Apply $\mathbf{v}_A = \mathbf{v}_{A/W} + \mathbf{v}_W$ to draw the velocity diagram.



Using the sine rule,

$$\frac{\sin \theta}{85} = \frac{\sin 140^\circ}{350}$$

$\sin \theta = 0.156$ and $\theta = 9^\circ$

(b) $$\frac{v_A}{\sin 31^\circ} = \frac{350}{\sin 140}$$

$$v_A = 280.4 \text{ km h}^{-1}$$

$$\text{Time taken} = \frac{AB}{v_A} = \frac{252}{280} \text{ h}$$

$$= \mathbf{54\ min}$$

Example 10

(a) In a wind blowing at 3 m s⁻¹ from the direction 060°, a cyclist travels due west at 4.5 m s⁻¹. Find the velocity of the wind relative to the cyclist.

(b) A boat travels due north at a speed of 20 km h⁻¹. To the man in the boat, the wind appears to blow at 10 km h⁻¹ from the bearing 030°. Find the actual velocity of the wind.

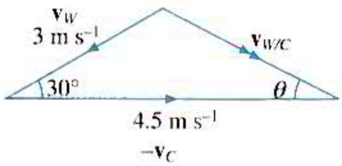
Solution:

(a) Let C denote the cyclist and W the wind.

$\mathbf{v}_C = 4.5 \text{ m s}^{-1}$ ← and $\mathbf{v}_W = 3 \text{ m s}^{-1}$ ↙

Apply $\mathbf{v}_W = \mathbf{v}_{W/C} + \mathbf{v}_C$

$\mathbf{v}_{W/C} = \mathbf{v}_W + (-\mathbf{v}_C)$



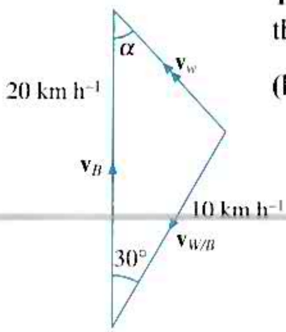
$$v_{W/C} = \sqrt{3^2 + 4.5^2 - 2(3)(4.5)\cos 30^\circ}$$

$$= 2.42 \text{ m s}^{-1}$$

$$\frac{\sin \theta}{3} = \frac{\sin 30^\circ}{2.42}$$

$$\theta = 38.3^\circ \text{ (to 1 dec. pl.)}$$

The velocity of the wind relative to the cyclist is **2.42 m s⁻¹** from the direction **308.3°** (270° + 38.3°).



(b) Let B denote the boat and W the wind.

$\mathbf{v}_B = 20 \text{ km h}^{-1}$ ↑ and $\mathbf{v}_{W/B} = 10 \text{ km h}^{-1}$ ↙

Apply $\mathbf{v}_W = \mathbf{v}_{W/B} + \mathbf{v}_B$ to draw a velocity diagram as shown.

Using cosine rule:

$$v_W = \sqrt{(10)^2 + (20)^2 - 2(10)(20)\cos 30°}$$
$$= 12.4 \text{ km h}^{-1} \quad \text{(to 1 dec. pl.)}$$

Using sine rule:

$$\frac{\sin\alpha}{10} = \frac{\sin 30°}{12.4} \quad \Rightarrow \quad \alpha = 23.8°$$

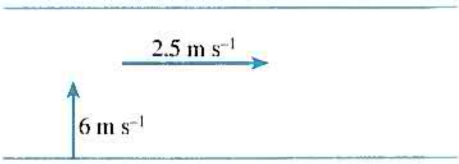
The actual velocity of the wind is **12.4 km h^{-1}** from the bearing **156.2°**.

Exercise 24.2

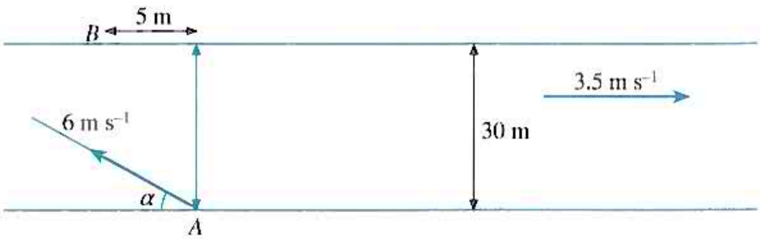
1. A river is flowing at 4 m s^{-1} due south. A boat, whose speed in still water is 3 m s^{-1}, is steered in the direction due east. Find the true speed and direction of the motion of the boat.

2. A river is flowing at 3 m s^{-1} due east. A speedboat, whose speed in still water is 5 m s^{-1}, is steered in the direction on a bearing of 330°. Find the resultant velocity of the speedboat.

3. In the diagram, a river is flowing at a speed of 2.5 m s^{-1} due east. A boat, whose speed in still water is 6 m s^{-1}, is steered in the direction due north. Find the true velocity of the boat.



4. A soldier who swims at 1.2 m s^{-1} in still water wishes to cross a river 20 m wide. The water is flowing between straight parallel banks at 1.8 m s^{-1}. He swims upstream in a direction making an angle of 70° with the bank. Find
   (a) the resultant velocity,
   (b) the time taken for the crossing, to the nearest second.

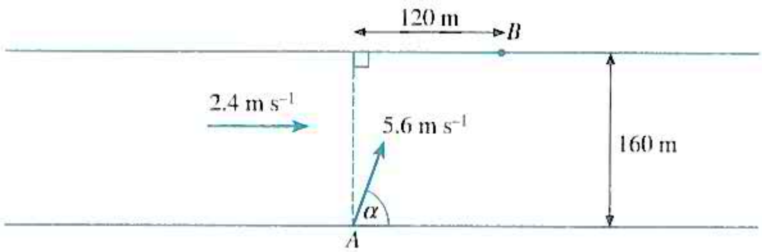
5. 

   The diagram shows a river, 30 m wide, flowing at a speed of 3.5 m s^{-1}, between straight parallel banks. A boat, whose speed in still water is 6 m s^{-1}, crosses the river from a point A on one bank to a point B on the opposite bank, 5 m upstream. In order to travel directly from A to B, the boat is steered in a direction making an angle α to the bank as shown. Find
   (a) the value of α,
   (b) the resultant speed of the boat,
   (c) the time taken for the crossing, to the nearest second.
   In order to make the return journey from B to A, what is the course taken by the boat?

6. The speed of an aircraft in still air is 300 km h^{-1}. The wind velocity is 60 km h^{-1} from the east. The aircraft is steered on the course in the direction 060°. Find the true velocity of the aircraft.

7. An aircraft flies due east from A to B where AB = 200 km. The wind is blowing from the direction 030° at 60 km h^{-1}. The speed of the aircraft in still air is 300 km h^{-1} and the pilot sets the course on the bearing $\theta°$. Find
   (a) the value of θ,
   (b) the time taken, in minutes, for the journey from A to B.

8. **(a)** An aircraft is flying due south at 350 km h^{-1}. The wind is blowing at 70 km h^{-1} from the direction $\theta°$, where $\theta°$ is acute. Given that the pilot is steering the aircraft in the direction 170°, find
   (i) the value of θ,
   (ii) the speed of the aircraft in still air.
   (b) A man who swims at 1.2 m s^{-1} in still water wishes to cross a river which is flowing between straight parallel banks at 2 m s^{-1}. He aims downstream in a direction making an angle 60° with the bank. Find
   (i) the speed at which he travels,
   (ii) the angle which his resultant velocity makes with the bank. *(C)*

9.


The diagram shows a river, 160 m wide, flowing at a speed of 2.4 m s^{-1}, between straight parallel banks. A boat crosses the river from a point A on one bank to a point B on the opposite bank, 120 m downstream. The speed of the boat in still water is 5.6 m s^{-1}. In order to travel directly from A to B, the boat is steered in a direction making an angle α to the bank as shown. Find
(a) the value of α, **(b)** the resultant speed of the boat,
(c) the time taken for the crossing.
The boat then makes the return journey from B to A. Find the resultant speed of the boat on this return journey. *(C)*

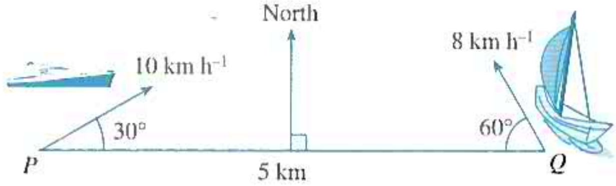
24.3 Relative Motion of Two Moving Objects

For two objects P and Q moving with true velocities $\mathbf{v}_P$ and $\mathbf{v}_Q$ respectively, we shall apply the same relative velocity equation:

$$\mathbf{v}_P = \mathbf{v}_{P/Q} + \mathbf{v}_Q$$
$$\text{or} \quad \mathbf{v}_{P/Q} = \mathbf{v}_P + (-\mathbf{v}_Q)$$

Apparent (relative) Path

Example 11



At a particular instant, two ships P and Q are 5 km apart and move with constant speeds and directions as shown. Find

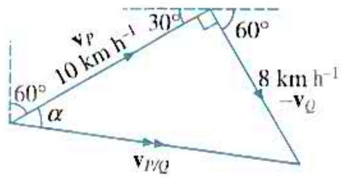
(a) the speed and direction of P relative to Q,

(b) the distance apart, in metres, when P is due south of Q.

Solution:

(a) Using $\mathbf{v}_P = \mathbf{v}_{P/Q} + \mathbf{v}_Q$

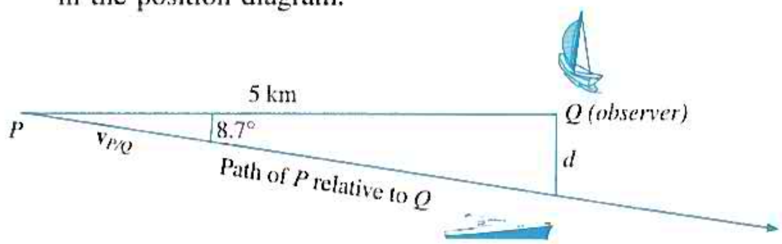
$$\mathbf{v}_{P/Q} = \mathbf{v}_P + (-\mathbf{v}_Q)$$



$$v_{P/Q} = \sqrt{8^2 + 10^2} = \mathbf{12.8\ km\ h^{-1}}$$

$\tan \alpha = \dfrac{8}{10}$, $\alpha = 38.7°$ and so the motion of P relative to Q is in the direction 098.7° (60° + α).

(b) The path of the motion of P relative to Q (observer) is shown in the position diagram.



When the ship P is due south of Q, the distance apart is

$$d = 5 \tan 8.7° \text{ km}$$
$$= \mathbf{765\ m}$$

In the previous example, the path of P observed by Q is known as the **apparent path** and the velocity $\mathbf{v}_{P/Q}$ is in the direction of the motion of P along the path.

Interception

Example 12

At an instant, two cyclists A and B, are 120 m apart with B due east of A. Cyclist A is travelling at 3 m s^{-1} in a direction 030° and cyclist B is travelling at 3.8 m s^{-1} in a direction α. Find

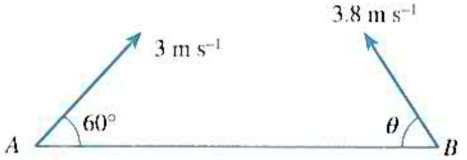
(a) the value of α for which cyclist B should travel in order to intercept cyclist A,

(b) the time taken for the interception to occur.

Assuming that cyclist A is travelling at a speed of 4 m s^{-1} in the same direction and the speed of cyclist B remains unchanged, show that there are two possible directions in which B should travel in order to intercept A.
Indicate the direction in which B should travel in order to intercept A as quickly as possible.

Solution:

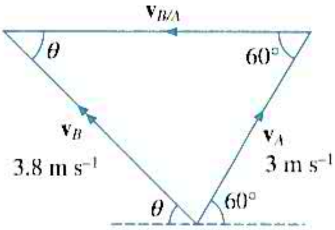
(a) For interception, B appears to move towards A as shown.



Then $\mathbf{v}_{B/A} = v_{B/A} \longleftarrow$

Applying $\mathbf{v}_B = \mathbf{v}_{B/A} + \mathbf{v}_A$
$= \mathbf{v}_A + \mathbf{v}_{B/A}$

we draw a velocity diagram as shown.



$$\frac{\sin\theta}{3} = \frac{\sin 60^\circ}{3.8}$$

$\sin\theta = 0.684$

So $\theta = 43.1^\circ$ and $\alpha = 270^\circ + \theta =$ **313.1°**.

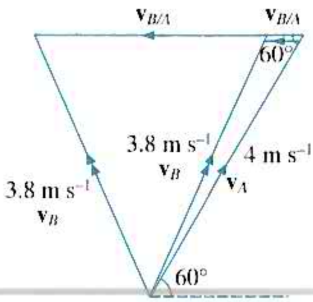
(b) From the velocity diagram,

$$\frac{v_{B/A}}{\sin 76.9^\circ} = \frac{3.8}{\sin 60^\circ}$$

$$v_{B/A} = 4.274 \text{ m s}^{-1}$$

$$\text{Time taken} = \frac{\text{distance apart}}{v_{B/A}} = \frac{120}{4.274} = \textbf{28.1 seconds}$$

Applying $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$ with $v_A = 4$ m s^{-1}, we draw the velocity diagram as shown.



Notice that there are two possible triangles and so there are two possible directions of $\mathbf{v}_B$ as shown.
For interception to occur as quickly as possible, we choose the course of B for which $v_{B/A}$ is larger.

Note: Since $\mathbf{v}_A$ is a known vector, we write $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$ so that we can draw $\mathbf{v}_A$ first.

For two objects P and Q moving in a plane, it is important to obtain the information:

(i) $\overrightarrow{QP}$, the initial position vector of P relative to Q

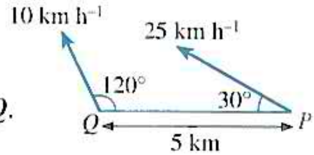
(ii) $\mathbf{v}_{P/Q}$, the velocity of P relative to Q

$\mathbf{v}_{P/Q}$ and $\overrightarrow{QP}$ are in opposite directions
$\Leftrightarrow$ To Q (observer), P appears to move towards Q
$\Leftrightarrow$ P will intercept Q

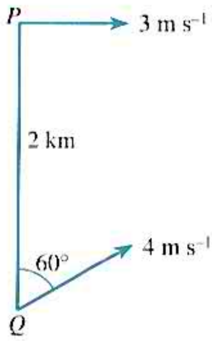
Exercise 24.3

1. Two particles P and Q, are 30 m apart with Q due north of P. Particle Q is moving at 5 m s^{-1} in a direction 090° and P is moving at 7 m s^{-1} in a direction 030°. Find
(a) the magnitude and direction of the velocity of Q relative to P,
(b) the time taken for Q to be due east of P, to the nearest second.

2. At a particular instant, two ships P and Q are 5 km apart. P is due east of Q and they move with constant speeds and directions as shown. Find
(a) the speed and direction of P relative to Q,
(b) the distance apart, in metres, when P is due north of Q.

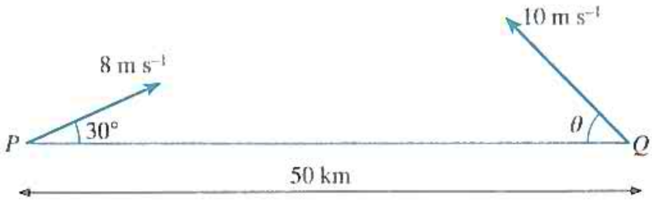


3. At a particular instant, two boats P and Q are 2 km apart and P is due north of Q. The ships move with constant speeds and directions as shown. Find
(a) the speed and direction of Q relative to P,
(b) the distance apart, in metres, when Q is due east of P.



4. At a particular moment, two ships A and B, are 5 km apart with A due west of B. Ship A is sailing due south at 5 km h^{-1} and ship B is sailing due west at 8 km h^{-1}. Find
(a) the velocity of A relative to B,
(b) the distance between the two ships when A is on the bearing of 225° from B.

5. Two aircraft A and B fly at the same height with constant velocities. At noon, aircraft B is 50 km due east of aircraft A and is flying due west at 450 km h^{-1}. Aircraft A is flying on the bearing 120° at 300 km h^{-1}. Find
(a) the velocity of B relative to A,
(b) the time when B is due north of A.

6. 

In the diagram, two particles P and Q, moving with speeds 8 m s^{-1} and 10 m s^{-1} respectively, leave simultaneously when they are 50 km apart with P due west of Q. Particles P and Q are moving in the directions as shown. Given that P and Q are on the path of collision, find

(a) the value of θ,

(b) the time that elapses before the collision, to the nearest second.

7. At a given instant, an airship is moving due north with a speed of 6 m s^{-1}. A helicopter which is 500 m due east of the airship flies at a speed of 12 m s^{-1} and steers on a bearing θ in order to intercept the airship. Find

(a) the value of θ, **(b)** the time that elapses before interception.

8. Two particles, A and B, are 50 m apart with A due north of B. Particle A is travelling at 10 m s^{-1} in a direction 075° and B is travelling at V m s^{-1} in a direction 015°.

(a) Given that $V = 20$, find

(i) the magnitude and direction of the velocity of B relative to A,

(ii) the time taken for B to be due west of A.

(b) Given that B collides with A, find

(i) the value of V, **(ii)** the time taken for B to collide with A. *(C)*

Important Notes

1. $\mathbf{v}_P$: True (actual) velocity of a moving object P relative to the Earth.
   v_P : True speed (ground speed) of P.

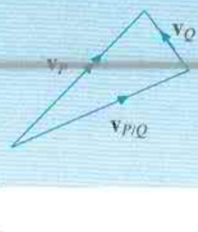
2. $\mathbf{v}_{P/Q}$: Relative (apparent) velocity of a moving object P relative to a moving object Q (observer).
   $\mathbf{v}_{Q/P}$: Relative (apparent) velocity of a moving object Q relative to a moving object P (observer).

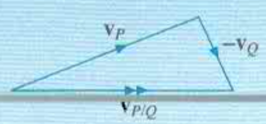
$$\mathbf{v}_{P/Q} = -\mathbf{v}_{Q/P}$$

3. **Relative velocity equation**

$$\mathbf{v}_P = \mathbf{v}_{P/Q} + \mathbf{v}_Q \quad \text{or} \quad \mathbf{v}_{P/Q} = \mathbf{v}_P + (-\mathbf{v}_Q)$$

Apply Triangle Law of Addition to draw velocity diagram.

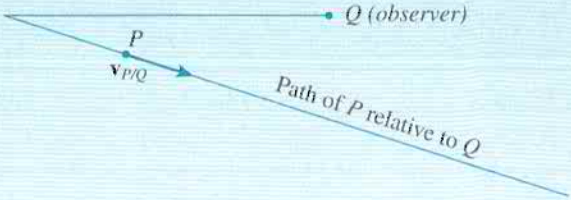




4. **Motion in the water**
   $\mathbf{v}_W$: True velocity of W (water)
   $\mathbf{v}_{P/W}$: Apparent velocity of an object P relative to W (still or moving water).
   (a) $v_{P/W}$ remains unchanged
   (b) course taken by P is the direction of $\mathbf{v}_{P/W}$

5. **Motion in the air**
   $\mathbf{v}_W$: True velocity of W (wind or air)
   $\mathbf{v}_{P/W}$: Apparent velocity of an object P relative to W (still air or wind)
   (a) $v_{P/W}$ remains unchanged
   (b) course taken by P is the direction of $\mathbf{v}_{P/W}$

6. **Apparent path of P relative to an observer Q**

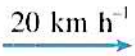


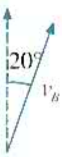
7. **Interception (collision)**
   Two moving objects P and Q will intercept each other $\Leftrightarrow$ $\mathbf{v}_{P/Q}$ is in the direction $\overrightarrow{PQ}$.

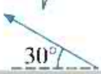
Miscellaneous Examples

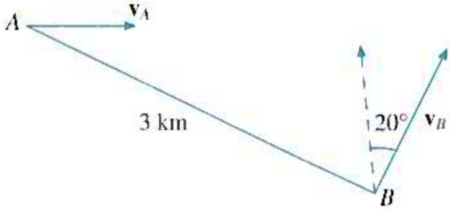
Example 13 A ship A is travelling at a speed of 20 km h^{-1} due east. At a certain instant, the captain of ship A sights another ship B which is 3 km away on the bearing 120° and moving in the direction of bearing 020°. The captain senses that they are on the path of collision. Find the speed of B relative to A and the time that elapses before collision.

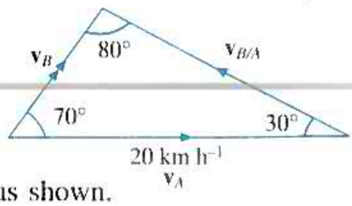
Solution:

$\mathbf{v}_A =$ 

$\mathbf{v}_B =$ 

$\mathbf{v}_{B/A} =$ 





Apply $\mathbf{v}_B = \mathbf{v}_{B/A} + \mathbf{v}_A$
$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$
to construct the velocity diagram as shown.

$$\frac{v_{B/A}}{\sin 70°} = \frac{v_A}{\sin 80°}$$

$$v_{B/A} = 20 \times \frac{\sin 70°}{\sin 80°}$$

$$= \mathbf{19.1\ km\ h^{-1}}$$

The time that elapses before collision $= \dfrac{\text{initial distance between } A \text{ and } B}{v_{B/A}}$

$$= \frac{3}{19.1}\ \text{h}$$

$$= \mathbf{9.42\ min}$$

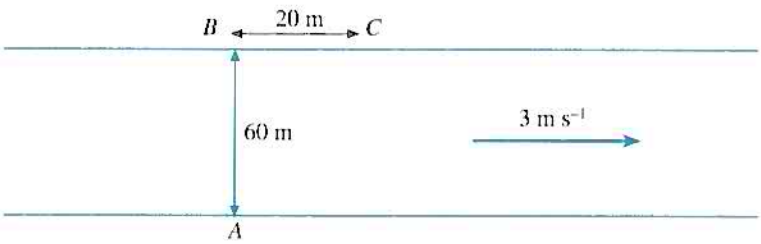
Miscellaneous Exercise 24

1. To an observer on a train which is heading in the direction 135° at 50 km h^{-1}, a car 3 km due east appears to be travelling due north at 20 km h^{-1}. Find
(a) the actual speed and direction of the car,
(b) the distance apart of the train and the car after 3 minutes.

2. To an officer on a patrol boat which is sailing due south at 20 km h^{-1}, a ship 1 km due west seems to travel on the bearing 150° at 20 km h^{-1}. Find
(a) the actual speed and direction of the ship,
(b) the distance apart when the ship is due south of the boat.

3. A wind is blowing from the direction 060° at 20 km h^{-1}. Find the speed and direction of the velocity of the wind relative to a boy who is cycling due north at 30 km h^{-1}. At a certain instant, a boy makes a right turn and cycles due east with the same speed, find the speed and direction of the velocity of the wind relative to the boy.

4.



The diagram shows a river 60 m wide, flowing at 3 m s^{-1} between straight banks. Point B is directly opposite point A and C is 20 m downstream from B. A canoeist, who can paddle with 4 m s^{-1} in still water, travels directly from A to C. Find
(a) the angle to the bank at which the canoeist should steer,
(b) the time taken to travel from A to C, to the nearest second.

If the canoeist crosses the river from A in the shortest possible time, find
(c) the time taken, to the nearest second,
(d) the distance of the landing point from B.

5. A plane is scheduled to fly due north from Singapore to Bangkok. Bangkok is 1400 km from Singapore. There is a wind blowing from the bearing 060° at constant speed of 96 km h^{-1}. Given that the speed of the plane in still air is 800 km h^{-1}, find
   (a) the course set by the pilot, (b) the flight time, to the nearest minute.

   On the return journey, the wind is blowing at the same speed from the bearing 030° and the speed of the plane in still air remains unchanged, find
   (c) the course set by the pilot, (d) the flight time, to the nearest minute.

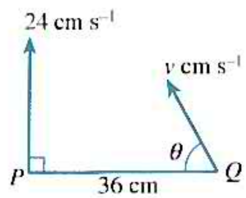
6. Two small marbles A and B are 160 cm apart with B due east of A. Marble A is moving at 8 cm s^{-1} in a direction 030° and marble B is moving at 12 cm s^{-1} in the direction 330°. Find, by calculation or drawing,
   (a) the velocity of B relative to A,
   (b) the time taken for B to be due north of A,
   (c) the distance between A and B when B is on the bearing 045° from A.

7. A ship A is steaming at 15 km h^{-1} in the direction 045° and is 10 km due west of another ship B.
   (a) Given that the speed of B is 20 km h^{-1}, find, by drawing or by calculation, the direction in which B must travel in order to intercept A and the time taken, to the nearest minute.
   (b) Given that the speed of B is 12 km h^{-1}, find, by drawing or by calculation, the two possible directions in which B must travel in order to intercept A.

8. (a)

   The diagram shows a river flowing at 1.5 m s^{-1} between parallel banks. A rower, whose speed through still water is 2 m s^{-1}, rows from A to the point B, immediately opposite A. The rower points the boat in a direction inclined at an angle α to the bank. Find the value of α.
   Given that AB = 120 m, find the time taken for the crossing, to the nearest second.

   (b) At a particular instant, particles P and Q are 36 cm apart and are moving in a horizontal plane with constant speeds and directions as shown in the diagram.



   Given that they collide 3 seconds later, find
   (i) the velocity, in magnitude and direction, of P relative to Q,
   (ii) the value of v and of θ.

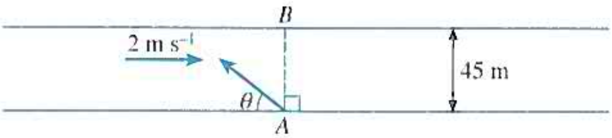
   (c) A car is travelling at 80 km h^{-1} in a direction due north, and the wind is blowing at 30 km h^{-1} from a direction 240°. Find the angle that a streamer, tied to the car roof, makes with the direction of motion of the car. *(C)*

9. **(a)** Two trains, A and B, are 10 km apart with B due east of A. Train A is travelling at 120 km h^{-1} along a straight track in a direction 045°. Train B is travelling at 100 km·h^{-1} along a straight track in a direction 300°. Find
   (i) the magnitude and direction of the velocity of A relative to B,
   (ii) the time taken for A to be due north of B.
   (b) A motorcyclist travels due west at 100 km h^{-1}. To the motorcyclist the wind appears to blow at 80 km h^{-1} from 300°. Find the magnitude and direction of the actual velocity of the wind. *(C)*

10. **(a)** The speed of an aircraft in still air is 300 km h^{-1}. In a wind of speed 140 km h^{-1}, blowing from a bearing of 230°, the aircraft flies due east. Find
   (i) the course set by the pilot,
   (ii) the time taken for the aircraft to travel 100 km.
   (b) To a cyclist travelling due east at 18 km h^{-1} the wind appears to be blowing at 12 km h^{-1} from a bearing of 150°. Find the true wind-speed and the direction from which the wind is blowing. *(C)*

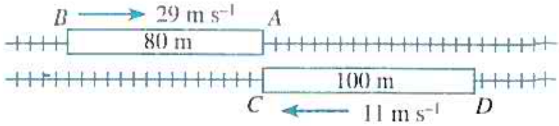
11. **(a)** A plane, whose speed in still air is 320 km h^{-1}, flies directly from London to Brussels, a distance of 320 km. The bearing of Brussels from London is 110° and there is a wind of 120 km h^{-1} blowing from the west. Find
   (i) the course set by the pilot,
   (ii) the time, in minutes, for the flight.
   (b) At 08 00 hours, a coastguard station receives a distress signal from a tanker, which is at a distance of 16 km on a bearing of 090°. The tanker is travelling at 12 km h^{-1} on a bearing of 315°. A lifeboat is immediately launched from the coastguard station to intercept the tanker. This lifeboat travels at constant speed in a straight line and intercepts the tanker at 08 20 hours. Calculate the speed of the lifeboat. *(C)*

12. **(a)** The diagram shows a river 45 m wide which flows at 2 m s^{-1}. A motor boat, of speed 2.5 m s^{-1} in still water, leaves a point A to cross the river. Given that the boat is steered at an angle θ to the bank as shown, find, by drawing or by calculation,
   (i) the value of θ for which the boat travels directly to B,
   (ii) the time taken for the journey from A to B.



   (b) At 08 00 hours, a lifeboat which has a maximum speed of 50 km h^{-1}, is 30 km to the west of a fishing boat. The fishing boat is sailing at 15 km h^{-1} on a bearing of 320°.
   Find, by drawing or by calculation,
   (i) the bearing on which the lifeboat should travel in order to intercept the fishing boat as quickly as possible,
   (ii) the time, to the nearest minute, at which the interception would occur. *(C)*

13. **(a)** The diagram shows two trains moving towards each other on parallel tracks; train AB of length 80 m is travelling at 29 m s^{-1} and train CD of length 100 m is travelling at 11 m s^{-1}.



Write down the velocity of train AB relative to train CD. Hence find the time taken from the instant when A and C are adjacent to the instant when B and D are adjacent.

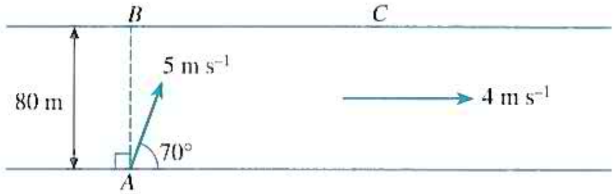
(b) Two towns S and T are 360 km apart and are such that T is on a bearing of 070° from S. A plane whose speed in still air is 300 km h^{-1} flies directly from S to T. Given that there is a wind blowing from north at 60 km h^{-1}, find

(i) the direction in which the pilot must steer the aircraft,

(ii) the time taken, to the nearest minute, for the journey.

When the aircraft returns from T to S, the speed of the wind has changed, but it is still blowing from the north. Given that the pilot steers the plane on a bearing of 270°, find the new speed of the wind. *(C)*

14. **(a)** A river flows between two parallel banks at a speed of 4 m s^{-1}. A boat, whose speed through still water is 5 m s^{-1}, leaves a point A, heading in a direction making an angle of 70° with the bank, as shown in the diagram.



The point B is directly opposite A on the other bank and AB = 80 m. At time t seconds after leaving A, the boat reaches the bank at the point C. Find

(i) the value of t, **(ii)** the distance BC,

(iii) the angle that the direction of motion of the boat makes with the bank.

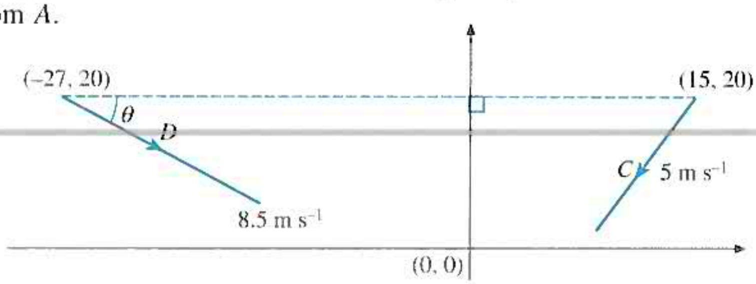
(b) An aircraft is travelling due north at 360 km h^{-1}. The pilot is steering a course of 350° and the aircraft's speed is 300 km h^{-1}. Find the speed of the wind and the direction from which it is blowing. *(C)*

15. **(a)** A straight horizontal moving walkway travels at 2 m s^{-1} in a direction from the fixed point A towards the fixed point B. A man P walks from A to B, on the moving walkway, at a speed of 1.5 m s^{-1} relative to the walkway. A man Q walks from B to A, on fixed horizontal ground alongside the walkway, at a speed of 1.6 m s^{-1}.

Calculate the speed of P relative to Q.

The distance AB is 200 m and P and Q pass each other at a point halfway between A and B. Find the time between Q's departure from B and P's departure from A.

(b)

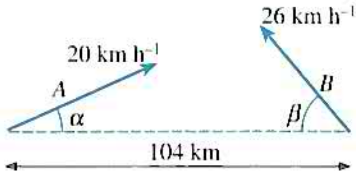


A footballer C runs at a speed of 5 m s^{-1}, with the ball close to his feet, in a straight line towards the point (0, 0). At the instant when he passes through the point (15, 20), another footballer D starts to run in a straight line, at 8.5 m s^{-1}, from the point (–27, 20) at an angle θ as shown in the diagram. Given that D intercepts the ball, and that the given coordinates are in metres, find

(a) the value of θ, **(b)** the velocity of D relative to C,

(c) the time taken by D to intercept the ball. *(C)*

16. Two ships, A and B, leave their ports simultaneously at 12 00 hours. The ports are 104 km apart with one port due west of the other. The speeds and directions of the two ships are shown in the diagram, where $\tan \alpha = \frac{3}{4}$ and $\tan \beta = \frac{12}{5}$. Find



(a) the speed and direction of the velocity of A relative to B,

(b) the time at which A is due south of B and the distance between A and B at this instant,

(c) the distance between the two ships at 17 00 hours. *(C)*

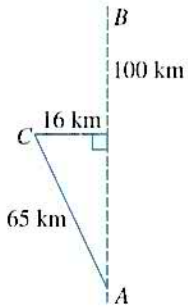
17. A tanker, travelling due north at 11 km h^{-1}, observes a liner 8 km due east at 07 00 hours. The liner is travelling at 20 km h^{-1} on a bearing of 037°. Find

(a) the magnitude and direction of the velocity of the liner with respect to the tanker

(b) the distance between the two ships at 08 00 hours.

Later, the tanker, still travelling due north at 11 km h^{-1}, sights a power-boat which appears to be travelling due west at 60 km h^{-1}. Find the magnitude and direction of the actual velocity of the power-boat. *(C)*

***18.** An aircraft leaves A to fly to B which is 100 km due North of A. The pilot sets a course due North but after 15 minutes he realises that, owing to a wind blowing from due East, the plane is at a point C, where C is 65 km from A and 16 km West of A. Find



(a) the speed, in km h^{-1}, of the wind,

(b) the speed, in km h^{-1}, of the aircraft in still air,

(c) the course the pilot should have set from A in order to have arrived directly at B.

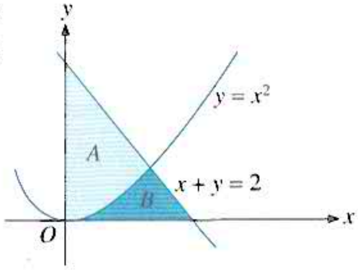
Find also the course the pilot should set from C in order to fly directly to B and, in this case, the time, to the nearest minute, taken for the journey from A to B. *(C)*

Revision Exercise 17

1. (a) Find the equation of the curve which passes through (4, –2) and for which $\frac{dy}{dx} = \frac{1}{\sqrt{x}} - 2x$.
   (b) Find
   (i) $\int (2x - 1)^3 \, dx$, (ii) $\int \sqrt{4 - x} \, dx$.
   (c) Integrate with respect to x
   (i) $e^{2x} + x$, (ii) $\sin 2x + \cos 3x$.

2. (a) Evaluate
   (i) $\int_{\frac{1}{2}}^{11} \sqrt{2x + 3} \, dx$, (ii) $\int_0^4 \frac{4}{\sqrt{2x + 1}} \, dx$.
   (b) Show that, if $y = \frac{x}{\sqrt{1 + x^2}}$, then $\frac{dy}{dx} = \frac{1}{(1 + x^2)\sqrt{1 + x^2}}$. Hence evaluate
   $$\int_0^2 \frac{1}{(1 + x^2)\sqrt{1 + x^2}} \, dx.$$

3. The diagram shows the shaded regions enclosed by the curve $y = x^2$, the line $x + y = 2$, the x-axis and the y-axis. Find
   (a) the area of the shaded region A,
   (b) the area of the shaded region B.



4. (a) Three collinear points A, B and C have position vectors $3\mathbf{p} - \mathbf{q}$, $\lambda\mathbf{p} + 2\mathbf{q}$ and $\mathbf{p} + 5\mathbf{q}$ respectively, relative to an origin O, and $\mathbf{p}$ is not parallel to $\mathbf{q}$. Find the value of λ and the ratio $AB : AC$.
   (b) Find the relationship between p and q if
   (i) the vector $\begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ has a magnitude of $2\sqrt{5}$.
   (ii) the vector $\begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} -1 \\ 5 \end{pmatrix}$ is parallel to the vector $\begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} 3 \\ -3 \end{pmatrix}$.

5. The position vectors of A and B relative to an origin O are $\mathbf{a}$ and $\mathbf{b}$ respectively. The line OA is produced to P where $2\overrightarrow{OP} = 3\overrightarrow{OA}$ and OB is produced to Q where $\overrightarrow{OQ} = 2\overrightarrow{OB}$. The lines AQ and BP meet at X. Express $\overrightarrow{AQ}$ and $\overrightarrow{BP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. Given that $\overrightarrow{AX} = \lambda\overrightarrow{AQ}$ and $\overrightarrow{BX} = \mu\overrightarrow{BP}$, express $\overrightarrow{OX}$ in terms of
   (a) λ, $\mathbf{a}$ and $\mathbf{b}$, (b) μ, $\mathbf{a}$ and $\mathbf{b}$.
   Hence determine the values of λ and μ and find the position vector of X.

6. A particle P moves in a straight line so that its displacement, s m, from a fixed point A is given by $s = 3t^2 - t^3$, where t is the time in seconds after leaving A. Calculate
   (a) the time when it is next at A,
   (b) the values of t when it is instantaneously at rest,
   (c) the acceleration when $t = 3$,
   (d) the total distance travelled during the first 4 seconds,
   (e) the maximum velocity of P.

7. **(a)** A motorist starting a car from rest accelerates uniformly to a speed of v m s^{-1} in 10 seconds. He maintains this speed for another 40 seconds and then applies the brakes and decelerates uniformly to rest. His deceleration is numerically equal to twice his previous acceleration.
   (i) Sketch a velocity-time graph for his motion.
   (ii) Calculate the time during which deceleration takes place.
   (iii) Given that the total distance moved is 855 m, calculate the value of v.
   (iv) Calculate the initial acceleration.
   (b) A stone is released from rest at a height of 80 m above the ground. Calculate the time taken for it to hit the ground. What is the speed with which it hits the ground?

8. A straight horizontal moving walkway travels at 1.5 m s^{-1} in a direction from the fixed point A towards the fixed point B. At the same instant, a woman P walks from A to B on the moving walkway, at a speed of 1 m s^{-1} and a man Q walks from A to B, on a fixed horizontal ground alongside the walkway, at a speed of 1.8 m s^{-1}. Calculate the speed of P relative to Q.
   Given that the distance AB is 90 m, find the time taken for Q to reach B and the distance between P and Q at that instant. (Assume that the woman P continues to walk at the same speed after she reaches B.)

Revision Exercise 18

1. **(a)** Find
   (i) $\int \sqrt{x}(2x + 1)\, dx$, **(ii)** $\int \frac{1}{(1 - 2t)^2}\, dt$.
   (b) Given that $\frac{dy}{dx}$ is directly proportional to the square root of x and that $y = 9$ and $\frac{dy}{dx} = 3$ when $x = 4$, find the value of y when $x = 9$.

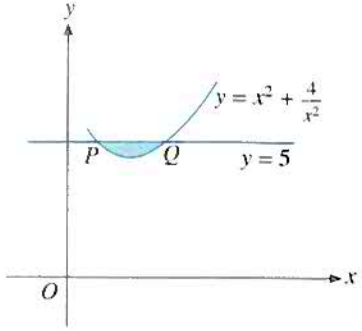
2. **(a)** Given that $\int_2^5 f(x)\, dx = 12$ and $\int_0^5 f(x)\, dx = 4$, evaluate
   (i) $\int_0^2 f(x)\, dx$, **(ii)** $\int_2^5 [2x - f(x)]\, dx$, **(iii)** $\int_0^5 [2f(x) - 1]\, dx$.
   Find the value of m for which $\int_0^2 [f(x) + mx^2]\, dx = 0$.

(b) Show that $\frac{d}{dx}\left[(x-1)\sqrt{2x+1}\right] = \frac{3x}{\sqrt{1+2x}}$. Hence evaluate $\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx$.

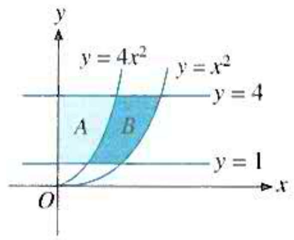
3. (a) The shaded region bounded by the curve $y = x^2 + \frac{4}{x^2}$ and the line $y = 5$ is shown in the diagram. Find

(i) the coordinates of P and Q,

(ii) the area of the shaded region.



(b) Calculate the ratio of the shaded areas A and B shown in the diagram.



4. (a) Find

(i) $\int \frac{1 + \cos 6x}{2}\,dx$ (ii) $\int \frac{e^{5x}\left(e^{1-x}\right)^2}{e^{x+3}}\,dx$.

(b) A particle P travels in a straight line from a fixed point O such that its acceleration, a m s^{-2}, is given by $a = 6t + k$, where t is the time in seconds after it passes O and k is a constant. If the velocity of P is -16 m s^{-1} after 1 second and P comes to instantaneous rest after 5 seconds, find

(i) the value of k,

(ii) the distance travelled in the first 6 seconds.

5. Relative to an origin O, the position vectors of A, B, C and D are $2\mathbf{i} + \mathbf{j}$, $-\mathbf{i} + 5\mathbf{j}$, $4\mathbf{i} + 3\mathbf{j}$ and $p\mathbf{i} - \mathbf{j}$ respectively. Find

(a) the value of m and of n such that $m\overrightarrow{OA} + n\overrightarrow{BC} = \mathbf{i} + 5\mathbf{j}$,

(b) the value of p such that $\overrightarrow{OB}$ is parallel to $\overrightarrow{CD}$,

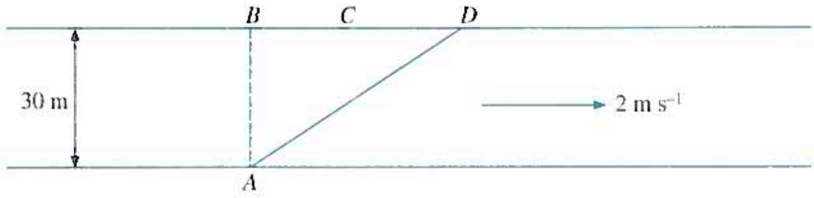
(c) the value of p such that $ABCD$ is a parallelogram,

(d) the values of p such that $|\overrightarrow{AD}| = \sqrt{13}$.

6. Relative to an origin O, the position vectors of two points A and B are $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$ respectively. The point C is on AB. Given that the position vector of C is $\begin{pmatrix} 4 - \lambda \\ 1 - 2\lambda \end{pmatrix}$, find the value of λ and the ratio $AC : CB$.

7. (a) A car accelerates uniformly from rest to a speed of 30 m s^{-1} in T s. It maintains this speed for a further $5T$ s. Sketch a velocity-time diagram for this motion. Given that the total distance travelled is 825 m, find
   (i) the value of T,
   (ii) the acceleration of the car in the first stage of the motion.
   (b) (Optional) A particle starts from rest at a point A and moves in a straight line with an acceleration of 0.8 m s^{-2}. Ten seconds later a second particle is also projected from A in the same direction at a constant speed of 16 m s^{-1}. Find
   (i) the distance between the two particles 15 seconds after the first started moving,
   (ii) the distance from A to the point at which the second particle collides with the first.

8.



Two soldiers P and Q who swim at 1.2 m s^{-1} in still water leave A to cross a river 30 m wide as shown. The water is flowing between straight parallel banks at 2 m s^{-1}.
   (a) P swims directly towards B and arrives at D. Find the time taken for the journey and the distance BD.
   (b) Q swims at an angle θ with the bank and arrives at C which is 10 m from D. Find the value of θ and the time taken to cross the river.
   Which soldier crosses the river first?

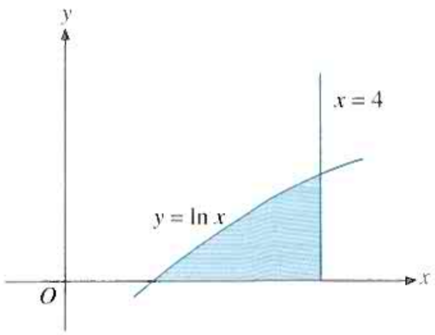
Revision Exercise 19

1. (a) The curve for which $\dfrac{dy}{dx} = \dfrac{2x - k}{\sqrt{x}}$, where k is a constant, has a stationary point at (1, 2). Find
   (i) the value of k,   (ii) the equation of the curve.
   (b) Find
   (i) $\displaystyle\int \left(2 + \sin \frac{1}{2}x\right) dx$,   (ii) $\displaystyle\int \frac{x - 1}{\sqrt{x} + 1}\, dx$.

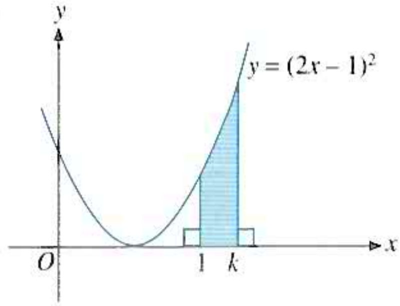
2. (a) Show that $\dfrac{d}{dx}(\tan x - x) = \tan^2 x$. Hence evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x \, dx$.
   (b) A function f is defined by $f(x) = 2x$ for $0 \leqslant x \leqslant 1$ and $f(x) = 4 - x$ for $1 < x \leqslant 2$. Sketch the graph of f and show that $\displaystyle\int_0^2 f(x)\, dx = 3\frac{1}{2}$.

3. (a) The diagram shows the shaded region bounded by the curve $y = \ln x$, the x-axis and the line $x = 4$.

Show that $\frac{d}{dx}[x(\ln x - 1)] = \ln x$ and hence find the area of the shaded region.



(b) In the diagram, the area of the shaded region is $20\frac{2}{3}$ square units. Find the value of k.

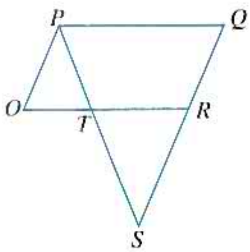


4. In the figure, $OPQR$ is a parallelogram. The point S lies on QR produced so that $RS = \lambda QR$ and PS intersects OR at T so that $OT = \mu OR$. Given that $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OR} = \mathbf{r}$, express

(a) $\overrightarrow{PT}$ in terms of $\mathbf{p}$, $\mathbf{r}$ and μ,

(b) $\overrightarrow{PT}$ in terms of $\mathbf{p}$, $\mathbf{r}$ and λ.

Hence deduce that $1 = \mu(1 + \lambda)$.



5. A particle moves in a straight line so that t seconds after passing a fixed point O, its velocity, v m s^{-1}, is given by $v = 5\left(1 - \frac{1}{2}e^{-2t}\right)$. Calculate

(a) the initial velocity of the particle,

(b) the acceleration of the particle when $t = \frac{1}{2}$,

(c) the displacement of the particle from O when $t = 2$.

6. (Optional) A particle A moves in a straight line with uniform acceleration from a point O towards a point P. After 3 seconds it is 33 m from O and after 5 seconds it 75 m from O. Calculate

(a) the initial speed and the acceleration,

(b) the speed when A is 52 m from O.

At the instant that A moves from O, a particle B leaves P and travels towards O at a constant speed of 12 m s^{-1}. Given that $OP = 174$ m, calculate

(c) the time it will take for A and B to collide,

(d) the speed of A at the moment of collision.

On the same velocity-time diagram, illustrate how the velocities of the two particles change with time up to the moment when they collide.

7. **(a)** A commercial aircraft, whose speed in still air is 800 km h^{-1}, flies directly from A to B, a distance of 1420 km. B is due north of A. There is a wind of 80 km h^{-1} from the bearing 045°. Find
   (i) the course set by the pilot,
   (ii) the time, in hours and minutes, for the flight.

   (b) At 08 00 hours, a patrol boat detects a speedboat that is at a distance of 16 km on a bearing of 120°. The speedboat is travelling at 18 km h^{-1} due north. The patrol boat leaves immediately and travels at constant speed in a straight line. The patrol boat intercepts the speedboat at 08 30 hours. Calculate the speed and the course of the patrol boat.

Revision Exercise 20

1. **(a)** Find the equation of the curve which crosses the y-axis at a point where $y = 2$ and for which $\frac{dy}{dx} = 2e^{3x}$.

   (b) Find

   (i) $\int \sin\left(2x - \frac{\pi}{3}\right) dx$, **(ii)** $\int (3 - 2x)^{-2}\, dx$.

2. **(a)** Evaluate, to 3 significant figures,

   (i) $\int_0^{\frac{\pi}{4}} (2 + \tan^2 x)\, dx$, **(ii)** $\int_1^2 \left(1 + \frac{2}{\sqrt{x}}\right) dx$.

   (b) Differentiate $\cos^3 2x$ with respect to x and hence find

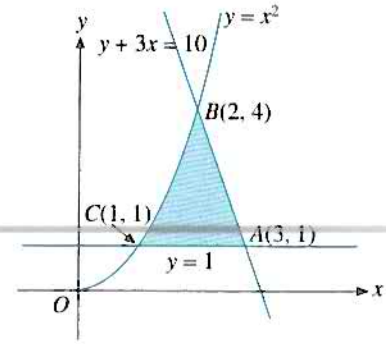
   (i) $\int \cos^2 2x \sin 2x\, dx$, **(ii)** $\int \sin^3 2x\, dx$.

3. **(a)** Find $\int_0^1 e^x e^{2x}\, dx$.

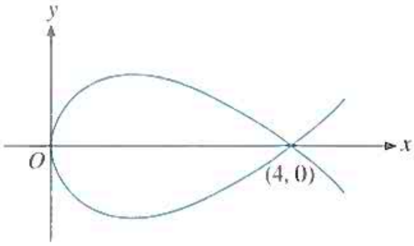
   (b) If $\int e^{2x} f(x)\, dx = e^{2x} \cos 3x + c$, find $f(x)$.

   (c) Differentiate $x^2 \ln x - x$ with respect to x. Hence evaluate $\int_1^2 x \ln x\, dx$.

4. **(a)** The diagram shows part of the curve $y = x^2$. The straight line $y + 3x = 10$ cuts the line $y = 1$ at $A(3, 1)$ and the curve at $B(2, 4)$. The line $y = 1$ cuts the curve at $C(1, 1)$. Calculate the area of the shaded region.



(b) The figure shows the loop of the curve $y^2 = x(x-4)^2$. Find the area of the loop.



5. Points A and B have position vectors **a** and **b** relative to O. Point C is such that $\overrightarrow{BC} = \frac{2}{3}\overrightarrow{OA}$. The lines OC and AB meet at D such that $OD = \lambda OC$ and $AD = \mu AB$.

   Express $\overrightarrow{OD}$ in terms of
   (a) λ, **a** and **b**, (b) μ, **a** and **b**.

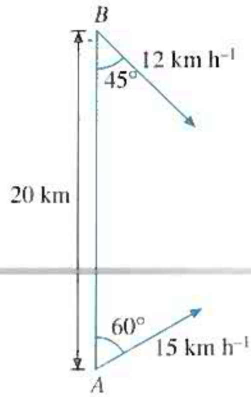
   Hence evaluate λ and μ, and find the ratio $\frac{AD}{DB}$.

6. A particle moves in a straight line and passes a fixed point O on the line so that, t seconds after passing O, its velocity, v m s^{-1}, is given by $v = 3t^2 + 2(1-t)^2$. Calculate
   (a) the initial velocity of the particle,
   (b) the acceleration of the particle when $t = 2$,
   (c) the displacement of the particle from O when $t = 3$.

7. (Optional) A helicopter, initially at rest on the ground, rises vertically with constant acceleration. When it is at a height of 80 m, its upward speed is 8 m s^{-1}. When it is at a height of 320 m, and still rising, an object A is released from the helicopter.
   (a) Calculate the initial velocity of A.
   (b) Calculate, to the nearest 0.1 s, the time that A takes to reach the ground.
   (c) Sketch the displacement-time graph for the motion of A from its release to its reaching the ground.

   After A is released the helicopter continues to rise with a different constant acceleration. When it is at a height of 520 m, it is rising with a speed of 24 m s^{-1}.
   (d) Find the value of this new acceleration and the time when it reaches this height.
   (e) Sketch the velocity-time graph for the motion of the helicopter.

8. At 12 00 hours, two ships, A and B, are 20 km apart with A due south of B. The speeds and directions of the two ships are shown in the diagram. Find
   (a) the speed and direction of the velocity of A relative to B,
   (b) the time at which A is due east of B and the distance between A and B at this instant,
   (c) the distance between the two ships at 13 30 hours.



PAPER 1 (2 hours)

Answer **all** *the questions.*

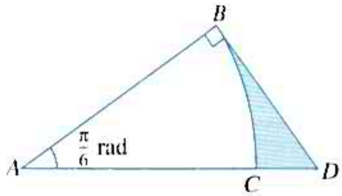
1. Solve the simultaneous equations $xy + y = 4$ and $3x + 2y = 11$. [4]

2. Three points have coordinates $A(-3, 5)$, $B(1, -2)$ and $C(5, \lambda)$. Find, in terms of λ, the coordinates of M, the midpoint of BC. [2]
Find the values of λ for which AM is perpendicular to BC. [2]

3. Find, in ascending powers of x, the first four terms of the expansion $(1 - 3x)^7$.
Hence obtain the coefficient of x^2 in the expansion of $\left(2x - 1 - \frac{7}{9x}\right)(1 - 3x)^7$. [5]

4. Seven friends queue up for a buffet. How many possible queues are there? [2]
After the buffet, they decide to split into two groups, of sizes 4 and 3, to go to their next destination in two taxis. In how many ways can this be done if two particular persons must be in the same taxi? [3]

5. ABC is a sector of a circle, centre A and $\angle BAC = \frac{\pi}{6}$ radians. Given that AB is perpendicular to BD and $AD = 10$ cm, calculate, to three significant figures,



(i) the perimeter of the shaded region, [3]
(ii) the area of the shaded region. [3]

6. **(a)** A curve is such that $\frac{dy}{dx} = 3x(x - 2)$. Given that it passes through $(1, 3)$, find its equation. [3]
(b) P is the point $(3, 2)$ on the curve $y = x^2 - 5x + 8$. Find the coordinates of the point of intersection of the tangent to the curve at P with the line $x + y + 3 = 0$. [5]

7. **(a)** Find the set of values of x for which $(x + 2)^2 < (2x + 1)^2$. [3]
(b) Express b in terms of a if the equation $3x^2 + 2ax + b = 3a$ has equal real roots. [3]

8. A man accelerates uniformly from rest to a speed of 20 m s^{-1} in T seconds. He then decelerates uniformly at 1 m s^{-2} until he comes to a stop. He covers 300 m during this time.
(i) Sketch the velocity-time graph for his motion. [2]
(ii) Calculate the value of T. [2]
(iii) For the part of his motion when he is accelerating, obtain an expression for the displacement, x m, from his starting point and sketch the displacement-time graph for this part of the motion. [3]

9. **(a)** Given that $y = x^3 - 3x$, find $\frac{dy}{dx}$. Hence obtain an expression for the approximate increase in the value of y when x increases from 2 to $2 + p$, where p is small. [3]

(b) The area of a rectangle has a constant value of 50 cm^2. One side, of length x cm, is increasing at a rate of 0.2 cm s^{-1}. Find the rate at which the other side is decreasing at the instant when $x = 5$. [4]

10. **(a)** Function f is defined by $f : x \mapsto \frac{x}{a - bx}$, $x \neq \frac{a}{b}$ where a and b are positive constants. If $f^2 : x \mapsto \frac{x}{4 - 3x}$, find

(i) the value of a and of b, [3]

(ii) the inverse function f^{-1}. [2]

(b) Sketch the graph of $y = |3x - 7|$ for $1 \leqslant x \leqslant 5$. Find the range of values of x for which $y < 1$. [3]

11. **(a)** Prove the identity $\frac{\sin^2 x - \cos^2 x}{1 + 2 \sin x \cos x} \equiv \frac{\tan x - 1}{\tan x + 1}$. [3]

(b) Find all the angles between 0° and 360° inclusive which satisfy the following equations.

(i) $\cos (2x - 40°) = \sin 70°$ [3]

(ii) $2 \operatorname{cosec}^2 x = 5(1 - \cot x)$ [4]

12. Answer EITHER **(a)** or **(b)**

(a) A particle P passed a fixed point O with velocity 3 m s^{-1} and moved in a straight line perpendicular to a wall. It hit the wall 6 seconds later. Before it hit the wall, its acceleration, a m s^{-2}, t seconds after passing O is given by $a = 5t - 2$. Calculate

(i) the velocity of P just before it hit the wall, [3]

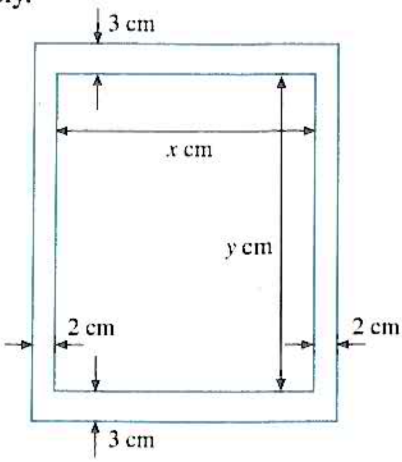
(ii) the least velocity of P, [3]

(iii) the distance of P from the wall after 3 seconds. [4]

(b) The diagram shows a rectangular poster of area 864 cm^2 with side margins of 2 cm and top and bottom margins of 3 cm. The width and height of the printing area are x cm and y cm respectively.

(i) Express y in terms of x and show that the area of printing, A cm^2, is given by $A = \frac{864x}{x + 4} - 6x$. [4]

(ii) Find the values of x and y for which A is a maximum. [6]



PAPER 2 (2 hours)

Answer **all** *the questions.*

1. Sketch the graph of the function $g : x \mapsto |3 \cos x + 2|$ for the domain $0 \leqslant x \leqslant \frac{3\pi}{2}$. State the range corresponding to this domain. [4]

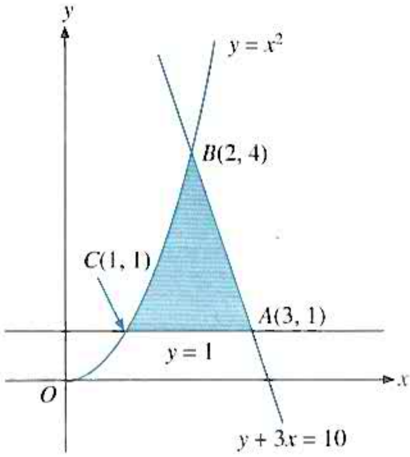
2. (a) Given that $\mathbf{A} = \begin{pmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & -4 \\ 1 & 2 \\ -1 & 1 \end{pmatrix}$, find $\mathbf{AB}$.

State, with reason, whether $\mathbf{A}^{-1}$ and $(\mathbf{AB})^{-1}$ exist. [4]

(b) Use a matrix method to solve the simultaneous equations:

$$x - 2y = 4$$
$$5x + 2y = 8$$ [3]

3. The diagram shows part of the curve $y = x^2$. The straight line $y + 3x = 10$ cuts the line $y = 1$ at $A(3, 1)$ and the curve at $B(2, 4)$. The line $y = 1$ cuts the curve at $C(1, 1)$. Calculate the area of the shaded region. [5]



4. Given that $f(x) = 4x^3 + 24x^2 - 6$,

(i) calculate the remainder when $f(x)$ is divided by $x + 6$. [2]

(ii) Hence solve the equation $f(x) = x$. [4]

5. (a) Solve the equation $\lg (7x^2 + 8x + 3) - \lg x^2 = 1$. [3]

(b) By means of the substitution $y = 3^x$, find the value of x such that $7 \times 3^{x-1} + 5 = 3^{x+1}$. [3]

6. Find the values of x between 0 and π for which the curve $y = e^{3x} \sin 2x$ has stationary points. [5]

7. Given that $\varepsilon = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$, $A = \{x : x \text{ is a multiple of } 4\}$ and $B = \{x : 15 < 3x < 30\}$, list the elements of A and B. [2]

(i) List the elements of $A \cup B$, $A \cap B$ and $A \cap B'$. [3]

(ii) Find $n(A' \cup B)$. [2]

(iii) Find an element x such that $x \notin A'$ and $x \in B$. [2]

8. (Optional) Two particles A and B are 114 m apart on a smooth horizontal surface. A is moving directly towards B with a speed of 8 m s^{-1} and retardation 2 m s^{-2}. B is moving directly towards A with a speed of 5 m s^{-1} and acceleration 4 m s^{-2}.

(i) Calculate the time taken before they meet. [5]

(ii) Calculate their speeds at this instant. [2]

9. **(a)** Differentiate $x^2 \ln x - x$ with respect to x. Hence evaluate $\int_1^2 x \ln x \, dx$. [4]

(b) Find

(i) $\int_0^1 \sqrt{e^{3x}} \, dx$, **(ii)** $\int_1^2 \frac{8}{(2x+3)^2} \, dx$. [4]

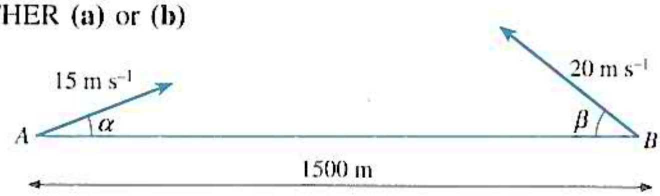
10. Variables x and y are connected by the equation $px^2 + qy = 1$. The table below shows experimental values of x and y. One value of y is subject to an abnormally large error.

x	1	2	3	4	5	6
y	0.47	0.87	1.53	2.47	3.87	5.14

Plot y against x^2 and use the graph to [5]
(a) identify the abnormal reading and estimate its correct value. [2]
(b) estimate the value of p and of q. [4]

11. Answer EITHER **(a)** or **(b)**

(a)



At 12 00 hours, a boat B is due east of boat A and their distance apart is 1500 m. The speeds and directions of the two boats are shown in the diagram where $\alpha = 30°$ and $\beta = 60°$.

(i) Find the speed and direction of the velocity of A relative to B. [3]

(ii) Find the time at which A is due south of B and the distance between A and B at this instant. [5]

(iii) If B changes its course immediately at 12 00 hours in order to intercept A, find the value of β. [4]

(b) Points A and B have position vectors **a** and **b** respectively relative to an origin O. The point C lies on OA produced such that $OC = 2OA$, and D lies on OB such that $OD = \frac{1}{3}OB$. Express $\overrightarrow{BC}$ and $\overrightarrow{DA}$ in terms of **a** and **b**. [2]

The lines AD and BC intersect at P.

(i) Given that $\overrightarrow{AP} = \mu\overrightarrow{DA}$, express $\overrightarrow{OP}$ in terms of μ, **a** and **b**. [2]

(ii) Given also that $\overrightarrow{BP} = \lambda\overrightarrow{BC}$, express $\overrightarrow{OP}$ in terms of λ, **a** and **b**. [2]
Hence calculate the value of λ and of μ and find the position vector of P. [4]

Another point Q is such that $APCQ$ is a parallelogram. Find its position vector. [2]

ANSWERS

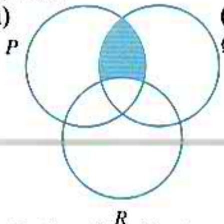
Exercise 1.1A

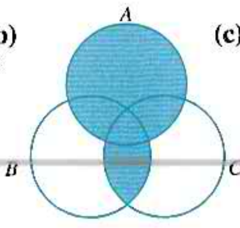
1. **(a)** {Saturday, Sunday} **(b)** {April, August} **(c)** {2, 3, 5, 7, 11}
2. **(a)** $A = \{2n - 1 : 2n - 1 < 12, x \in \mathbb{Z}^+\} = \{1, 3, 5, 7, 9, 11\}$
   (b) $B = \{n : n < 5, n \in \mathbb{N}\} = \{1, 2, 3, 4, 5\}$
   (c) $C = \{5n : 5n < 19, n \in \mathbb{Z}^+\} = \{5, 10, 15\}$
   (d) $D = \{3n : 12 < 3n < 20, n \in \mathbb{Z}^+\} = \{15, 18\}$
   (e) $E = \{x : -2 < x < 3, x \in \mathbb{R}\}$
3. **(a)** $A = \{0, 1, 2, 3, 4, 5, 6, 7\}$ **(b)** $B = \{6, 9, 12, \ldots\}$
   (c) $C = \{2, 3, 7\}$ **(d)** $D = \{2\}$
4. **(a)** $A = \{\ \}$ **(b)** $B = \{\ \}$ **(c)** $C = \{4\}$ **(d)** $D = \{7, 14, 21, 28\}$
5. $F = \{1, 2, 3, 5, 6, 10, 15, 30\}$
   $G = \{6, 12, 18, 24, 30 \ldots\}$, 6 or 30
6. $C = \{1, 6, 7, 8\}$, $T = \{2, 3, 4, 5, 6, 7, 8\}$, 1
7. $\{9, 16, 25, 36\}$
8. **(a)** $A = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$
   (b) $B = \{3, 4, 5, 6, 7, 8\}$ **(c)** $C = \{1, 2, 9, 12, 18, 36\}$
9. $D = \{3, 6, 9, 12\}$, $E = \{6, 12\}$, $C = \{6, 12\}$
10. $A = \{1, 3, 5\}$
11. $S = \{1, 2, 3\}$, $T = \{1, 3, 5, 9, 15, 45\}$
    (a) 2 **(b)** $U = \{1, 2, 3, 5, 9, 15, 45\}$

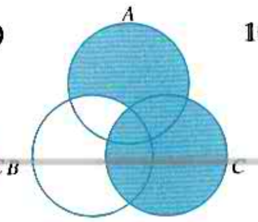
Exercise 1.1B

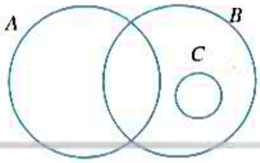
1. **(a)** $A = \{8, 9, 10\}$ **(b)** $B = \{1, 2, 3, 4, 5, 6\}$ **(c)** $C = \{4, 5, 6, 7, 8\}$
   (d) $D = \{1, 2, 3, 4, 5, 6\}$
2. **(a)** $A = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$ **(b)** $B = \{5, 10, 15, 20\}$
   (c) $C = \{10, 20\}$; True, False, True.
3. **(a)** $A = \{5, 6, 10, 12\}$ **(b)** $B = \{5, 7, 11\}$ **(c)** $C = \{6, 9, 12\}$
4. **(a)** $A = \{3, 6, 9, 12, 15, 18\}$ **(b)** $B = \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20\}$
5. $A = \{8, 10, 12\}$, $B = \{9, 12\}$
   (a) 11 **(b)** A is the set of multiples of 2, B is the set of multiples of 3.
6. $A = \{4, 5\}$, $B = \{5, 6, 7, 8\}$ **(a)** 4 **(b)** No
7. $A' = \{b, d, f, g, h\}$, $B' = \{a, c, e, g\}$, $x = g$, Yes.
8. $A = \{6, 12, 18\}$, $B = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$. Yes.
9. **(a)** $A = \{3, 5, 7, 9, 11, 13, 15, 17, 19\}$, $B = \{3, 5, 7, 11, 13, 17, 19\}$
   (b) $C = \{3, 5, 7, 11, 13, 17, 19\}$, $D = \{3, 5, 7, 9, 11, 13, 15, 17, 19\}$,
   C is the set of common elements of C and D.
   D is the set of elements which are either members of A or B (or both).

Exercise 1.2

1. **(a)** 8 **(b)** 51, 24
2. **(a)** 12 **(b)** 21, 21
3. 7, 4
4. 0, 3
5. **(a)** 4 **(b)** 6 **(c)** 28
6. **(a)** 8 **(b)** 16, 7 **(c)** 9
7. **(a)** $P = \{2, 3, 4, 6, 8\}$, $Q = \{3, 5\}$, $P' \cap Q' = \{7, 9, 10\}$ **(b)** 1, 3
8. **(a)** $P = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$, $Q = \{3, 6, 9, 12, 15, 18\}$, $P \cap Q = \{6, 12, 18\}$
   (b) 3, 6
9. **(a)**


(b)


(c)


10.


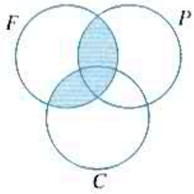
11. $A \cap B = \{x : 2 < x < 3, x \in \mathbb{R}\}$; $A \cup B = \{x : 1 \leq x \leq 5, x \in \mathbb{R}\}$;
    $A' \cap B' = \{x : x < 1 \text{ or } x > 5, x \in \mathbb{R}\}$; $A' \cup B = \{x : x < 3 \text{ or } x > 5, x \in \mathbb{R}\}$

12. $A \cap B = \{x : 3 \leqslant x \leqslant 4\}$; $A' \cap B = \{x : 4 < x < 6\}$;
$A \cap B' = \{x : 1 < x < 3\}$; $A' \cap B' = \{x : x \leqslant 1 \text{ or } x \geqslant 6\}$

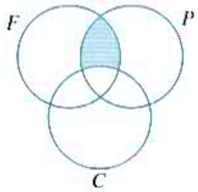
Exercise 1.3

1. **(a)** $P \cap H$ **(b)** 48 **2.** 13

3. **(a)** **(i)** $x \in (F \cap P) \cup (F \cap C)$

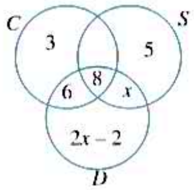


(ii) $x \in F \cap P \cap C'$



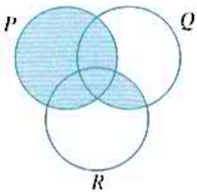
(b) 14 **4.** **(a)** 2 **(b)** 0

5. **(a)**



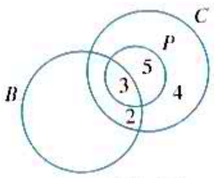
(b) 4 **(c)** 8

6. **(a)**



(b) **(i)** 12 **(ii)** $C \cap F'$

7. **(a)**



(b) 6 **(c)** 2

8. **(a)** 5 **(b)** 25 **9.** $23 + x$, $32 - x$, 32, 5

10. **(a)** $35 - x$ **(b)** 5 **11.** **(a)** 13 **(b)** 5 **(c)** $P' \cap C$

12. **(a)** 9 **(b)** 24 **13.** **(a)** $x - 3$ **(b)** 3, 9

14. $22 - x$, $x + 13$, $15 - x$, 0, 15

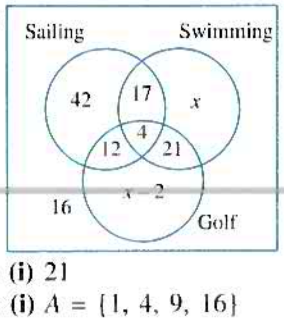
Miscellaneous Exercise 1

1. **(a)** $A = \{3, 6, 9, \ldots\}$ **(b)** $B = \{3, 4, 5, 6, 7\}$ **(c)** $C = \{3, 6, 9, 12\}$
(d) $A \cup (B \cap C) = A = \{3, 6, 9, \ldots\}$, $(A \cup B) \cap C = C = \{3, 6, 9, 12\}$; No

2. 4 or 8 **3.** $\{6, 12, 18, 24, 30\}$, $\{6, 12, 18, 24, 30\}$

4. $x = 3$, $y = 2$ **5.** 6

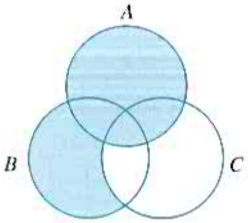
6. **(a)**



(b) **(i)** 70 **(ii)** 138

7. **(a)** **(i)** 21 **(ii)** 25
(b) **(i)** $A = \{1, 4, 9, 16\}$ **(ii)** 0 **(iii)** $\{8, 10, 14\}$

8. **(a)**



(b) **(i)** 4 **(ii)** $S' \cap P'$

9. **(a)** 30 **(b)** 30 **(c)** 80 **10.** **(a)** 18 **(b)** 5, 150

11. **(a)** 8 **(b)** 16 **(c)** 40

2

Exercise 2.1

1. $x = 1,\ y = 3$ **2.** $x = 3,\ y = -2$ **3.** $x = -1,\ y = 4$

4. not correct **5.** $a = 2,\ b = -1$ **6.** $p = 0,\ q = -\frac{2}{3}$

Exercise 2.2

1. $x = 1, y = 2$ or $x = 4, y = -4$ **2.** $x = -1, y = 3$

3. $x = 1, y = -2$ or $x = -\frac{2}{5}, y = \frac{11}{5}$ **4.** $s = -1, t = 2$ or $s = \frac{7}{5}, t = -\frac{8}{5}$

5. $x = 2, y = 3$ or $x = 3, y = 2$ **6.** $x = 1, y = 1$ or $x = -\frac{9}{5}, y = -\frac{16}{5}$

7. $x = 0, y = -1$ or $x = -4, y = -3$ **8.** $x = 4, y = -1$ or $x = 7, y = -3$

9. $x = -1, y = \frac{2}{3}$ or $x = -\frac{3}{2}, y = \frac{1}{2}$ **10.** $x = \frac{2}{3}, y = 4$ or $x = 3, y = \frac{1}{2}$

11. $x = 1, y = -4$ or $x = -2, y = 2$ **12.** $x = 3, y = -1$ or $x = -2, y = -2$

13. $\left(\frac{1}{2}, -1\right), \left(-\frac{3}{2}, \frac{1}{3}\right)$ **14.** $(5, -3), \left(-6, \frac{13}{3}\right)$

15. $\left(\frac{5}{2}, 8\right), (1, 5)$

Miscellaneous Exercise 2

1. $x = 1, y = 2$ or $x = \frac{5}{2}, y = \frac{5}{4}$ **2.** $x = -5, y = \frac{5}{3}$

3. $x = 1, y = 1$ or $x = 2, y = 4$ **4.** $x = 3, y = 2$ or $x = -\frac{9}{2}, y = \frac{19}{2}$

5. $\left(5, \frac{5}{2}\right), (6, 2)$ **6.** $\left(-\frac{2}{5}, -\frac{14}{5}\right), (2, 2)$

7. $a = 1, b = 7$ or $x = -7, y = 7$ **8.** $p = -1; x = -\frac{3}{2}, y = -2$

9. 7 m × 11 m

3

Exercise 3.1

1. **(a)** 1 **(b)** 4 **(c)** 144 **(d)** $\frac{1}{2}$ **(e)** 5 **(f)** $\frac{1}{3}$

2. **(a)** 54 **(b)** 4

3. **(a)** $8y$ **(b)** $\frac{1}{2}y^2$ **(c)** $\frac{12}{y}$ **(d)** $\frac{32}{y^3}$ **(e)** $\frac{1}{2}y^2$ **(f)** $y^3 - \frac{1}{y^2}$

4. **(a)** $18y^2$ **(b)** $\frac{1}{6}yz$ **(c)** $\frac{z^2}{y}$

6. **(a)** $3\sqrt{2}$ **(b)** $4\sqrt{3}$ **(c)** $2\sqrt{3}$ **(d)** $4\sqrt{3}$

(e) $7\sqrt{2}$ **(f)** -1 **(g)** $\sqrt{3} - 1$ **(h)** $29 - 4\sqrt{7}$

7. (a) $2\sqrt{3}$ (b) $3\sqrt{2} - 3$ (c) $6\sqrt{3} + 9$ (d) $30 + 12\sqrt{5}$
(e) $20 - 8\sqrt{6}$ (f) $17 - 12\sqrt{2}$ (g) $\frac{-11 - 4\sqrt{6}}{5}$ (h) $\frac{7\sqrt{15} - 3}{22}$

Exercise 3.2

1. (a) $\frac{3}{2}$ (b) $\frac{5}{2}$ (c) -2 (d) $-\frac{5}{2}$ (e) $\frac{1}{2}$
(f) 2 (g) 2, -2 (h) 2 (i) $\frac{2}{3}$ (j) $\frac{5}{4}$
(k) 2 (l) -6, 2 (m) -2, 4 (n) -2, 1
2. $a = 3$, $n = 2$
3. (a) $x = -\frac{4}{9}$, $y = \frac{1}{9}$ (b) $x = 2$, $y = 1$ (c) $x = 1$, $y = -1$
4. $m = 4$, $n = 3$ 5. 3 6. (a) 0, 1 (b) 0, 2
7. (a) 0, 1 (b) 1, 3 (c) $-\frac{1}{2}$, $\frac{1}{2}$ (d) 0 (e) -2, 0 (f) 3
9. (a) 2 (b) ± 2 (c) 6 (d) 3 (e) 2 (f) 2
10. $r = 3$, $k = \frac{2}{3}$ 11. $x = 1$, $y = -1$ or $x = 2$, $y = 1$ 12. 4

Exercise 3.3

1. (a) $4 = \log_2 16$ (b) $-2 = \log_3 \left(\frac{1}{9}\right)$ (c) $2 = \log_{10} 100$
(d) $3 = \log_a y$ (e) $x = \log_2 p$ (f) $4 = \log_x (2 - k)$
2. (a) $5^3 = 125$ (b) $2^{-2} = \frac{1}{4}$ (c) $4^3 = 64$
(d) $x^4 = 3$ (e) $3^n = y$ (f) $2^{p+1} = 4y$
3. (c), (d) and (e) are not defined
4. (a) 8 (b) 3 (c) $\frac{3}{2}$ (d) 5
(e) $-\frac{7}{16}$ (f) $\frac{3}{4}$ (g) 2, 4 (h) 4
5. (a) -2 (b) 4 (c) 8 (d) $\frac{25}{16}$ (e) 3 (f) 6
6. 2 7. 3^{a+b}
8. (a) $x = 2$, $y = 4$ (b) $x = 4$, $y = 2$ 9. 9

Exercise 3.4

1. 1.37 2. 6.48 3. 0.834 4. 0.282 5. 1.95 6. 0.880
7. 0.829 8. 1.21 9. 1.26 10. 2.58 11. 1.94 12. -0.713
13. 0.811 14. 2.58 15. ± 2.40
16. (a) 1.29 (b) -9.85 (c) 1.71 17. 1.86
18. (a) 0.633 (b) 0.112 (c) -1.16 (d) -1.70
19. (a) 0.693 (b) 1.10, 1.39 (c) 0, 0.693
20. (a) 1.26 (b) -1.39, 2.20
21. $x = 0.631$, $y = 1.631$
22. (a) 9.11 (b) 4.38
23. (a) $y = \lg (x + 1)$ (b) $y = 1 - \ln 3x$ (c) $y = e^x - 1$
(d) $y = 10^{\frac{1}{2}x - 1}$ (e) $y = \frac{1}{2}\ln (x - 4)$ (f) $y = e^{4x} - x$

24. **(a)** 4.07 **(b)** 5.65, 0.177 **(c)** 1.35 **(d)** 3.33×10^8 **(e)** 18.9
(f) 3.69 **(g)** 0.349 **(h)** 4.5 **(i)** 72.5

Exercise 3.5

1. **(a)** 1.21 **(b)** 2.18 **(c)** 1.20 **(d)** –3.17 **(e)** 0.639 **(f)** 2.08

2. **(a)** 3 **(b)** 8

4. **(a)** 2 **(b)** –1 **(c)** 3 **(d)** 3 **(e)** –2 **(f)** $\frac{1}{4}$

5. **(a)** 1 **(b)** 1 **(c)** 2 **(d)** –2 **(e)** 1 **(f)** 0

6. **(a)** $-\log_a 2$ **(b)** $\log_x \left(\frac{25}{2}\right)$ **(c)** $\lg \left(\frac{3}{2}\right)$ **(d)** $\lg (x + 2)$

7. **(a)** $\frac{1}{2}$ **(b)** –3 **(c)** $\frac{1}{3}$ **(d)** 4

8. **(a)** 1.176 **(b)** 0.826 5 **(c)** –0.222 **(d)** 0.947 **(e)** 2.699 **(f)** –0.745

9. **(a)** $2a + b$ **(b)** $1 + b$ **(c)** $a + 2b$ **(d)** $a - b$ **(e)** $a - 1$ **(f)** $2a - b$

10. **(a)** $p + 2q$ **(b)** $1 + p - q$ **(c)** $\frac{1}{2}(1 + 3p + q)$
(d) $2 + \frac{1}{2}p - 2q$ **(e)** 10^{p+q} **(f)** $q(10^p)$

11. **(a)** $y = 10x^3$ **(b)** $y = \frac{100}{\sqrt{x}} - 1$ **(c)** $y = 9(x + 2)^{\frac{3}{2}}$ **(d)** $y = -0.7x$

12. **(a)** $y = x^2 - 1$ **13.** **(a)** $\log_3 45$ **(b)** $\lg 40$ **(c)** $\log_a \left(\frac{8}{a}\right)$

14. **(a)** $a = 2,\ b = -\frac{1}{3}$ **(b)** $a = 1,\ b = 0.9$ **(c)** $a = -1.5,\ b = 1.7$

Exercise 3.6

1. 2 **2.** 3 **3.** $\frac{1}{5}$ **4.** 1 **5.** $\frac{9}{8}$

6. 6 **7.** –1, 7 **8.** 1, 7 **9.** 12 **10.** $1\frac{1}{2}$

11. **(a)** 3 **(b)** 10 **12.** $x = 3,\ y = 5$ **13.** $p = \frac{13}{2},\ q = \frac{3}{2}$

14. **(a)** 3, $\frac{1}{27}$ **(b)** 1, $3^{\sqrt{3}}$, $3^{-\sqrt{3}}$

15. **(a)** 27, $\frac{1}{27}$ **(b)** 8, $\frac{1}{8}$ **(c)** 3 **(d)** –1, 1

16. $2a$, $5a$ **17.** 40

Miscellaneous Exercise 3

1. **(a)** 24 **(b)** $-\frac{3}{2}$ **2.** $a = \frac{1}{3},\ b = 3.17$

3. **(a)** 0.631 **(b)** 1, 1.262 **4.** **(a)** –0.453 **(b)** –0.916

5. **(a)** 1 **(b)** $3\frac{1}{2}$ **(c)** 1.10

6. **(a)** $x = 3,\ y = 2$ **(b)** **(i)** $\frac{2}{9}$ **(ii)** 3.82

7. **(a)** 45 **(b)** $x = 4,\ y = 2$

8. **(a)** –2 **(b)** 9 **(c)** $-\frac{1}{2}$, 1 **(d)** 3
(e) 64, $\frac{1}{64}$ **(f)** –3, 2 **(g)** 5 **(h)** $\frac{1}{2}$
(i) –2, 10 **(j)** 4, $\frac{1}{4}$

9. **(a)** 0.893 **(b)** 2.16 **(c)** 1.71 **(d)** 3.52 **(e)** –2.56 **(f)** 1.26

10. 2^{2a+3b}, 2^{a-3b}; $a = \frac{4}{3}$, $b = \frac{7}{9}$ **11.** 1.631; 0.738; 3.262 **12.** $2 + 3p$; $\frac{1}{6}p$

13. **(a)** $p + 2q$ **(b)** $\frac{1}{2}p - q$ **(c)** $\frac{2}{p}(1 + q)$ **(d)** 2^{2p+2q}

14. $\frac{1}{3}(a + b + 1)$; $\frac{\sqrt{5}}{2}$ **15.** $\frac{1}{5}(4m - 3n)$; $\frac{1}{10}(2m + n)$

16. **(a)** 9 **(b)** **(i)** 73.5 **(ii)** 59.6 **(c)** (1, 7.39)

17. **(a)** **(i)** 13.2 **(ii)** 34.7 **(b)** $5\frac{1}{3}$ **(c)** 2

19. **(a)** **(i)** 80 **(ii)** 26.4 **(iii)** 5.81; 18 **(b)** **(i)** 4 **(ii)** –1.73

20. **(a)** 3 **(b)** –3

21. **(a)** **(i)** $\frac{5}{2}$ **(ii)** $\sqrt{10}$ **(b)** $y = \frac{x}{20}$

22. **(a)** 8, $\frac{1}{27}$ **(b)** $\frac{1-m}{3m}$ **23.** **(a)** 3 **(b)** 1

24. **(a)** $\frac{4}{n-1}$ **(b)** $\frac{3}{2}$ **25.** **(a)** 4 **(b)** $\sqrt{a}$

26. **(a)** $4a$ **(b)** $\frac{1}{12}$

4

Exercise 4.1

2. **(a)** min. $y = 1$, $x = 3$ **(b)** max. $y = 4$, $x = 2$ **(c)** min. $y = 2$, $x = 0$

(d) min. $y = -5$, $x = -1$ **(e)** max. $y = 1$, $x = -3$ **(f)** max. $y = 9$, $x = \frac{1}{2}$

3. 0, $x = -2$ **4.** $\frac{93}{4} - (x - \frac{9}{2})^2$ **(a)** $\frac{93}{4}$, $x = \frac{9}{2}$

5. $a = 4$, $p = 2$, $q = -1$; $(2, -1)$ **6.** 1, $x = 1$

7. 1 **9.** $y = 2x^2 + 8x + 11$

11. **(a)** $x = 0, -4$; $y = 0$; $(-2, -4)$ **(b)** $x = -4, 3$; $y = 12$; $(-\frac{1}{2}, \frac{49}{4})$

(c) $x = -1, 7$; $y = -7$; $(3, -16)$ **(d)** $x = 0, 6$; $y = 0$; $(3, 9)$

(e) $x = -1, \frac{3}{2}$; $y = -3$; $(\frac{1}{4}, -\frac{25}{8})$ **(f)** $x = -1, 6$; $y = -6$; $(\frac{5}{2}, -\frac{49}{4})$

13. **(a)** $(-a, 1 - 4a^2)$ **14.** 5, $y = -2x^2 + 12x - 10$

17. $k\left(x - \frac{2}{k}\right)^2 + 3k - \frac{4}{k}$; $-\frac{2}{3}$

Exercise 4.2

1. **(a)** real and distinct, $1 \pm \sqrt{6}$ **(b)** real and equal, $-\frac{1}{2}, -\frac{1}{2}$ **(c)** real and distinct, $-\frac{1}{3}$, 3

(d) not real **(e)** real and distinct, 1, –1.5 **(f)** real and distinct, $\frac{1}{2}\left(1 \pm \sqrt{7}\right)$

2. $k < 9$ **3.** $\pm\sqrt{6}$

4. **(a)** ±3 **(b)** $p \geqslant 1$ **(c)** $p > \frac{2}{3}$ **(d)** $p > \frac{1}{4}$

5. $m > \frac{17}{8}$ **6.** $-\frac{13}{4}$ **7.** $k < -\frac{13}{4}$; $k \geqslant -\frac{13}{4}$

8. $k \geqslant -3,\ k \pm \sqrt{2k+6}$ **10.** $q = \frac{5}{4}p$ **11.** $k > -1$

14. $x = 1 \pm \sqrt{t-1},\ t \geqslant 1$ **15.** $c < \frac{4}{3}$

16. $p = -6, 2$; $(3, 0)$, $(-1, 0)$ **17. (a)** $-\frac{1}{2}$ **18.** $-\frac{1}{3}$

19. (a) $m > 2$ **(b)** $m = 2$ **(c)** $m < 2$

20. $p < -\frac{1}{24}$ **21.** $c < 7$ **23.** $k \leqslant \frac{1}{4}$; $\frac{1}{4}$

Exercise 4.3

1. (a) $\{x : x < -3 \text{ or } x > 3, x \in \mathbb{R}\}$ **(b)** $\{x : -1 < x < 3, x \in \mathbb{R}\}$

(c) $\{x : -6 < x < 1, x \in \mathbb{R}\}$ **(d)** $\left\{x : x \leqslant \frac{1}{2} \text{ or } x \geqslant 3, x \in \mathbb{R}\right\}$

(e) $\{x : x < -2 \text{ or } x > 6, x \in \mathbb{R}\}$ **(f)** $\left\{x : -\frac{3}{2} \leqslant x \leqslant \frac{1}{2}, x \in \mathbb{R}\right\}$

(g) $\{x : x \leqslant -4 \text{ or } x \geqslant 4, x \in \mathbb{R}\}$ **(h)** $\{x : -4 \leqslant x \leqslant 0, x \in \mathbb{R}\}$

(i) $\left\{x : x < -\frac{2}{5} \text{ or } x > 1, x \in \mathbb{R}\right\}$ **(j)** $\{x : -\sqrt{18} < x < \sqrt{18}, x \in \mathbb{R}\}$

2. $\left\{x : x < -\frac{1}{2} \text{ or } x > 3, x \in \mathbb{R}\right\}$ **3.** $x < 2 - \sqrt{10}$ or $x > 2 + \sqrt{10}$ **4.** $m < -2$ or $m > 8$

5. $p < -16$ or $p > 0$ **6.** $\{p : -5 < p < 4, p \in \mathbb{R}\}$ **7.** $\{m : m \leqslant 0 \text{ or } m \geqslant 8, m \in \mathbb{R}\}$

8. $\{k : -15 < k < 1, k \in \mathbb{R}\}$ **10.** $-6 < c < 6$ **11.** $\{k : -2 < k < 2, k \in \mathbb{R}\}$

12. $6 < k < 8$ **13.** $p = -12, k = 4$ **14.** $m < 0$ or $m \geqslant 1$

Miscellaneous Exercise 4

2. (a) $p \geqslant 2$ **(b)** $-1 < p < 3, p \neq 2$

3. (a) $x \leqslant -2$ or $x \geqslant 7$ **(b)** $-\sqrt{3} < x < \sqrt{3}$

4. (a) $p = -4, q = 2$ **(b)** $9, -2$ **(c)** $\{x : x < -5 \text{ or } x > 1, x \in \mathbb{R}\}$

5. (a) $(1, 3)$; $\frac{1}{2}$ **(b)** -12

6. (a) $2, 10$ **(b)** $k < -8$ or $k > 8$

7. (a) $\left\{x : x < -1 \text{ or } x > \frac{3}{2}, x \in \mathbb{R}\right\}$ **(b)** $k < \frac{3}{4}$

8. (a) 4 **(b)** $7 : 6, 1$

9. (a) $-2 < p < 8$; $-2, 8$ **(b)** $(x-1)^2 + 36$; $36, x = 1$

10. (a) $\frac{q^2}{r - 2q}$ **(b)** $\left\{k : -\frac{4}{7} \leqslant k \leqslant 4, k \in \mathbb{R}\right\}$

11. (a) $\{x : -4 \leqslant x \leqslant 1, x \in \mathbb{R}\}$ **(c)** $m \geqslant -2$

12. (a) $-2 < k < 8$ **(b)** $p = -4, x = 1$

14. (a) one real root, $x = -1$ **(b)** $-1 \leqslant x < 1$ or $3 < x \leqslant 4$

15. (a) 9 **(b)** $\{k : k < -10 \text{ or } k > 2, k \in \mathbb{R}\}$; $k = -10, 2$

16. $2\frac{7}{8}, x = -\frac{3}{4}$; $k \leqslant 2\frac{7}{8}$ **17. (a)** $x < -1$ or $x > 2$ **(b)** $c > 9$

18. (a) $k \geqslant 10\frac{1}{4}$ **(b)** $m > 2$ **19.** $k > 8$

21. (a) $\left\{x : 1 - \frac{1}{\sqrt{2}} \leqslant x \leqslant 1 + \frac{1}{\sqrt{2}}, x \in \mathbb{R}\right\}$ **(b)** $p > 1$

22. $(m-1)^2 = 4c$ **23.** $y = 5 + 6x - x^2$ **24.** $\frac{3}{2}, 5$

5 Exercise 5.1

1. $a = 3, b = -2$
2. $a = 3, b = 6$
3. $a = 4, b = 9$
4. $a = 1, b = -10$
5. $a = 6, b = 1, c = -2$
6. $A = 2, B = 4$
7. $p = 3, q = -1, r = -7$
8. $a = 1, b = -3, c = -4$
9. $A = 3, B = 7, C = 6$
10. $A = 2, B = -1, C = 6$
11. $A = 12, B = -2$
12. $a = 6, b = -3$

Exercise 5.2

1. **(a)** 6 **(b)** 19 **(c)** 47 **(d)** -12
2. 8
3. $0, \frac{1}{3}$
4. ± 2
5. $32 - 8a$
6. $a = -6, b = -1$
7. 3
8. $a = -2, p = 2$
11. **(a)** 1 **(b)** $a = 5, b = -14$
12. 3
13. $p = -1, q = 3$
14. 26

Exercise 5.3

1. **(a)** Yes **(b)** No **(c)** Yes
3. 2
4. $3, -1$
5. $-5, 7$
6. $-8, x + 2$
7. $a = 2, b = -7; (x - 4)(x + 1)(2x - 1)$
8. 7
9. -3
10. $b = \frac{3}{a} - 2a$
12. $a = 2, b = -2$
13. **(a)** $-1, -\frac{1}{2}$ **(b)** $-1, -2; p = -2$
14. $6b^3$

Exercise 5.4

1. $(x - 1)(x - 2)(3x - 1), (x - 2)^2(x + 2)$
2. **(a)** $-1, 0, 4$ **(b)** $-2, 2, 3$
3. **(a)** $-1, 2, 3$ **(b)** $-2, -\frac{3}{4}, 2$ **(c)** $-2, \frac{3}{4}, 3$ **(d)** $-2, 1, 2$ **(e)** $1, 1, 4$ **(f)** $-2, -2, 2$
4. $-2, -2, \frac{3}{2}$
5. **(a)** $-2, -0.43, 0.77$ **(b)** $0.72, 2.78, 3$
6. $-3, -2, \frac{1}{2}$
7. $(x + 1)^2(x - 4)$ **(a)** $-1, -1, 4$ **(b)** $-1, 0, 4$ **(c)** $-2, -1, 5$
8. $-3, -2, -1, 2$
9. $x + a$ is a factor; $-a, -\frac{3}{2}a, 2a$

Miscellaneous Exercise 5

1. $3, x - 2$
2. $A = 3, B = -5; 3x - 5$
3. $42; 0; (x + 3)(x - 1)(x - 2); -\frac{1}{3}, \frac{1}{2}, 1$
4. $-1, 3$
5. 6
6. $3, a = -2, b = 1; -5$
7. $p = -5, q = 19; -\frac{5}{3}$
8. **(a)** $-4, -1, 2$ **(b)** $-\frac{4}{3}, -\frac{1}{3}, \frac{2}{3}$
9. 4; 5
10. $p = -2\sqrt{3}; -\sqrt{2}, \sqrt{2}, 2\sqrt{3}$
11. $-7, \frac{5}{9}$
12. $\frac{3}{4}$
13. $-4, 1, 2; \pm\sqrt{2}, \pm 1$
14. **(b)** $-2, 3, 6$
15. 4
16. $-\frac{3}{2}, -\frac{1}{2}, 2$
17. **(a)** $4; b = -5, c = 6$ **(b)** $-1, -0.45, 4.45$
18. **(a)** $4; 4, \pm\sqrt{\frac{5}{2}}$ **(b)** $x + y, x + 3y$
19. **(a)** 9 **(b)** $3, -2, -4; \frac{3}{5}, -\frac{2}{5}, -\frac{4}{5}$

20. -6

21. $p = -4$, $q = -4$, $r = 16$, $D(0, 16)$

22. (a) $-2, -1, 3$ (b) $b = -3$, $c = 3$, $d = 22$

23. (a) $x(x - 2)(2x - 1)$; $x < 0$ or $\frac{1}{2} < x < 2$ (b) $A = 3$, $B = -2$, $C = 2$

24. (a) $a = 11$, $b = -21$ (b) -10; $-1, 2, \frac{5}{2}$

25. $2x^2 - 2ax + 3a$; $0 < a < 6$

28. -2, $x^3 - x^2 - 2x$

Revision Exercise 1

1. (a) (i) $A = \{1, 3, 5, 7\}$, $B = \{2, 3, 5\}$ (ii) 1 or 7 (iii) 2
   (b) $x = 3$, $y = -3$ or $x = 1$, $y = 1$
2. (a) 3 (b) 3 (c) 0.377
3. (a) 2, 64 (b) $3^{2a + \frac{1}{2}b}$, 3^{a-b}; $a = -1$, $b = 4$
4. (a) $\frac{1}{3} < x < 3$ (b) $y = 7 - 2(x - 1)^2$; $y = 7$, $x = 1$
5. (a) $-2 < p < 6$ (b) $-8, 4$
6. (a) 25 (b) $-2, 4$
7. (a) -3 (b) $-3, 2, \frac{5}{2}$

Revision Exercise 2

1. (a) $A = \{3, 5, 7, 9\}$, $B = \{1, 2, 4, 5, 7, 8\}$, $B' = \{3, 6, 9\}$
   (b) $A \cap B = \{5, 7\}$, $A \cup B = \{1, 2, 3, 4, 5, 7, 8, 9\}$
   (c) 5
2. (a) $k \leqslant 1$ or $k \geqslant 8$, $k \neq 0$; 7 (b) $2p^3 = q^2 - q^3$
3. $x = 6$, $y = \frac{1}{2}$ or $x = 1$, $y = 3$; $a = 1.63$, $b = -1$ or $a = 0$, $b = 1.58$
4. (a) (i) 24 (ii) $\frac{1}{2}$ (b) (i) 7.39 (ii) 5.68
5. (a) 2 (b) $a = 5$, $b = \frac{1}{8}$
6. (a) $A = 2$, $B = -4$, $C = 7$ (b) $-\frac{3}{2}, 3, 4$
7. (a) $p = 6$, $q = -19$; $(2x - 1)(x - 2)(3x + 5)$
   (b) (i) $a = -3$, $b = -2$ (ii) $c \geqslant -\frac{17}{4}$

Revision Exercise 3

1. (a) (i) 3 (ii) 15, 9 (b) (i) 7 (ii) 9
2. (a) $x = -3$, $y = -\frac{3}{2}$ or $x = 7$, $y = \frac{9}{2}$ (b) $\frac{2a + 5b}{3}$, $\frac{80}{81}$
3. (a) $\frac{11}{24}$ (b) (i) 20 (ii) 0.829 (iii) 0, 1
4. (a) $\frac{1}{2}, -3$ (b) $a = 2$, $b = -2$
5. (a) (i) -6 (ii) 69 (iii) Does not exist (iv) -15
   (b) $-2, -\frac{3}{4}, 2$; 0.693
6. (a) $-1 < x < 1$ or $2 < x < 4$ (b) $(2x - 5)(x + 1)$ (c) $-2(x - 2)^2 + 17$; 17

Revision Exercise 4

1. **(b)** 8 **(c)** 2 **(d)** 9 **2.** **(a)** 1, 3 **(b)** 0.217

3. **(a)** $x = 1,\ y = 1$ or $x = -\frac{25}{2},\ y = -8$ **(b)** **(i)** $a = 7,\ b = -4$ **(ii)** -2

4. **(a)** $k \leqslant -\frac{4}{3}$ or $k \geqslant \frac{4}{3}$ **(b)** $-16 < a < 0$

5. **(a)** $p = \frac{1}{3}q$ **(b)** $-1, -\frac{1}{3}, 2;\ x < -1$ or $-\frac{1}{3} < x < 2$

6. **(a)** 3 **(b)** 3.44 **(c)** $x = 2,\ y = 1$

7. **(a)** $x \leqslant -2$ or $x \geqslant 6;\ x \leqslant -\sqrt{6}$ or $x \geqslant \sqrt{6}$ **(b)** $k < 1$

6

Exercise 6.1

1. **(a)** 1×3 **(b)** 2×4 **(c)** 3×1 **(d)** 4×2 **(e)** 1×1 **(f)** 2×1

2. $\begin{pmatrix} 65 & 35 \\ 55 & 45 \\ 52 & 48 \end{pmatrix}$; 3×2 **3.** $\begin{pmatrix} 7 & 9 \\ 8 & 6 \end{pmatrix}$; 2×2

4.

$$\begin{array}{r c} & \begin{array}{ccc} \text{coffee} & \text{tea} & \text{soft drinks} \end{array} \\ \begin{array}{r} \text{Monday} \\ \text{Tuesday} \\ \text{Wednesday} \end{array} & \begin{pmatrix} 160 & 125 & 210 \\ 145 & 130 & 275 \\ 120 & 155 & 325 \end{pmatrix} \end{array};\ 3 \times 3$$

5.

$$\begin{array}{r c} & \begin{array}{cc} \text{Malaysia} & \text{Thailand} \end{array} \\ \begin{array}{r} \text{June} \\ \text{July} \\ \text{August} \end{array} & \begin{pmatrix} 143 & 105 \\ 65 & 46 \\ 122 & 89 \end{pmatrix} \end{array};\ 3 \times 2$$

6. **(a)** \$1.80 **(b)** Stall A **(c)** \$8.90

7. **(a)** $a = 4,\ b = -2,\ c = 5$ **(b)** $a = 0,\ b = 4,\ c = 1,\ d = -2$
(c) Different orders **(d)** $a = 5,\ b = -8,\ c = 9,\ d = 4$
(e) Not all corresponding elements are equal **(f)** Different order

Exercise 6.2

1. **(a)** $\begin{pmatrix} 8 & 1 \\ 1 & 0 \\ 11 & 9 \end{pmatrix}$ **(b)** Different orders **(c)** $\begin{pmatrix} 2 & 0 \\ 7 & 6 \end{pmatrix}$

(d) $\begin{pmatrix} 8 \\ 7 \end{pmatrix}$ **(e)** $\begin{pmatrix} 7 & 5 & 8 \\ 2 & -2 & c \end{pmatrix}$ **(f)** Different orders

2. **(a)** $\begin{pmatrix} 4 & -4 & 0 \\ 13 & -6 & -6 \end{pmatrix}$ **(b)** $(11 \quad 4 \quad -10 \quad 6)$ **(c)** Different orders **(d)** $\begin{pmatrix} 2 & 4 \\ 9 & -1 \end{pmatrix}$

3. **(a)** $\begin{pmatrix} -3 \\ 8 \end{pmatrix}$ **(b)** $(2 \quad -3)$ **(c)** $\begin{pmatrix} -4 & 4 \\ 4 & 8 \end{pmatrix}$ **(d)** $\begin{pmatrix} 12 & 8 \\ -4 & 16 \end{pmatrix}$

4. **(a)** $\begin{pmatrix} 18 & 75 \\ 39 & 2 \\ 2 & 33 \end{pmatrix}$ **(b)** Does not exist **(c)** $\begin{pmatrix} 8 & -67 \\ 73 & 50 \end{pmatrix}$ **(d)** $\begin{pmatrix} 15 & -48 \\ -91 & 58 \\ -33 & -5 \end{pmatrix}$

(e) $\begin{pmatrix} -11 & -61 \\ 109 & 25 \end{pmatrix}$ **(f)** Does not exist

5. (a) $\begin{pmatrix} 6 & 45 & -18 \\ 21 & -9 & 9 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & -22.5 & 13 \\ -10 & 2.5 & -5 \end{pmatrix}$ (c) $\begin{pmatrix} 5 & 37.5 & -15 \\ 17.5 & -7.5 & 7.5 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 30 & -20 \\ 13 & -2 & 7 \end{pmatrix}$ (e) $\begin{pmatrix} 4 & 90 & -52 \\ 40 & -10 & 20 \end{pmatrix}$ (f) $\begin{pmatrix} 6 & 0 & 12 \\ 1.5 & -6 & -1.5 \end{pmatrix}$

6. $\begin{pmatrix} 335 & 360 & 260 & 120 \\ 415 & 395 & 310 & 170 \\ 170 & 250 & 130 & 165 \end{pmatrix}$

7. (a) 18 (b) 8 (c) 2 (d) $\begin{pmatrix} 18 & 8 & 6 \\ 7 & 11 & 2 \\ 8 & 8 & 4 \end{pmatrix}$

8. $\mathbf{D} = \begin{pmatrix} 6.63 & 5.95 & 5.53 & 6.46 \\ 6.38 & 6.12 & 5.95 & 6.55 \end{pmatrix}$

9. $\begin{pmatrix} 13 & 6 & 8 \\ 11 & 12 & 3 \end{pmatrix}$

10. $\mathbf{S} = \begin{pmatrix} 90 & 75 & 30 & 60 \\ 75 & 105 & 45 & 45 \end{pmatrix}$; $\begin{pmatrix} 480 & 400 & 160 & 320 \\ 400 & 560 & 240 & 240 \end{pmatrix}$;

The number of each type of papers delivered to each vendor in a week.

11. $\mathbf{M} = \begin{pmatrix} 8.90 \\ 6.50 \end{pmatrix}$; $\mathbf{S} = \begin{pmatrix} 5.20 \\ 3.20 \end{pmatrix}$; $\begin{pmatrix} -0.305 \\ 0.375 \end{pmatrix}$;

The numbers show how the prices of each item differ in Singapore dollars.

12. (a) $\begin{pmatrix} 94.80 \\ 72.60 \\ 105.50 \end{pmatrix}$ (b) The numbers show how much each child spent during the vacation.

13. (a) $\begin{pmatrix} 283 & 388 \\ 142 & 295 \end{pmatrix}$ (b) $\begin{pmatrix} 180 & 250 \\ 200 & 320 \end{pmatrix}$

(c) $\begin{pmatrix} 157 & 409 \\ 213 & 508 \end{pmatrix}$; The numbers represent the numbers of hardbacks and paperbacks sold at each branch.

Exercise 6.3

1. (a) $\begin{pmatrix} 8 \\ 1 \end{pmatrix}$ (b) not possible (c) $(18 \quad -18 \quad -27)$ (d) $\begin{pmatrix} 3 & -9 & -1 \\ 7 & -24 & -1 \end{pmatrix}$

(e) $(0 \quad 10)$ (f) not possible (g) not possible (h) $\begin{pmatrix} -28 & 16 & -26 \\ -9 & 6 & -9 \\ -2 & 8 & -7 \end{pmatrix}$

(i) not possible (j) $\begin{pmatrix} 3 & -9 & 1 \\ 1 & 12 & 17 \end{pmatrix}$ (k) $\begin{pmatrix} 11 & 12 \\ 27 & 16 \end{pmatrix}$ (l) $\begin{pmatrix} -19 & 11 & -22 \\ -6 & 4 & -8 \\ 30 & -21 & 42 \end{pmatrix}$

2. (a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

3. (a) $\begin{pmatrix} 8 & -13 \\ 1 & 9 \\ -1 & 25 \end{pmatrix}$ (b) $\begin{pmatrix} 8 & 0 & -4 \\ 5 & 34 & -45 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 9 \\ -27 & 46 \end{pmatrix}$

(d) $\begin{pmatrix} -1 & -10 & 13 \\ 8 & 0 & -4 \\ 18 & 4 & -14 \end{pmatrix}$ (e) $\begin{pmatrix} 3 & -5 \\ -14 & -18 \end{pmatrix}$ (f) $\begin{pmatrix} 21 & -32 \\ 26 & -140 \end{pmatrix}$

4. **(a)** -2 **(b)** $a = 2$, $b = 2$ **(c)** $p = 10$, $q = 3$ **(d)** -2, 4

5. **(a)** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; No

6. **(a)** $\begin{pmatrix} -2 & 4s+9 \\ 4r & 0 \end{pmatrix}$ **(b)** $\begin{pmatrix} 4 & -30 \\ 0 & 64 \end{pmatrix}$ **(c)** $r = -2$, $s = -3$

7. **(a)** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$ **(d)** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

8. **(a)** $\begin{pmatrix} 1 & 0 & 13 \\ -10 & 34 & 6 \end{pmatrix}$ **(b)** $\begin{pmatrix} -1 & 4 & 3 \\ -8 & 26 & 0 \end{pmatrix}$; $a = -8$, $b = 238$

9. all $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ **10.** all $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **11.** 15.3 kg **12.** $\begin{pmatrix} 8.60 \\ 10.75 \end{pmatrix}$

13. **(a)** $\begin{pmatrix} 280 \\ 200 \end{pmatrix}$ **(b)** The sales, in cents, from lemonade and orangeade.

14. $\begin{pmatrix} 232 \\ 360 \\ 216 \end{pmatrix}$; The number of each type of toys produced.

15. $\begin{pmatrix} 720 \\ 650 \end{pmatrix}$; The amount spent, in cents, by each library for the newspapers.

16. **(a)** $\begin{pmatrix} 1.20 \\ 2.00 \\ 2.35 \end{pmatrix}$ **(b)** $\begin{pmatrix} 12 & 40 & 25 \\ 3 & 30 & 10 \end{pmatrix}$

(c) $\begin{pmatrix} 153.15 \\ 87.10 \end{pmatrix}$; The total costs of the bulbs for the workshop as well as the offices.

Exercise 6.4

1. **(a)** 11 **(b)** -16 **(c)** 10 **(d)** -55
(e) 52 **(f)** -5 **(g)** $-2 + 2\pi$ **(h)** -4

2. **(a)** $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 1 \\ 5 & 6 \end{pmatrix}$ **(d)** $\begin{pmatrix} -\frac{1}{2} & \frac{3}{10} \\ -1 & \frac{2}{5} \end{pmatrix}$ **(e)** $\begin{pmatrix} \frac{1}{12} & \frac{1}{20} \\ \frac{1}{6} & -\frac{1}{10} \end{pmatrix}$

(g) $\begin{pmatrix} 1 & -\frac{1}{5} \\ \frac{1}{2} & 0 \end{pmatrix}$ **(h)** $\begin{pmatrix} -2 & -\frac{1}{2} \\ -3 & -1 \end{pmatrix}$ **(i)** $\begin{pmatrix} \frac{3}{13} & \frac{-2}{13} \\ \frac{2}{13} & \frac{3}{13} \end{pmatrix}$

3. 3; $\begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ -\frac{1}{3} & -1 \end{pmatrix}$ **4.** -6 **5.** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

6. **(a)** 10 **(b)** $\begin{pmatrix} \frac{1}{5} & \frac{1}{10} \\ -\frac{2}{5} & \frac{3}{10} \end{pmatrix}$ **(c)** $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$

7. **(a)** $-\frac{3}{2}$ **(b)** $-\frac{7}{2}$ **(c)** $\begin{pmatrix} \frac{5}{2} & -1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$

8. **(a)** $\begin{pmatrix} -1 & -3 \\ 10 & -8 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 & \frac{1}{2} \\ -1 & \frac{3}{2} \end{pmatrix}$ **(c)** $\begin{pmatrix} -1 & 2 \\ -8 & 6 \end{pmatrix}$

9. **(a)** $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ **(b)** $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ **(c)** $\begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix}$

10. **(a)** $p = 1,\ q = -2$ **(b)** $a = 3,\ b = 7$ **(c)** $a = -3,\ b = 2$

11. 3×2; $\begin{pmatrix} 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$; $\begin{pmatrix} 1 & -2 \\ 2 & -1 \\ 0 & 3 \end{pmatrix}$ **12.** $\begin{pmatrix} 1 & \frac{1}{2} \\ -3 & -1 \end{pmatrix}$ **(a)** $\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ **(b)** $\begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix}$

Exercise 6.5

1. **(a)** $\begin{pmatrix} 4 & 2 \\ 12 & 6 \end{pmatrix}$; not unique **(b)** $\begin{pmatrix} 2 & 3 \\ 3 & -1 \end{pmatrix}$; $x = 5, y = 1$

   (c) $\begin{pmatrix} 4 & 2 \\ 7 & 3 \end{pmatrix}$; $x = 1, y = -1$ **(d)** $\begin{pmatrix} 4 & -5 \\ -8 & 10 \end{pmatrix}$; not unique

2. **(a)** $x = 3,\ y = -2$ **(b)** $x = -3,\ y = 7$ **(c)** $x = 5,\ y = -3$
   (d) $x = 3,\ y = -1$ **(e)** $x = 7,\ y = 4$ **(f)** $x = -1,\ y = 2$

Miscellaneous Exercise 6

1. **(a)** $\begin{pmatrix} 11 \\ -3 \end{pmatrix}$ **(b)** $(0 \quad 8 \quad 12)$ **(c)** $\begin{pmatrix} 40 & -6 \\ -11 & -9 \end{pmatrix}$ **(d)** (41)

2. $p = 1,\ q = -3,\ r = 5$

3. **(a)** $\begin{pmatrix} 31 & -2 \\ -6 & -5 \end{pmatrix}$ **(b)** $\begin{pmatrix} -20 & 2 \\ -28 & 64 \end{pmatrix}$ **(c)** $\begin{pmatrix} -1 & -2 \\ 3 & 5 \end{pmatrix}$ **(d)** $\begin{pmatrix} 7 & -2 \\ 0 & -1 \end{pmatrix}$

4. **(a)** $\begin{pmatrix} 16 & 14 \\ 0 & 9 \end{pmatrix}$ **(b)** $-\frac{1}{6}$ **(c)** -2

5. **(a)** $\begin{pmatrix} 4 & -1 \\ 6 & 2 \end{pmatrix}$ **(b)** 4

6. **(a)** $h = 2 - 2k$ **(b)** Not possible because not all corresponding elements are equal.

   (c) $\begin{pmatrix} 0 & -\frac{1}{4} \\ 1 & 1 \end{pmatrix}$ **7. (a) (i)** $\begin{pmatrix} 7 & -6 \\ 6 & -3 \end{pmatrix}$ **(ii)** $(29 \quad 7)$ **(b) (ii)** 2, -5

8. $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$; $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

9. (a) $\begin{pmatrix} 5 & -1 \\ -11 & 1 \end{pmatrix}$, $-\frac{1}{6}\begin{pmatrix} 1 & 1 \\ 11 & 5 \end{pmatrix}$ (b) $-\frac{1}{3}\begin{pmatrix} -4 & -1 \\ 5 & 2 \end{pmatrix}$, $\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$, $-\frac{1}{6}\begin{pmatrix} 1 & 1 \\ 11 & 5 \end{pmatrix}$; Yes

10. (a) $x = 3$, $y = 2$ (b) $x = 4$, $y = 1$ (c) $x = -1$, $y = 6$ (d) $x = 8$, $y = -2$

11. $\frac{1}{23}\begin{pmatrix} 2 & 3 \\ -3 & 7 \end{pmatrix}$; $x = -2$, $y = 5$ 12. $-\frac{1}{5}\begin{pmatrix} -1 & -3 \\ -1 & 2 \end{pmatrix}$; $x = 5$, $y = 1$

13. fifteen 22¢ stamps and eleven 40¢ stamps

14. $\begin{pmatrix} 61 & 32 \\ 72 & 29 \\ 65 & 22 \end{pmatrix}$; The numbers of sacks and boxes of vegetables delivered to the shop.

15. (a) $\begin{pmatrix} 3 \\ 8 \\ 5 \end{pmatrix}$ (b) $\begin{pmatrix} 205 & 160 & 70 \\ 310 & 200 & 65 \end{pmatrix}$

(c) $\begin{pmatrix} 2245 \\ 2855 \end{pmatrix}$; The sales from tickets sold on Saturday and on Sunday.

16. (a) $\mathbf{AB} = \mathbf{C}$ (b) $\frac{1}{2}\begin{pmatrix} 4 & -3 \\ -6 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} 12 \\ 8 \end{pmatrix}$

(d) The selling price of a large tin and of a small tin of paint.

17. (a) $\begin{pmatrix} 1815 \\ 1365 \end{pmatrix}$; The cost of making the tables and chairs at each outlet.

(b) $\begin{pmatrix} 95 \\ 7 \end{pmatrix}$; The profit from each table and each chair.

(c) $\begin{pmatrix} 2633 \\ 1979 \end{pmatrix}$; The profit made at each outlet.

18. (a) (i) $\begin{pmatrix} 58 \\ 32 \end{pmatrix}$ (ii) The cost of making each type of toy.

(b) (i) $(12\ \ 200)$ (ii) The cost of making 100 toy buses and 200 toy lorries.

19. $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$; The numbers of spray I and spray II containers to order.

20. 114 ounces of mix **A** and 62 ounces of mix **B**.

21.
$$\begin{array}{l} \\ \text{Style 1} \\ \text{Style 2} \\ \text{Style 3} \end{array} \begin{array}{c} \text{Plywood Hours} \\ \begin{pmatrix} 4 & 14 \\ 5 & 10 \\ 3 & 8 \end{pmatrix} \end{array} \qquad \begin{array}{l} \\ \text{Cost of} \\ \text{Plywood} \\ \text{Labour Cost} \end{array} \begin{array}{cc} \text{Grade} & \text{Grade} \\ A & B \\ \end{array} \begin{pmatrix} 26 & 20 \\ 35 & 31 \end{pmatrix}; \begin{pmatrix} 594 & 514 \\ 480 & 410 \\ 358 & 308 \end{pmatrix}$$

7 Exercise 7.1

1. 28 units; 49 sq. units 2. (a) 6 sq. units (b) $2(a + 2)$ sq. units

3. 18 sq. units 4. $BC = \sqrt{10}$ units, $AB = AC = \sqrt{65}$ units

5. 10 sq. units, $\frac{20}{13}$ units 6. 14

7. (a) (1, 0) (b) (0, −3) 8. 3 9. $B(7, 6)$

Exercise 7.2

1. **(a)** (5, 7) **(b)** $(3a, 2a)$ **(c)** $(t + 2, 3 - t)$ **(d)** $\left(\frac{a-b}{2}, \frac{a-b}{2}\right)$

2. $p = 9$, $q = 3$ **3.** $p = -1$, $q = 3$ or $p = 3$, $q = -1$

4. **(a)** $\left(-1, 2\frac{1}{2}\right)$ **(b)** 5 units **5.** **(a)** $\left(\frac{17}{2}, \frac{9}{2}\right)$, $D(19, 8)$

6. $p = -2$, $r = 3$ **7.** **(b)** $AC = BD = \sqrt{20}$ units, Yes

Exercise 7.3

1. **(a)** -1 **(b)** $\frac{7}{4}$ **(c)** $t + 1$ **(d)** 2

2. $M(3, 1)$, $N\left(\frac{9}{2}, \frac{9}{2}\right)$ **3.** 8 **5.** $\frac{1}{2}$, 1 **6.** -1, 2 **7.** 1, 2

Exercise 7.4

1. $y = 2x + 8$ **2.** $y = 3x - 12$

3. **(a)** $y = -x + 6$ **(b)** $y = -\frac{1}{2}x + \frac{5}{2}$ **(c)** $y = \frac{1}{5}x - 1$ **(d)** $y = 3x - 3a$

4. **(a)** $y = -3x + 2$ **(b)** $y = -2x + 5$

5. **(a)** (2, 0), (0, -4) **(b)** (-2, 0), (0, 3) **(c)** (3, 0), (0, -4)

6. $y = 2x + 5$; $\frac{5\sqrt{5}}{2}$ units **7.** $y = t(x - t)$, $t = 2, 3$ **8.** $y = \frac{1}{t}x + t$, $t = -1, 2$

9. $5\frac{1}{2}$ **10.** **(a)** $y = 3x + 9$, $y = 4 - 2x$ **(b)** 15 sq. units

Exercise 7.5

1. $y = 4x + 7$ **2.** $2x - 3y - 12 = 0$ **3.** $y = 3x - 2$

4. $4x + 3y - 1 = 0$; $p = 4$ **5.** $y = 2x - 5$

6. **(a)** (1, -2) **(b)** $\left(4, \frac{1}{2}\right)$

7. (3, -1) **8.** $3x + 2y + 5 = 0$ **9.** $P(4, 1)$

10. $x + 3y + 2 = 0$ **11.** $a = 2$, $b = 9$

Exercise 7.6

2. $\angle ABC$ **3.** -1 **4.** **(a)** -12 **(b)** -3, 5

5. $y = 2x - 3$, $F(2, 1)$ **6.** $a = -\frac{1}{3}$, $b = -\frac{5}{3}$

7. $F(5, 5)$; $\sqrt{5}$ units **8.** -3, 15

9. **(a)** 2 units **(b)** $3\sqrt{2}$ units **(c)** $\frac{16\sqrt{5}}{5}$ **(d)** $\frac{3}{5}$

10. $\frac{11\sqrt{5}}{5}$ units **11.** **(a)** 2 **(b)** $\frac{12\sqrt{5}}{5}$ units

12. **(a)** $a = 2$, $b = -\frac{1}{2}$ **(b)** $2\sqrt{5}$ units **13.** 1 : 12

14. **(a)** $y = -2x + 21$ **(b)** $4y = -x + 42$; $P(6, 9)$; 2 : 3

15. $x + 2y = 14$; 6 : 1

16. **(a)** $x + 2y - 11 = 0$ **(b)** (1, 5) **(c)** (2, 7)

Exercise 7.7

1. (a) $y = -x + 5$ (b) $y = \frac{1}{2}x$ (c) $y = \frac{1}{2}x + \frac{23}{4}$

(d) $y = -3x + 7$ (e) $y = \frac{1}{2}x - \frac{5}{4}$ (f) $y = \frac{1}{3}x + \frac{10}{3}a$

2. $x + 3y - 7 = 0$, $P(7, 0)$
3. $(-4, 0)$, $(0, 1)$
4. $x - 2y + 3 = 0$, $P(7, 5)$, 10
5. (a) $4y = 14x - 45$ (b) $x + y = 9$ (c) $\left(4\frac{1}{2}, 4\frac{1}{2}\right)$
6. $t = 2$, $D(1, 8)$
7. 0, 2

Exercise 7.8

1. $(4, 12)$, $(-2, 0)$
2. $(2, -4)$, $(18, 12)$
3. $\left(-\frac{3}{2}, -\frac{1}{2}\right)$
4. (a) $(2, 7)$ (b) $6\sqrt{2}$ units
5. (a) $2\sqrt{10}$ units (b) $x - 3y + 4 = 0$
6. $(1, 1)$, $\left(-\frac{7}{4}, -\frac{3}{8}\right)$
7. $\left(\frac{4}{3}, -4\right)$
8. $\left(2\frac{1}{2}, 4\frac{1}{2}\right)$

Miscellaneous Exercise 7

1. $\left(\frac{7}{2}, \frac{3}{2}\right)$, $D(5, 6)$
3. $a = 5$ or 15, $D(-7, -11)$ or $D(3, -11)$
4. $a = 2$ or $5\frac{1}{3}$
5. $x = 3$, $y = 4$
6. (b) $\frac{1}{4}(a^2 + b^2)$
7. $6\sqrt{2}$ units, 18 sq. units, $3\sqrt{2}$ units
8. $P(-2, 0)$, $Q(4, 4)$
9. (a) $\alpha = 6$, $\beta = 7$ (b) $2\sqrt{10}$ units (c) $4\sqrt{10}$ units (d) 40 sq. units
10. (a) $p = \frac{1}{2}$, $q = 8\frac{1}{2}$, $r = -3$ (b) 25 sq. units
11. (a) 2 (b) -3, 5
12. (a) $3x - 2y + 9 = 0$, $t = -5$ (b) $3x - 2y - 11 = 0$, $a = -3$
13. $P(1, -1)$, $2\frac{1}{4}$ sq. units
14. $y = m(x - 2) + 3$ (a) $A\left(\frac{2m - 3}{m}, 0\right)$, $B(0, 3 - 2m)$ (b) $-\frac{1}{2}$, $-4\frac{1}{2}$
15. (a) $7x - 4y + 25 = 0$ (b) $3x - y - 5 = 0$ (c) $(9, 22)$ (d) 25 sq. units
16. (a) $a = \frac{1}{2}$, $b = -3$ (b) $(1, 4)$
17. (a) $y + 2x = 10$ (b) $2y = x - 5$; $(5, 0)$
18. (a) $3y + 2x = 5$ (b) $y = 5$; $(-5, 5)$
19. (a) Equation of BC is $y = \frac{3}{4}x - 4$, Equation of AD is $4x + 3y - 18 = 0$

(b) $(8, 2)$ (c) $(16, 8)$ (d) 5 units (e) 50 unit^2

20. (a) 22 (b) $x - 2y + 14 = 0$ (c) $y + 2x + 3 = 0$

(d) $(-4, 5)$ (e) 270 unit^2

21. (a) $(8, 4)$

(b) Equation of AB is $y = x + 4$, Equation of BD is $y = 3x - 10$, Equation of CD is $2y = x$

(c) $B(7, 11)$, $D(4, 2)$ (d) 30 unit^2

22. (a) $y = 3x + 2$ (b) $D(0, 2)$ (c) $y = 3x - 8$ (d) $C(2, -2)$; 40 unit^2

23. **(a)** $AD: y = \frac{1}{3}x + 2$, $CD: y = -3x + 52$

(b) $(15, 7)$ **(c)** $\left(\frac{15}{2}, \frac{59}{2}\right)$ **(d)** $5 : 3$ **(e)** $9 : 16$

24. $3x + 2y = 22$, $B(0, 11)$, $D(8, -1)$; $\frac{4\sqrt{65}}{5}$ units

25. $3y + 4x = 17$, $p = -2$; $D(6, 6)$; 50 sq. units

26. **(a)** $2y = x - 5$, $2y + x = 13$ **(b)** $C(9, 2)$ **(c)** $D\left(\frac{27}{5}, \frac{1}{5}\right)$ **(d)** $\frac{12\sqrt{5}}{5}$

27. **(a)** $(7, 5)$ **(b)** $(8, 3)$ **(c)** 20 sq. units

28. **(a)** $(6, 4)$ **(b)** 15; $T(9, 5)$

29. **(a)** 1 **(b)** $3x + 4y = 4$; $\left(\frac{4}{3}, 0\right)$ **(c)** $D\left(\frac{11}{2}, 0\right)$; $14\frac{7}{12}$ sq. units

30. $AB = \frac{\lambda + 1}{\lambda^2}\sqrt{2\lambda^2 - 2\lambda + 1}$, $\lambda = \frac{1}{2}$

31. **(a)** $\frac{a^2(1 + t^2)}{t}$ **(b)** $\frac{a(1 + t^2)^2}{t^2}$

8 Exercise 8.1

1. **(a)** $y = 2x^2 + 1$ **(b)** $y = -x^2 + \frac{1}{x}$ **(c)** $y = \frac{2x}{4x - 1}$ **(d)** $y = x^2 - 3x$
2. $a \approx -0.2$, $b \approx 6.4$
3. $a \approx 0.5$, $b \approx -0.1$
4. $h \approx 1.5$, $k \approx 0.5$
5. $p \approx -1.5$, $q \approx 42$
6. $a \approx 3.2$, $b \approx 1.6$
7. **(a)** Plot xy against y, b = gradient, $-a$ = xy-intercept

   (b) Plot $\frac{y}{\sqrt{x}}$ against $\sqrt{x}$, q = gradient, p = $\frac{y}{\sqrt{x}}$-intercept

   (c) Plot $(y - x^2)$ against x, a = gradient, b = $(y - x^2)$-intercept

   (d) Plot $(y^2 - x)$ against y, a = gradient, $-b$ = $(y^2 - x)$-intercept

   (e) Plot $\frac{y^2}{x}$ against x, $-\frac{m}{n}$ = gradient, $\frac{1}{n} = \frac{y^2}{x}$-intercept

   (f) Plot lg y against lg x, b = gradient, $-$lg a = (lg y)-intercept

   (g) Plot lg y against x, $-$lg q = gradient, lg p = (lg y)-intercept

   (h) Plot $\frac{e^y}{x}$ against x; p = gradient, $-q$ = $\left(\frac{e^y}{x}\right)$-intercept
8. Plot 10^y against x, a = gradient, b = 10^y-intercept
9. $h = 10$, $k = -2$
10. $a \approx 1.58$, $b \approx 2.67$
11. $a \approx 2$, $b \approx -0.4$
12. $C \approx 1.2$, $D \approx 1.5$
13. Plot lg y against x, $k \approx 12$, $b \approx 2.5$

Miscellaneous Exercise 8

1. **(a)** $y = \frac{15}{2x+3}$ **(b)** $y = 1000x^2$
2. $h = 4$, $k = -16$, $r = 4$
3. $a = 3$, $b = -4$
4. Plot y^2 against x, a = gradient, b = y^2-intercept

5. (a) 4 (b) $y = \frac{3x + 8}{2x^2}$ 6. (a) $p \approx -3$, $q \approx 1.5$ (b) $x \approx 3.4$

7. $xy = ax^2 + b$ (a) $a \approx -0.67$, $b \approx 7.47$ (b) $y \approx 3.26$

8. (a) 25.1; 17.78 (b) $C = 100$, $a \approx 1.78$ (c) $x \approx 8$

9. (a) Plot y^2 against x^2. If the graph is a straight line, then the equation is valid and $-\frac{a}{b}$ = gradient, $\frac{1}{b}$ = y^2-intercept.

(b) Plot lg y against lg x. If the graph is a straight line, then the equation is valid and d = gradient, lg c = lg y-intercept.

10. $A \approx 21$, $b \approx 0.56$

11. (a) $a \approx 2$, $b \approx 1.3$ (b) ± 0.92 (c) 0.28

12. $k = 7.39$, $p = 0.472$

13. (a) $a \approx 2$, $b \approx 0.5$ (b) 1.90 s (c) 0.25 m

14. (a) 0.91, 1.00 (b) 0.5 (c) $p \approx -1$, $q \approx 0.5$

15. (a) 2.83 (b) $a \approx -0.4$, $b \approx -2$

16. (a) $m \approx 3$, $n \approx 2$ (b) 1.26

9

Exercise 9.1

1. The relation in (b) is not a function as $2 \mapsto -3$ and $2 \mapsto 1$.

2. (a) function (b) function

(c) Not a function, the vertical line $x = \frac{1}{2}$ cuts the graph at two points.

3. (a) $\{-2, 14, 22\}$ (b) $\frac{3}{2}$, 1 4. (a) 1, -9, 27 (b) $-\frac{1}{2}$, 3

5. (a) $a = 3$, $b = -5$ (b) -2, 4, 10

6. (a) $a = 2$, $b = 5$, $c = -1$ (b) $\frac{1}{2}$

7. (a) $a = 3$, $b = 1$ (b) $\frac{2}{3}$ (c) $2\frac{2}{3}$, -3

8. (a) $a = 1$, $b = -1$ (b) -1, 2 9. (a) $a = 7$, $b = 6$ (b) 13

10. (a) (i) $4 \leqslant f(x) \leqslant 14$ (ii) $0 \leqslant x \leqslant \frac{6}{5}$

(b) (i) $3 \leqslant f(x) \leqslant 7$ (ii) $-1 \leqslant x \leqslant 2$

(c) (i) $2 \leqslant f(x) \leqslant 4$ (ii) $0 \leqslant x \leqslant 6$

(d) (i) $1 \leqslant f(x) \leqslant 5$ (ii) $1 \leqslant x \leqslant 4$

11. (a) $0 \leqslant f(x) \leqslant 8$ (b) $2 \leqslant x \leqslant 4$

12. (a) $1 < x < 2$ (b) (i) $1 \leqslant g(x) \leqslant 9$ (ii) $0 \leqslant g(x) \leqslant 9$

13. (a) $1 \leqslant f(x) \leqslant 9$ (b) $2 \leqslant x \leqslant \sqrt{2} + 1$ or $1 - \sqrt{2} \leqslant x \leqslant 0$

14. $-\frac{1}{2}$, $-\frac{2}{5}$ 15. $a = 2$, $b = 1$

16. (a) $3a + 2$ (b) $9a - 4$ (c) $6a^2 - 4$ (d) $\frac{2a - 1}{a + 1}$

17. $a = 1$, $b = -1$; $a = -1$, $b = -1$

Exercise 9.2

1. (a) gf: $x \mapsto 3 - 6x$; fg: $x \mapsto 9 - 6x$ (b) gf: $x \mapsto 1 - 4x - 4x^2$; fg: $x \mapsto 5 - 2x^2$

(c) gf: $x \mapsto \frac{2}{x - 4}$, $x \neq 4$; fg: $x \mapsto \frac{2}{x} - 4$, $x \neq 0$

(d) gf: $x \mapsto \dfrac{1 + 2x}{2x}$, $x \neq 0$; fg: $x \mapsto \dfrac{3x - 1}{x - 1}$, $x \neq 1$

2. **(a)** f^2: $x \mapsto 4x + 9$; f^3: $x \mapsto 8x + 21$

(b) f^2: $x \mapsto \dfrac{9x}{2x + 1}$; $x \neq 1, -\dfrac{1}{2}$; f^3: $x \mapsto \dfrac{27x}{7x - 1}$, $x \neq 1, -\dfrac{1}{2}, \dfrac{1}{7}$

(c) f^2: $x \mapsto x$, $x \neq 1$; f^3: $x \mapsto \dfrac{x}{x - 1}$, $x \neq 1$

(d) f^2: $x \mapsto \dfrac{3(2x - 1)}{7 - 2x}$, $x \neq \dfrac{1}{2}, \dfrac{7}{2}$; f^3: $x \mapsto \dfrac{3(7 - 2x)}{14x - 13}$, $x \neq \dfrac{1}{2}, \dfrac{7}{2}, \dfrac{13}{14}$

3. fg: $x \mapsto 3x^2 + 22$; gf: $x \mapsto 9x^2 + 24x + 22$ **(a)** 1, 2 **(b)** 0, -4

4. **(a)** f^2: $x \mapsto 9x + 8$ for $-1 \leqslant x \leqslant 2$, $-1 \leqslant f^2(x) \leqslant 26$

(b) f^2: $x \mapsto x^4$ for $-1 \leqslant x \leqslant 16$

5. $q = 2p$ **(a)** -3 **(b)** -3

6. fg: $x \mapsto \dfrac{x - 1}{x + 1}$, $x \neq 1$; gf: $x \mapsto \dfrac{2x - 1}{2x}$, $x \neq 0$; $x = \dfrac{1}{3}$ **7.** $g(x) = x^2 + 4$

8. **(a)** $a = 3$, $b = 2$ **(b)** f^4: $x \mapsto 81x + 80$

9. **(a)** $f^2(x) = x$, $x \neq 1$ **(b)** $a = 3$, $b = -1$

10. **(a)** $a = 2$, $b = -3$ **(b)** gf: $x \mapsto \dfrac{8}{(x - 3)^2} - 3$, $x \neq 3$

11. fg: $x \mapsto \dfrac{x - 2}{2(x - 1)}$; $x \neq 1, 2$; gf: $x \mapsto -\dfrac{1}{1 + 2x}$, $x \neq -1, -\dfrac{1}{2}$ **(b)** 0, $\dfrac{5}{2}$

12. $f^5(x) = \dfrac{x + 1}{x - 1}$, $x \neq 1$; $f^{10}(x) = x$, $x \neq 1$

13. **(a)** f^2: $x \mapsto x - 4$ **(b)** f^5: $x \mapsto x - 10$ **(c)** g^2: $x \mapsto x$, $x \neq 0$

(d) g^5: $x \mapsto \dfrac{2}{x}$, $x \neq 0$ **14.** $a = 1$, $b = 2$ **15.** 1, 2

16. $f^3(x) = \dfrac{x}{3x + 1}$, $x \neq -1$, $x \neq -\dfrac{1}{2}$, $x \neq -\dfrac{1}{3}$; $f^n(x) = \dfrac{x}{nx + 1}$, $x \neq -1$, $x \neq -\dfrac{1}{2}$, $x \neq -\dfrac{1}{3}$,

$x \neq -\dfrac{1}{4}, \ldots, x \neq -\dfrac{1}{(n - 1)}, x \neq -\dfrac{1}{n}$

Exercise 9.3

2. **(a)** $-1 < f(x) < 3$ **(b)** $-1 < x < 3$

3. **(a)** f^{-1}: $x \mapsto \dfrac{1}{3}(x + 2)$ **(b)** f^{-1}: $x \mapsto \dfrac{3 + x}{x}$, $x \neq 0$

(c) f^{-1}: $x \mapsto \dfrac{2x}{x - 2}$, $x \neq 2$ **(d)** f^{-1}: $x \mapsto \dfrac{x + 3}{2(x - 1)}$, $x \neq 1$

4. **(a)** $\dfrac{11}{5}, \dfrac{7}{3}$ **(b)** 6 **5.** **(a)** 5 **(b)** $\dfrac{p + 2}{p(p - 2)}$

6. **(a)** $\dfrac{1}{2}$ **(b)** $\dfrac{1}{3}; \dfrac{2}{3}$ **7.** **(a)** $a = 2$, $b = 1$ **(c)** $f^{-1}(x) = \dfrac{x + 1}{x - 1}$, $x \neq 1$

8. fg: $x \mapsto \dfrac{2}{x} + 3$, $x \neq 0$; gf: $x \mapsto \dfrac{1}{2x + 3}$, $x \neq -\dfrac{3}{2}$; f^{-1}: $x \mapsto \dfrac{x - 3}{2}$;

g^{-1}: $x \mapsto \dfrac{1}{x}$, $x \neq 0$; $x = -\dfrac{1}{3}$

9. $f^{-1}: x \mapsto \frac{x+1}{x-2}, x \neq 2$ (a) 4 (b) $\frac{1}{2}, -2$

10. (a) -2 (b) $fg: x \mapsto \frac{3}{2x+3}, x \neq -\frac{3}{2}$ (c) $\frac{1}{7}$

11. $f(5) = 7.5$ $f^{-1}(5) = 7.5$

12. -5

13. (a) $f^{-1}: x \mapsto \frac{a+x}{x}, x \neq 0$ (b) $\frac{1}{3}$

14. $f^{-1}: x \mapsto \frac{x}{x-2}$; $x \neq 2$; $a = 9$

15. $\frac{9}{2}, \frac{11}{2}, \frac{11}{2}$

16. (a) $f^{-1}: x \mapsto \frac{3}{2x+1}, x \neq -\frac{1}{2}$ (b) 1

17. (a) $f^{-1}: x \mapsto \frac{x+1}{1-x}, x \neq 1$ (b) $m = -3$, $c = 12$

18. (b) 3

19. (a) $ff: x \mapsto 4x + 3$ (b) $g^{-1}: x \mapsto 5 - x$ (c) $fg: x \mapsto 11 - 2x$
(d) $(fg)^{-1}: x \mapsto \frac{1}{2}(11 - x)$ (e) $f^{-1}: x \mapsto \frac{1}{2}(x - 1)$ (f) $g^{-1}f^{-1}: x \mapsto \frac{1}{2}(11 - x)$

20. (a) $f^2: x \mapsto \frac{2x+1}{2x+3}, x \neq -\frac{1}{2}, -\frac{3}{2}$; $f^{-1}: x \mapsto \frac{1-x}{2x}, x \neq 0, x \neq -\frac{1}{2}$

21. $fg: x \mapsto 6x - 1$; $f^{-1}: x \mapsto \frac{x-3}{2}$; $g^{-1}: x \mapsto \frac{x+2}{3}$

22. (a) $fg: x \mapsto \frac{6}{x} + 1, x \neq 0$; $(fg)^{-1}: x \mapsto \frac{6}{x-1}, x \neq 1$
(b) $f^{-1}: x \mapsto \frac{x-1}{3}$; $g^{-1}: x \mapsto \frac{2}{x}, x \neq 0$; $g^{-1}f^{-1}: x \mapsto \frac{6}{x-1}, x \neq 1$

23. (a) $fg: x \mapsto \frac{5-3x}{x-1}, x \neq 1$; $(fg)^{-1}: x \mapsto \frac{5+x}{x+3}, x \neq -3, x \neq 1$
(b) $f^{-1}: x \mapsto \frac{x+3}{2}$; $g^{-1}: x \mapsto \frac{1}{x} + 1, x \neq 0$; $g^{-1}f^{-1}: x \mapsto \frac{x+5}{x+3}, x \neq -3$; Yes

24. $f^{-1}: x \mapsto \frac{5-x}{2}$; $g^{-1}: x \mapsto \frac{1+x}{1-x}, x \neq 1$; $g^{-1}f^{-1}: x \mapsto \frac{7-x}{x-3}, x \neq 3$;
$(fg)^{-1}: x \mapsto \frac{7-x}{x-3}, x \neq 3$

Exercise 9.4

1. (a) $7, 5; -\frac{2}{3}, \frac{4}{3}$ (b) 1, 3

2. (a) 1, 3 (b) $\frac{2}{3}, 4$ (c) $-1, 3$ (d) $\frac{4}{3}, 6$

3. (a) $0 \leqslant f(x) < 3$ (b) $0 \leqslant f(x) \leqslant 5$ (c) $0 \leqslant f(x) \leqslant 5$
(d) $0 \leqslant f(x) < 5$ (e) $1 \leqslant f(x) \leqslant 3$ (f) $-2 \leqslant f(x) \leqslant 3$

4. (a) $1 \leqslant f(x) \leqslant 3$ (b) $-4 \leqslant f(x) \leqslant 0$ (c) $0 \leqslant f(x) \leqslant 2$ (d) $0 < f(x) \leqslant 5$

5. (a) 1, 3 (b) f is not a one-one function

6. (a) $0 \leqslant f(x) \leqslant 3$; 1 (b) $1 \leqslant f(x) < 8$; $\frac{3}{2}, 0$

7. (a) $\frac{1}{3} \leqslant x \leqslant 3$ (b) $x < -2$ or $x > 4$

8. (a) $-4 \leqslant y \leqslant 0$ (b) $4 \leqslant x \leqslant 5$

9. $x = 1, y = 1$

10. $x = 1, y = 1$

11. $x = \frac{5}{3}$

12. 0, 2, 4

13. $a = 1$, $b = 1$

Miscellaneous Exercise 9

1. (a) $p = 3,\ q = -1$ **(b)** 8 **(c)** 3 **2.** $a = 3;\ x = -1,\ 3$

3. (a) $f^{-1}: x \mapsto 2 - x$ **(b)** $ff: x \mapsto x$ **(c)** $gg: x \mapsto x,\ x \neq 0$

(d) $fg: x \mapsto \dfrac{2x - 3}{x},\ x \neq 0$ **(e)** $gfg: x \mapsto \dfrac{3x}{2x - 3},\ x \neq 0,\ \dfrac{3}{2}$

4. (a) $ff: x \mapsto 16x - 15$ **(b)** $gf: x \mapsto \dfrac{8x - 11}{4x - 3},\ x \neq \dfrac{3}{4}$

(c) $f^{-1}: x \mapsto \dfrac{1}{4}(x + 3)$ **(d)** $g^{-1}: x \mapsto \dfrac{5}{2 - x},\ x \neq 2$

(e) $(fg)^{-1}: x \mapsto \dfrac{20}{5 - x},\ x \neq 5$

5. (a) 7; –5; 2 **(b) (i)** $g^{-1}: x \mapsto \dfrac{3 - x}{x + 1},\ x \neq -1$

(ii) $fg: x \mapsto -\dfrac{2x}{x + 1},\ x \neq -1$ **(iii)** $gfg: x \mapsto \dfrac{5x + 3}{1 - 3},\ x \neq -1,\ 1$

6. (b) $f^3: x \mapsto \dfrac{x}{1 - 3x},\ x \neq 1,\ \dfrac{1}{2},\ \dfrac{1}{3}$ **7. (b)** $q = 5$ **8.** $p = 2,\ q = -1$

9. $fg: x \mapsto \dfrac{3a}{x - 2} + b,\ x \neq 2;\ gf: x > \dfrac{3}{ax + b - 2},\ x \neq \dfrac{2 - b}{a};\ a = -\dfrac{3}{2},\ b = \dfrac{13}{2}$

11. (a) $0 \leqslant f(x) \leqslant 7$ **(b)** $-2\frac{1}{4} \leqslant g(x) \leqslant 4$ **(c)** $0 \leqslant h(x) \leqslant 6$ **(d)** $-4 \leqslant k(x) \leqslant 3$

12. (a) $0 \leqslant f(x) \leqslant 9$ **(b)** $-3 \leqslant g(x) \leqslant 6$ **(c)** $0 \leqslant h(x) < 10$

13. The graphs are reflective images of each other in the line $y = x$; $1\frac{1}{2}$

14. $x = -\dfrac{1}{2},\ y = \dfrac{1}{2}$ or $x = -2,\ y = 1$

15. (a) $f^{-1}: x \mapsto \dfrac{2(x + 1)}{x - 1},\ x \neq 1$ **(b)** $m = 3,\ c = -13$

16. $a = 4,\ b = 3,\ 3 \leqslant y \leqslant 8$

17. (a) None **(b) (i)** 1.41 **(ii)** $0 \leqslant x \leqslant \dfrac{8}{5}$

Revision Exercise 5

1. (a) (i) $\begin{pmatrix} 45 & 32 \\ 26 & 18 \end{pmatrix}$ **(ii)** $\begin{pmatrix} -1 & 2 \\ -1 & 1.5 \end{pmatrix}$ **(iii)** $\begin{pmatrix} 5 & 2 \\ -3 & -1 \end{pmatrix}$

(b) $-\dfrac{1}{4}\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$; $x = -2,\ y = 4$

2. (a) $y = 2x - 6$ **(b)** (6, 6) **(c)** 15 sq. units; 15 sq. units

3. (a) $y = x - 3$ **(b)** (9, 6) **(c)** (6, 9) **(d)** 42 sq. units

4. $a = 9.5,\ b = 4$ **5.** Plot $xy - x$ against y; gradient $= -q$, $(xy - x)$-intercept $= p$

6. (a) (i) 2 **(ii)** 1

(b) (i) –2, 8 **(ii)** $gf: x \mapsto -\dfrac{2}{3x},\ x \neq 0;\ gg: x \mapsto \dfrac{2x - 2}{x + 1},\ x \neq -1,\ 1$

7. (a) $0 \leqslant y \leqslant 7$ **(b)** $x < 1$ or $x > \dfrac{11}{3}$

Revision Exercise 6

1. (a) $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ (all same answer)

(b) (ii) $a = -6, 1$ (iii) $\frac{1}{4}\begin{pmatrix} -6 & 2 \\ 13 & -5 \end{pmatrix}$

2. $(2, 1)$, $\sqrt{13}$; $(5, -1)$

3. (a) $2y + x = 6$, $y = x - 1$ (b) $D\left(\frac{8}{3}, \frac{5}{3}\right)$ (c) $E(3, 1)$ (d) -2

4. (a) $a = 4$, $b = -6$; 6 (b) $g: x \mapsto \frac{3x + 1}{2 - x}, x \neq 2$

5. $gf: x \mapsto 4x^2 + 1$; $fg: x \mapsto 2x^2 - 4x + 5$; $h: x \mapsto |8x - 9|$

6. $\lg y = \left(\frac{1}{2}\lg b\right)x + \lg a$ (a) $a = 4$, $b = 2.7$ (b) 3.69

7. $a = -\frac{5}{2}$, $b = 6$

Revision Exercise 7

1. (a) (i) $\begin{pmatrix} 6 & 6 & -4 \\ -5 & -2 & 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & 9 & -4 \\ -2 & -17 & 8 \end{pmatrix}$, $a = 20$, $b = -12$, $c = 10$

   (b) $x = 3$, $y = -1$

2. $(7, 6)$

3 (a) $fg: x \mapsto \frac{4}{x} + 3, x \neq 0$ (b) $gf: x \mapsto \frac{2}{2x + 3}, x \neq -\frac{3}{2}$

4. (a) 4, 6 (b) $ff: x \mapsto \frac{16x}{x + 9}, x \neq -9, 3$ (c) $k > 2$ or $k = 0$

5. $0 \leqslant x \leqslant 4$

6. $y = \frac{18 - 3x}{2x + 3}$

7. (a) $a = 2$, $b = -0.4$ (b) 1.64 (c) 2.49

Revision Exercise 8

1. (a) $\mathbf{PQ} = \mathbf{R}$

   (b) The elements give the amounts Mary would spend by getting her supply from shop A and shop B.

   (c) $\begin{pmatrix} 0.3 & -0.2 \\ -0.4 & 0.6 \end{pmatrix}$

   (d) $a = 3.80$, $b = 6.80$; A box of diskettes costs \$3.80 at shop B while a ream of paper costs \$6.80 at shop A.

2. (a) $y = 2mx - 4m - 1$, $y = mx + m + 1$

   (b) $\left(\frac{2 + 5m}{m}, 6m + 3\right)$ (c) $\left(7\frac{2}{3}, 7\frac{1}{2}\right)$

3. (a) $(6, 8)$ (b) $y = x + 2$, $(3.6, 5.6)$

4. (a) 0.11; 0.26 (b) $A = 0.26$, $n = 3$

5. 3

6. $-1 < x < 2$ or $3 < x < 6$

7. $16x - 5y = xy$

Exercise 10.1

1. (a) $\frac{3}{5}, \frac{4}{5}, \frac{4}{3}$ (b) $\frac{2}{3}, \frac{\sqrt{5}}{3}, \frac{\sqrt{5}}{2}$ (c) $\frac{12}{13}, \frac{5}{13}, \frac{5}{12}$

2. (a) $\frac{15}{17}, \frac{15}{8}$ (b) $\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}$ (c) $\frac{24}{25}, \frac{7}{24}$ (d) $\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}$

3. $\frac{1}{4}$ 4. $4\frac{1}{2}$ 5. (a) $\frac{1}{\sqrt{6}}$ (b) 2

6. (a) 1 (b) 1

7. (a) 3rd, 70° (b) 1st, 30° (c) 4th, 60° (d) 3rd, 80°

8. (a) 130° (b) 300° (c) 210° (d) –135°

9. (a) 20°, 160°, 200°, 340° (b) 70°, 110°, 250°, 290° (c) 35°, 145°, 215°, 325°

10. (a) –170°, –10°, 10°, 170°

11. (a) –260°, 100°, 460° (b) –100°, 260°, 620° (c) –280°, 80°, 440°

Exercise 10.2

1. (a) negative (b) negative (c) positive (d) positive
   (e) negative (f) positive (g) positive (h) positive

2. (a) 1st, 3rd (b) 4th (c) 1st, 2nd (d) 2nd, 3rd

3. (a) $-\sqrt{3}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) -1
   (e) $-\frac{1}{\sqrt{2}}$ (f) $-\frac{\sqrt{3}}{2}$ (g) $\sqrt{3}$ (h) $\frac{1}{\sqrt{2}}$

4. $\frac{3}{5}, \frac{3}{4}$ 5. $-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}$ 6. $\frac{5}{12}, -\frac{12}{13}$

7. $-\frac{1}{\sqrt{5}}, -2$ 8. $-\frac{\sqrt{3}}{2}, \sqrt{3}$ 9. $\frac{12}{13}, -\frac{5}{13}$

10. (a) $-\frac{1}{\sqrt{3}}$ (b) $\sqrt{\frac{2}{3}}$ (c) $-\sqrt{2}$

11. (a) $-k$ (b) $\sqrt{1-k^2}$ (c) $-\frac{k}{\sqrt{1-k^2}}$ (d) $\sqrt{1-k^2}$

12. (a) 135.2°, 224.8° (b) 60°, 240° (c) 60°, 120° (d) 116.6°, 296.6°
    (e) 72.5°, 287.5° (f) 62.2°, 242.2° (g) 188.6°, 351.4°
    (h) 45°, 135°, 225°, 315° (i) 35.3°, 144.7° (j) 63.0°, 297.0°

13. (a) –150°, –30° (b) –330°, –30°, 30° (c) –315°, –135°, 45°
    (d) –225°, –135°, 135°

14. $x = 64.0°$, $y = 118.0°$

Exercise 10.3

1. (a) 90°, 270° (b) 0°, 180°, 360°
   (c) 0°, 180°, 360° (d) 0°, 60°, 180°, 300°, 360°
   (e) 0°, 104.0°, 180°, 284.0°, 360° (f) 19.5°, 135°, 160.5°, 315°

2. (a) –10, 4 (b) –3, 7 (c) 1, 7

3. (a) $\{y : -3 \leqslant y \leqslant -1, y \in \mathbb{R}\}$ (b) $\{y : -5 \leqslant y \leqslant 5, y \in \mathbb{R}\}$
   (c) $\{y : -6 \leqslant y \leqslant 2, y \in \mathbb{R}\}$ (d) $\{y : -1 \leqslant y \leqslant 3, y \in \mathbb{R}\}$
   (e) $\{y : -6 \leqslant y \leqslant 0, y \in \mathbb{R}\}$ (f) $\{y : 0 \leqslant y \leqslant 5, y \in \mathbb{R}\}$
   (g) $\{y : y \geqslant 0, y \in \mathbb{R}\}$ (h) $\mathbb{R}$ (i) $\{y : y \geqslant 0, y \in \mathbb{R}\}$

7. 2 8. 4

9. (a) 2 (b) $180° < x < 270°$ 10. –5

Exercise 10.4

1. (a) $\frac{5}{4}, \frac{5}{3}, \frac{4}{3}$ (b) $\frac{5}{4}, -\frac{5}{3}, -\frac{4}{3}$ (c) $-\frac{2}{\sqrt{3}}, 2, -\sqrt{3}$ (d) $-\sqrt{10}, -\frac{\sqrt{10}}{3}, \frac{1}{3}$

2. (a) $-\frac{1}{2}$ (b) $-\frac{2}{\sqrt{5}}$ (c) $\sqrt{5}$

3. **(a)** $\frac{1}{3}$ **(b)** $\frac{3}{\sqrt{8}}$ **(c)** $\frac{1}{\sqrt{8}}$

4. **(a)** $\sqrt{2}$ **(b)** $-\sqrt{3}$ **(c)** $-\sqrt{3}$

5. **(a)** 2nd, 4th **(b)** 1st, 2nd **(c)** 1st, 3rd **(d)** 3rd, 4th

6. **(a)** 210°, 330° **(b)** 36.9°, 216.9° **(c)** 54.2°, 305.8° **(d)** 136.0°, 316.0°
(e) 90°, 270° **(f)** 45°, 135°, 225°, 315° **(g)** 30°, 150°, 210°, 330°
(h) 45°, 135° **(i)** 90°, 270° **(j)** 65.2°, 294.8°

7. **(a)** 2 **(b)** −2 **(c)** 1 **(d)** $-\frac{2}{\sqrt{3}}$ **(e)** $-\sqrt{3}$ **(f)** $-\frac{2}{\sqrt{3}}$

8. **(a)** 3 **(b)** 5

Miscellaneous Exercise 10

1. **(a)** 97.5°, 277.5° **(b)** 210°, 330°
(c) 53.1°, 126.9°, 233.1°, 306.9° **(d)** 26.6°, 39.8°, 206.6°, 219.8°

2. **(a)** $-\frac{\sqrt{3}}{4}$ **(b)** $-3\sqrt{3}$

3. **(a)** $\frac{t}{\sqrt{1+t^2}}$ **(b)** $\sqrt{1-t^2}$ **(c)** $\frac{1}{t}$ **(d)** $-\frac{1}{t}$

4. **(a)** $-\frac{3}{5}$ **(b)** $-\frac{3}{4}$ **(c)** $-\frac{5}{3}$

5. max. $y = 10$, $x = 180°$, min. $y = 4$, $x = 0°$, 360°

6. **(a)** $\{y : -7 \leqslant y \leqslant 3,\ y \in \mathbb{R}\}$ **(b)** $\{y : -1 \leqslant y \leqslant 3,\ y \in \mathbb{R}\}$ **(c)** $\mathbb{R}$
(d) $\{y : -1 \leqslant y \leqslant 3,\ y \in \mathbb{R}\}$ **(e)** $\mathbb{R}$
(f) $\{y : 0 \leqslant y \leqslant 12,\ y \in \mathbb{R}\}$ **7.** 3 **8.** ±136.7°

9. $\frac{12}{13}$, $-\frac{5}{12}$ **10.** k, $\frac{\sqrt{1-k^2}}{k}$ **11.** 2 **12.** $135° < \frac{\alpha}{2} < 180°$

13. −194.0°, −14.0°, 166.0°, 346.0° **14.** $a = 5$, $b = 2$

15. **(a)** 210°, 330° **(b)** $0° \leqslant x \leqslant 210°$ or $330° \leqslant x \leqslant 360°$

Exercise 11.1

1. **(a)** $\pm\frac{\sqrt{8}}{3}$ **(b)** $\pm\frac{1}{\sqrt{8}}$ **3.** $\frac{1}{6}xy$ **4.** $\pm\frac{4}{3}$

6. **(a)** 1 **(b)** $\cos x$ **(c)** $\tan^2 x$ **(d)** $\operatorname{cosec} x$ **7.** $\frac{1}{3}$, $\frac{4}{5}$

Exercise 11.2

1. **(a)** 111.8°, 291.8° **(b)** 53.1°, 233.1°

2. **(a)** 14.5°, 90°, 165.5°, 270° **(b)** 0°, 120°, 240°, 360°
(c) 45°, 108.4°, 225°, 288.4° **(d)** 0°, 45°, 135°, 180°, 225°, 315°, 360°

3. **(a)** 30°, 150°, 210°, 330° **(b)** 90°, 270°
(c) 110.9°, 159.1°, 290.9°, 339.1° **(d)** 1.6°, 38.4°, 181.6°, 218.4°
(e) 53.3°, 143.3°, 233.3°, 323.3° **(f)** 52.8°, 157.2°, 232.8°, 337.2°

4. **(a)** 60°, 300° **(b)** 35.3°, 144.7°, 215.3°, 324.7° **(c)** 14.0°, 45°, 194.0°, 225°

5. **(a)** 7.5°, 97.5°, 187.5°, 277.5° **(b)** 60°, 180°, 300° **(c)** 70.5°, 289.5°

6. **(a)** 39.0°, 141.0°, 219.0°, 321.0° **(b)** 70.5°, 90°, 270°, 289.5°
(c) 30°, 90°, 150° **(d)** 0°, 131.8°, 180°, 228.2°, 360°
(e) 120.5°, 159.5°, 300.5°, 339.5° **(f)** 45°, 135°, 225°, 315°
(g) 0°, 63.4°, 180°, 243.4°, 360° **(h)** 176.9°, 356.9°

(i) 180° (j) 60°, 300° (k) 146.9° (l) 71.6°, 153.4°, 251.6°, 333.4°

7. (a) 55°, 125° (b) 32.8°, 100.5°, 152.8° (c) 35.3°, 144.7° (d) 135°, 153.4°

8. $(\cos x + 2)(2 \sin x - 1)$; –330°, –210°, 30°, 150°

9. $-\frac{\sqrt{5}}{2}$

10. (a) $\{y : -2 \leqslant y \leqslant 0, y \in \mathbb{R}\}$ (b) $\{y : 0 \leqslant y \leqslant 3, y \in \mathbb{R}\}$
(c) $\{y : -2 \leqslant y \leqslant 6, y \in \mathbb{R}\}$ (d) $\{y : -1 \leqslant y \leqslant 7, y \in \mathbb{R}\}$

12. 4 13. (a) 7 (b) 4 14. 3 15. (a) 3 (b) 5

Miscellaneous Exercise 11

1. (a) $x^2 + y = 3$ (b) $y^2 = 4x^2 + 4$

2. (a) $\pm\sqrt{1 - s^2}$ (b) $\pm\frac{\sqrt{1 - s^2}}{s}$ (c) $\frac{1}{s}$

3. (a) $a^2 \cos^2 \theta$ (b) $\cos^3 \theta$

4. 26.6°, 206.6°

5. (a) 30°, 150°, 210°, 330° (b) 0°, 60°, 120°, 180°, 240°, 300°, 360°
(c) 56.3°, 116.6°, 236.3°, 296.6°

6. (a) 85.7°, 175.7°, 265.7°, 355.7° (b) 90°, 116.6°, 270°, 296.6° (c) 90°

7. (a) 90°, 221.8°, 318.2° (b) 0°, 38.9°, 180°, 321.1°, 360°
(c) 5°, 65°, 185°, 245°

8. (a) 32.0°, 90°, 212.0°, 270° (b) 60°, 300° (c) 90°, 170°, 330°

9. $\pm\frac{\sqrt{7}}{3}, \pm\sqrt{3}$

10. 146.8°, 326.8°

12. 262.8°

13. $-\frac{1}{2}, -\frac{1}{4}, 1$; 45°, 153.4°, 166.0°

15. 26.6°, 153.4°

16. 22.5°, 112.5°, 202.5°, 292.5°

17. 69.4°, 290.6°

18. 30°, 150°

19. (a) 21.1°, 338.9° (b) 22.5°, 90°, 112.5°, 202.5°, 292.5
(c) 30°, 90°, 150° (d) 68.2°, 135°, 248.2°, 315°

21. a

22. (a) $\frac{15}{32}$ (b) $\frac{8}{15}$

23. $-5 \leqslant y \leqslant 1$

24. (a) 1 (b) 2

26. $x = 15°$, $y = 135°$ or $x = 105°$, $y = 45°$

27. $\frac{k^2 - 1}{k^2 + 1}, \frac{2k}{k^2 + 1}$

28. –1

29. $\frac{1}{5}(x + 2y)$, $\frac{1}{5}(y - 2x)$, $x^2 + y^2 = 5$

30. $P(0°, 4)$, $Q(90°, 1)$

31. (a) $(0°, 0)$, $(60°, \sqrt{3})$, $(180°, 0)$, $(300°, -\sqrt{3})$, $(360°, 0)$
(b) $0° < x < 60°$ or $90° < x < 180°$ or $270° < x < 300°$

12

Exercise 12.1

1. (a) 22.5° (b) 120° (c) 135° (d) 150°

2. (a) $\frac{7\pi}{6}$ (b) $\frac{4\pi}{3}$ (c) $\frac{7\pi}{4}$ (d) $\frac{11\pi}{6}$

3. (a) $\{y : -3 \leqslant y \leqslant 1, y \in \mathbb{R}\}$ (b) $\{y : 1 \leqslant y \leqslant 2, y \in \mathbb{R}\}$ (c) $\{y : 0 \leqslant y \leqslant 3, y \in \mathbb{R}\}$
(d) $\{y : 2 \leqslant y \leqslant 4, y \in \mathbb{R}\}$ (e) $\{y : -5 \leqslant y \leqslant 5, y \in \mathbb{R}\}$ (f) $\mathbb{R}$

5. (a) $-\frac{1}{\sqrt{2}}$ (b) $\frac{\sqrt{3}}{2}$ (c) –1 (d) $-\frac{1}{2}$

6. (a) 2.49 (b) 1.93 (c) 1.78

7. (a) $\frac{\pi}{6}, \frac{11\pi}{6}$ (b) $\frac{\pi}{4}, \frac{5\pi}{4}$ (c) $\frac{7\pi}{6}, \frac{11\pi}{6}$

(d) 0.421, 2.72, 3.56, 5.86 (e) 0.464, 3.61 (f) 2.55, 5.70

(g) $\frac{\pi}{2}$, $\frac{3\pi}{2}$, 0.393, 2.75 (h) 0.398, 2.74, 3.54, 5.89 (i) 1.09, 1.68, 4.23, 4.82

(j) $\frac{\pi}{2}$, $\frac{7\pi}{6}$, $\frac{11\pi}{6}$

8. $\frac{2}{3}$, $3\frac{1}{3}$, $8\frac{2}{3}$

Exercise 12.2

1. (a) 4 cm (b) 9.37 cm
2. (a) 1.4 (b) 6 cm (c) 25 cm^2 (d) 6 cm (e) 40 cm^2 (f) 1
3. 9 cm^2 **4.** $\frac{4}{3}$ rad.; 24 cm^2 **5.** (a) 1.40 rad. (b) 20.7 cm^2
6. (b) 11.1 cm (c) 17.3% **7.** (a) 6 cm (b) 16 cm^2
8. 14.6 cm^2 **9.** 1.38 cm **10.** 12.5 cm^2; 16.2 cm
11. (a) 1.29 rad. (b) 5.89 cm^2 **12.** (a) 16.5 m (b) 2.28 m^2
13. 47.7 cm^2, 26.0 cm^2 **14.** 2.42 rad., 817 cm^2
15. (a) 10.5 cm (b) 9.1 cm^2 (c) 17.1 cm^2 (d) 5.32 cm

Miscellaneous Exercise 12

2. (a) 4 (b) 2 (c) 4 (d) 4
3. 1 **4.** 3 **5.** 1; 39
6. (a) 5 (b) 0 **7.** (a) 7.08 (b) 0.580
8. (a) 2.50 (b) $\frac{\pi}{12}$, $\frac{5\pi}{12}$ (c) 0, 2.42 (d) $\frac{\pi}{2}$, 0.381
9. (a) 120 cm^2 (b) 0.058 5 **10.** 0.644 rad., 32.7 cm^2
11. (a) 15.1 cm, 10.3 cm^2 (b) 25.9 cm, 15.3 cm^2 **12.** (a) 24 (b) 9 : 7
13. (b) 66.4 cm (c) 101 cm^2 (d) 294 cm^2 (e) 41.6%
14. (b) 38.7 cm (c) 99.0 cm^2
16. (a) 20 cm (b) 1.85 rad. (c) 77.1 cm^2
17. 12 cm (a) 10.6 cm (b) 7.86 cm (c) 14.6 cm^2 **18.** 2.64 cm^2
19. (a) 7.09 cm (b) 1.26 rad. (c) 0.388 cm (d) 2.55 cm^2
20. (a) 96π cm^2 (b) $64\sqrt{3}$ cm^2 **21.** 17.5 cm^2

13

Exercise 13.1

1. 8 **2.** 1024 **3.** 12 **4.** (a) 120 (b) 720 **5.** 24
6. 30 **7.** 20 **8.** 120 **9.** 96 **10.** 40 320

Exercise 13.2

1. (a) 720 (b) 362 880
2. 24 **3.** 24 **4.** 362 880 **5.** 120 **6.** 24
7. (a) $\frac{1}{5}$ (b) 56 (c) 210
8. (a) $\frac{1}{n}$ (b) $\frac{1}{n(n-1)(n-2)}$ (c) $n^2(n-1)$
9. (a) 210 (b) 90 (c) 120
10. 870 **11.** 1 663 200 **12.** 504 **13.** 5040 **14.** 720
15. 100 800 **16.** 120; 36 **17.** 336; 120 **18.** (a) 48 (b) 72
19. 5040 (a) 144 (b) 1440

20. (a) 576 (b) 144 (c) 1440
21. 120 22. 5040 (a) 720 (b) 1440
23. (a) 720 (b) 240 (c) 216
24. (a) 30 240 (b) 282 240 (c) 151 200

Exercise 13.3

1. 5, 84 2. (a) 1, 1 (b) $n, \frac{n(n-1)}{2}$ 3. 66
4. 2002 5. 15 6. 35 7. 28 8. 35 9. 700
10. 900 11. (a) 10 (b) 126; 1260
12. 2520 13. (a) 28 (b) 112 14. 5005; 1960; 315
15. (a) 330 (b) 315 (c) 215; 168 16. (a) 16 (b) 90; 112
17. 252 (a) 105 (b) 21 (c) 231
18. (a) 462 (b) 210 (c) 200 (d) 281

Miscellaneous Exercise 13

1. 35 2. 6720 3. 210 4. 70 5. 27 000 6. 14 400
7. 15 504; 2730 8. 144 9. (a) 5040 (b) 1440 (c) 2880
10. (a) 126 (b) 2730 (c) 336
11. 3 628 800 (a) 40 320 (b) 241 920 (c) 2 903 040
12. (a) 462 (b) 81 13. 525; 225; 24
14. (a) 420 (b) 525 (c) 27 (d) 756
15. 840 (a) 72 (b) 13
17. 5005; 720; 72 18. 15 (a) 225 (b) 180
19. (a) (i) 144 (ii) 2160
(b) (i) 210 (ii) 60 (iii) 105 (iv) 100

14

Exercise 14.1

1. (a) $1 - 8x + 24x^2 - 32x^3 + 16x^4$ (b) $1 + 15x + 90x^2 + 270x^3 + 405x^4 + 243x^5$
(c) $1 - 6ax + 15a^2x^2 - 20a^3x^3 + 15a^4x^4 - 6a^5x^5 + a^6x^6$ 2. $58\sqrt{2}$
3. (a) $1 + 10x + 45x^2 + 120x^3 + \ldots$ (b) $1 - 12x + 66x^2 - 220x^3 + \ldots$
(c) $1 - 16x + 112x^2 - 448x^3 + \ldots$ (d) $1 + 18x + 144x^2 + 672x^3 + \ldots$
(e) $1 - 24x + 252x^2 - 1512x^3 + \ldots$ (f) $1 + 9x^2 + 36x^4 + 84x^6 + \ldots$
(g) $1 - 14x^2 + 84x^4 - 280x^6 + \ldots$ (h) $1 - 8x^3 + 30x^6 - 70x^9 + \ldots$
(i) $1 + \frac{16x}{y} + \frac{112x^2}{y^2} + \frac{448x^3}{y^3} + \ldots$
4. $1 + 30x + 405x^2 + 3240x^3 + \ldots$
(a) $1 - 30x + 405x^2 - 3240x^3 + \ldots$ (b) $1 + 30x^2 + 405x^4 + 3240x^6 + \ldots$
5. $1 + 18x + 144x^2 + 672x^3 + \ldots$; 1.195 072
6. (a) $1 + 24x + 240x^2 + \ldots$ (b) $1 - 28x + 364x^2 + \ldots$; $1 - 4x - 68x^2 + \ldots$
7. $1 - 10x - 36x^2 + 424x^3 + \ldots$ 8. $2 + 7x + 10x^2 + 7x^3 + \ldots$; 2.807
9. $80x + 18\,240x^3 + 992\,256x^5 + \ldots$; 0.818
10. (a) $8960x^4$ (b) $-77\,520x^{14}$ (c) $-8064x^5$ (d) $1792x^{12}$
11. (a) -1512, 20 412 (b) 1760, 59 136 (c) 0, $-1\frac{7}{8}$
12. -6 13. (a) 5 : 156 (b) -253
14. $1 - 6x + 15x^2 - 20x^3 + \ldots$; $1 + 12x + 60x^2 + 160x^3 + \ldots$; $1 + 6x + 3x^2 - 40x^3 + \ldots$
15. $a = \frac{1}{6}$, $b = -\frac{7}{12}$ 16. -360 17. $1 - 4x + 6x^2 - 4x^3 + x^4$; x^{12}
18. (a) $1 + 11x + 26x^2 + \ldots$ (b) $1 - 11x^2 + \ldots$

Exercise 14.2

1. (a) $81 - 216x + 216x^2 - 96x^3 + 16x^4$ (b) $32 + 80x^2 + 80x^4 + 40x^6 + 10x^8 + x^{10}$

(c) $64 - \frac{96}{x} + \frac{60}{x^2} - \frac{20}{x^3} + \frac{15}{4x^4} - \frac{3}{8x^5} + \frac{1}{64x^6}$

2. (a) $64 + 576x + 2160x^2 + 4320x^3 + \ldots$ (b) $1024 - 640x + 160x^2 - 20x^3 + \ldots$

(c) $\frac{1}{256x^8} - \frac{1}{8x^5} + \frac{7}{4x^2} - 14x + \ldots$

3. (a) $128 - 224x + 168x^2 - 70x^3 + \ldots$; 125.776 7 4. $a = 3$, $b = 36$

5. $1 - 18x + 144x^2 + \ldots$; $32 + 80x + 80x^2 + \ldots$; $32 - 496x + 3248x^2 + \ldots$

6. (a) $3360x^6$ (b) $-489\ 888x^6$ (c) $3360x^4y^6$ (d) $14\frac{7}{16}x^{-6}$

7. (a) $3360x^{10}$ (b) $-15\ 360$ (c) 13 440

8. $\frac{1}{32} - \frac{5}{8}x + 5x^2 - 20x^3 + \ldots$; $-33\frac{1}{8}$ 9. $\frac{5}{8}$

Miscellaneous Exercise 14

1. $1 - 6x + 15x^2 - 20x^3 + \ldots$; $1 - 3x^2 + \frac{15}{4}x^4 - \frac{5}{2}x^6 + \ldots$

3. $64 + 48x + 15x^2 + \frac{5}{2}x^3 + \ldots$; 63.521 497 5 4. (a) 7 (b) $-1\ 760$

5. (a) $1 + 18x + 135x^2 + 540x^3 + \ldots$ (b) $1 - 20x + 160x^2 - 640x^3 + \ldots$; -65

6. $140x^3$ 7. $x^{16} - 16x^{13} + 112x^{10} - 448x^7 + \ldots$; -240

8. $1 + 4p + 6p^2 + 4p^3 + p^4$; $1 + 4x + 10x^2 + 16x^3 + \ldots$; 1.52

9. $64 - 192p + 240p^2 - 160p^3 + 60p^4 - 12p^5 + p^6$; $64 - 96x + 444x^2 + \ldots$

10. (a) $1 + \frac{15}{2}x + \frac{45}{2}x^2$ (b) $32 - 80x + 80x^2$; 200

11. ± 2 12. 8 13. -1, 1

14. $32 + 240x + 720x^2 + 1080x^3 + \ldots$

(a) 34.473 08 (b) $64 + 448x + 1296x^2 + \ldots$

15. 15, $13\frac{1}{8}$, $69\frac{3}{8}$ 16. -960 17. $-56a^7$

18. (a) 9405 (b) -280 (c) 4

19. $362 + 209\sqrt{3}$; $362 - 209\sqrt{3}$ 20. ${}^nC_2a^{n-2}b^2x^2$, ${}^nC_3a^{n-3}b^3x^3$

21. $a = \frac{1}{2}$, $A = 1$, $B = 4$, $C = 7$ 22. $a = -\frac{3}{2}$, $n = 8$ 23. 1.05

24. 1512 25. $a = 2$, $b = 2$ 26. 141

Revision Exercise 9

1. (a) 12 cm (b) 60 cm^2 (c) 27.4 cm^2

2. (a) 180° (b) 60°, 120°, 240°, 300° (c) 30°, 150°

3. (a) 4 (b) 4 (c) 7

4. (a) $-\frac{1}{\sqrt{2}}$, $-\sqrt{3}$ (b) $\sqrt{1-c^2}$, $\frac{c}{\sqrt{1-c^2}}$ 5. 10; 8

6. (a) 362 880 (b) 80 640 (c) 282 240 (d) 161 280

7. $1 - 16x + 112x^2$; 913

Revision Exercise 10

1. (b) $\{y : -1 \leqslant y \leqslant 5, y \in \mathbb{R}\}$ 2. $-\frac{\sqrt{3}}{2}$
3. (a) 90° (b) 9.4°, 99.4°, 189.4°, 279.4°
(c) 0°, 70.5°, 109.5°, 180°, 250.5°, 289.5°, 360° (d) 164.1°, 344.1°
4. (a) 1.2 rad. (b) 52.8 cm^2
5. (a) 252 (b) 30 240 (c) 195
6. (a) $14\ 784x^7$ (b) $a = -2$, $b = 1080$
7. $16 - 32p + 24p^2 - 8p^3 + p^4$; $16 + 128x + 368x^2 + \ldots$

Revision Exercise 11

1. (a) 112.5°, 157.5°, 292.5°, 337.5° (b) 63.4°, 135°, 243.4°, 315° (c) 11.3°, 45°
2. 2 4. (b) $P = 8 + 8\cos\theta + 8\sin\theta$ (c) 0.644
5. (a) 1.5 rad. (b) 435.9 cm^2
6. (a) $128 - 1344x + 6048x^2$; 127 (b) -1760
7. 360 (a) 120 (b) 180

Revision Exercise 12

1. 70 (a) 35 (b) 15 (c) 55; 6720
2. 40 320 (a) 10 080 (b) 30 240 (c) 2880
3. (a) (i) 82.5°, 172.5°, 262.5°, 352.5° (ii) 60°, 300°
(b) $\frac{1}{3}\pi$, 1.91
4. (a) $-\frac{\sqrt{5}}{2}$ (b) (i) $-\sqrt{1+t^2}$ (ii) $-\frac{t}{\sqrt{1+t^2}}$
5. (a) $\{y : -4 \leqslant y \leqslant 4, y \in \mathbb{R}\}$ (b) (i) 1 (ii) 3
6. 2; $-8\frac{9}{16}$ 7. $\frac{4}{3}\pi r$

15

Exercise 15.1

1. (a) $6x + 4$ (b) $4x^3 - 14x + 6$ (c) $6x^2 + 10x - 4$ (d) $4 - \frac{2}{x^2}$
(e) $18x + \frac{6}{x^3}$ (f) $-\frac{18}{x^4} + \frac{1}{x^2}$ (g) $2bx$ (h) $10x - \frac{4}{x^2}$
(i) $3 + \frac{1}{\sqrt{x}}$ (j) $16x + 3 - \frac{1}{2\sqrt{x}}$ (k) $5x^{\frac{3}{2}} - 6x^{\frac{1}{2}} - 6$
(l) $9x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}$ (m) $10x^{\frac{3}{2}} + 3x^{-\frac{3}{2}}$ (n) $a + \frac{b}{x^2}$
2. (a) 2 (b) $1 - \frac{4}{x^2}$ (c) $4x + \frac{3}{2x^2}$ (d) $-\frac{4}{x^3}$
(e) $\frac{9}{2}x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$ (f) $3 + \frac{1}{4}x^{-\frac{3}{2}} - x^{-2}$
3. (a) $4x + 1$ (b) $\frac{3}{2}\sqrt{x} - 2$ (c) -1 (d) $24x - 10x^{\frac{2}{3}}$
(e) $2 + \frac{2}{x^2}$ (f) $-6x^{\frac{1}{2}} + \frac{5}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$
4. (a) 10 (b) 9 (c) 2 (d) 5
5. (a) 10 (b) $-1\frac{1}{2}$ (c) $-4\frac{1}{6}$ (d) 5
6. (a) 0 (b) -5, 5 (c) 1 (d) -8, 8

7. **(a)** -5 **(b)** $\frac{1}{4}$ **(c)** $-8, 8$ **(d)** $-2, -\frac{1}{2}$

8. $(1, 6)$ 9. $a = -\frac{1}{7}, b = -\frac{16}{7}$ 10. $a = 1, b = -6$

11. $a = 9, b = -2$ 12. $a = \frac{1}{2}, b = \frac{5}{2}$ 13. $p = 2, q = -5$

14. **(a)** $8\frac{3}{4}$ **(b)** $\left(-\frac{1}{3}, -6\right), \left(\frac{1}{3}, 6\right)$ 15. **(a)** $-2, 4$ **(b)** $(-1, 16), (3, -16)$

16. $a = \frac{1}{2}, b = \frac{5}{4}$ 17. $a = \frac{7}{4}; -2$

Exercise 15.2

1. **(a)** $5(x + 2)^4$ **(b)** $8(2x - 1)^3$ **(c)** $\frac{5}{4}\left(\frac{1}{4}x + 2\right)^4$
   (d) $-40(1 - 4x)^9$ **(e)** $-24x(2 - 3x^2)^3$ **(f)** $3(2x - 1)(1 - x + x^2)^2$

2. **(a)** $\frac{36}{(3 - 4x)^4}$ **(b)** $-\frac{8}{(2x + 7)^2}$ **(c)** $\frac{12}{(2 - x)^3}$ **(d)** $\frac{-24x}{\left(6x^2 + 5\right)^2}$

3. **(a)** $\frac{1}{\sqrt{2x - 3}}$ **(b)** $-\frac{1}{\sqrt{6 - 2x}}$ **(c)** $\frac{x}{\sqrt{x^2 - 2}}$
   (d) $-\frac{3x}{\sqrt{5 - 3x^2}}$ **(e)** $\frac{2x - 1}{2\sqrt{x^2 - x + 1}}$ **(f)** $\frac{x + 1}{\sqrt{x^2 + 2x + 2}}$

4. **(a)** $-\frac{3\left(2 - \sqrt{x}\right)^5}{\sqrt{x}}$ **(b)** $\frac{-3}{x^2\left(1 - \frac{1}{x}\right)^4}$ **(c)** $-\frac{3}{(3x - 2)^3}$
   (d) $\frac{1}{2\sqrt{x\left(\sqrt{x} + 2\right)}}$ **(e)** $3\left(x - \frac{1}{x}\right)^2\left(1 + \frac{1}{x^2}\right)$ **(f)** $4\left(\sqrt{x} + 2x\right)^3\left(\frac{1}{2\sqrt{x}} + 2\right)$

5. **(a)** 96 **(b)** $-\frac{1}{2}$ **(c)** -2 **(d)** 108

6. $(0, 1)$ 7. $(1, 2)$ 8. $a = \frac{7}{3}$ or $\frac{5}{3}$ 9. **(a)** -12 **(b)** $-\frac{1}{16}$

Exercise 15.3

1. **(a)** $3x(x + 2)$ **(b)** $(2x - 1)^2(8x - 1)$ **(c)** $2(3x + 2)^3(4 - 15x)$
   (d) $2(1 + x)(2x^2 + x + 1)$ **(e)** $2x(1 - 4x)^2(1 - 10x)$ **(f)** $(1 - x^2)(1 - 5x^2)$
   (g) $2(2x + 1)(4x^2 - 5x + 3)$ **(h)** $2(1 + 4x)^2(6 - x - 10x^2)$ **(i)** $-12x^3(1 - x^2)$
   (j) $-2(x + 1)^3(1 - 2x)^2(1 + 7x)$ **(k)** $\left(1 - \frac{1}{2}x\right)\left(-2x^2 + \frac{13}{2}x - 7\right)$ **(l)** $-2(8x^3 + 3x^2 + 23x + 6)$

2. **(a)** $\frac{(1 - x)(1 - 5x)}{2\sqrt{x}}$ **(b)** $\frac{1 + 3x}{\sqrt{1 + 2x}}$ **(c)** $-\frac{4(3x + 1)}{\sqrt{1 - 4x}}$
   (d) $\frac{5x^2 + 4x + 1}{2\sqrt{x + 1}}$ **(e)** $\frac{2x\left(1 - 3x^2\right)}{\sqrt{1 - 2x^2}}$ **(f)** $\frac{24x^2 - 3x + 4}{\sqrt{3x^2 + 1}}$

3. **(a)** $(x + 2)^2(5x^2 + 8x + 2)$ **(b)** $\frac{x\left(21x^2 - 10\right)}{\sqrt{5 + 6x}}$

5. $3\frac{1}{4}$ **6.** 0, 8 **7.** $\frac{3x - 5}{2\sqrt{x - 1}}$, $\frac{5}{3}$

8. $\frac{dy}{dx} = \frac{(x - 4)^3(9x - 4)}{2\sqrt{x}}$; $x = 4$, $\frac{4}{9}$ **10.** −2, 2, −2

Exercise 15.4

1. (a) $\frac{1}{(x+1)^2}$ **(b)** $\frac{5}{(2x+1)^2}$ **(c)** $-\frac{5}{(1+x)^2}$ **(d)** $\frac{1}{(1-2x)^2}$

(e) $\frac{x^2+6x}{(x+3)^2}$ **(f)** $\frac{6x-12x^2}{(1-4x)^2}$ **(g)** $\frac{2(x^2-x-1)}{(2x-1)^2}$ **(h)** $\frac{2x^2(3-2x)}{(1-x)^2}$

(i) $-\frac{4x}{(1+x^2)^2}$ **(j)** $\frac{3+4x-4x^2}{(1-2x)^2}$ **(k)** $\frac{2x}{(1+x)^3}$ **(l)** $\frac{-2x^4+3x^2-4x}{(1-x^3)^2}$

2. (a) $\frac{1-x}{2\sqrt{x}(1+x)^2}$ **(b)** $\frac{2-x}{2(1-x)\sqrt{1-x}}$ **(c)** $\frac{2x+2}{(2x+1)\sqrt{2x+1}}$

(d) $\frac{3-2x}{(1-4x)\sqrt{1-4x}}$ **(e)** $\frac{5}{(1-x^2)\sqrt{1-x^2}}$ **(f)** $\frac{6x^3-18x}{(2x^2-3)\sqrt{2x^2-3}}$

3. (a) $\frac{14}{(1-4x)^2}$ **(b)** $\frac{6}{(1+x^2)\sqrt{1+x^2}}$

4. $\frac{2}{3}$ **5.** −1 **6.** $-\frac{1}{16}$ **7.** −1, 3

Exercise 15.5

1. (a) $y = 5x - 7$, $5y = 17 - x$ **(b)** $y = -3x - 1$, $3y = x + 7$ **(c)** $y = -x + 4$, $y = x + 2$

2. (a) $8x + 12y = 1$ **(b)** $y = x + 3$

3. $2x + 2y = -1$, $y = x - 2$, $\left(\frac{3}{4}, -\frac{5}{4}\right)$ **4.** $x + 4y = -19$, $x - 4y = 19$; $\left(0, -\frac{19}{4}\right)$

5. $3x + 13y + 168 = 0$ **6.** $2y = 7 - x$, $8y = 5x + 13$, $\left(\frac{5}{3}, \frac{8}{3}\right)$

7. $6y = x + 59$, $3y + 2x = 38$, $\left(\frac{17}{5}, \frac{52}{5}\right)$ **8.** $2y = x + 3$, $y + 2x = 4$

9. $y = x - 16$, $y = x + 16$ **10.** $y = 4x + 2$ **11.** $2y - x = 4$

12. $y = 3x - 3$, $\frac{11}{3}$ **13.** $a = 6$, $b = 15$ **14.** $a = 2$, $b = 1$ **15.** $a = 2$, $b = 4$ **16.** −2

17. $y = (1 + 2a)x - a^2$, $a = 5, -1$; $y = 11x - 25$, $y + x + 1 = 0$ **18.** $a = 2$, $b = 6$, $\left(\frac{3}{4}, \frac{19}{2}\right)$

Miscellaneous Exercise 15

1. (a) $-\frac{3x}{\sqrt{1-3x^2}}$ **(b)** $(1 + 2x)^3(1 + 10x)$ **(c)** $-\frac{1}{2}x^{-\frac{3}{2}} + x^{-\frac{1}{2}}$

2. (a) $-\frac{8}{(2x+1)^2}$ **(b)** $\frac{7}{(1-3x)^2}$ **(c)** $\frac{4-3x}{2\sqrt{2-x}}$

3. 30

4. (a) $\dfrac{5(\sqrt{x}+1)}{\sqrt{x}}(x+2\sqrt{x})^4$ (b) $x = \pm 2$

5. $a = 1$, $b = 5$

6. (a) $57y = 27x - 62$ (b) $(2, 3)$, $(-2, -3)$

7. $3y + 5 = 2x$

8. $y = 4x - 2$; $(-1, 6)$

9. $\dfrac{dy}{dx} = \dfrac{7}{(3-2x)^2}$, $y = 7x - 4$

10. $(2, 11)$

11. 156 sq. units

12. 32.5 sq. units

13. $y = -x$, $(0, 0)$

14. $k = 2$

15. $3x^2 + 2px + 2$; $p = -3$, $q = 5$

16. $a = 1$, $b = -\dfrac{5}{2}$

17. $a = 1$, $b = -10$

18. $\dfrac{1}{2}$; $-\dfrac{25}{8}$

19. $a = 2$, $b = -2$

20. $(-3, -5)$, $(1, 3)$

16

Exercise 16.1

1. (a) 6 cm s^{-1}, increasing (b) -3 cm s^{-1}, decreasing (c) -9.75 cm s^{-1}, decreasing

2. (a) $t = 3$ (b) $t = 0$

3. 5 cm, $-\dfrac{1}{8}$ cm s^{-1}

4. 363 cm^3 s^{-1}

5. $21\dfrac{7}{9}$ cm^3 s^{-1}

6. 84 cm^2 s^{-1}

7. $A = \pi(2t^2 + 1)^2$, 144π cm^2 s^{-1}

8. (a) 6 cm^3 s^{-1} (b) 72 cm^3

9. $\dfrac{2\pi}{3}\left(\dfrac{t}{10}+1\right)^3$, $\dfrac{\pi}{5}\left(\dfrac{t}{10}+1\right)^2$ cm^3 s^{-1}

Exercise 16.2

1. (a) 24 units s^{-1} (b) 6 units s^{-1} (c) -36 units s^{-1}
   (d) 30 units s^{-1} (e) 24 units s^{-1} (f) 2 units s^{-1}

2. (a) $\dfrac{4}{15}$ units s^{-1} (b) $\dfrac{3}{2}$ units s^{-1} (c) 12 units s^{-1} (d) $\dfrac{2}{3}$ units s^{-1}

3. (a) 16π cm^2 s^{-1} (b) 12π cm^2 s^{-1}

4. $\dfrac{1}{6}$ cm s^{-1}

5. 0.4π cm^2 s^{-1}

6. $\dfrac{5}{2}$ m s^{-1}

7. 72π cm^3 s^{-1}

8. (a) $\dfrac{75}{4\pi}$ cm s^{-1} (b) $\dfrac{25}{3\pi}$ cm s^{-1}

9. $\dfrac{5}{2}$ cm s^{-1}

10. 0.001 cm s^{-1}

11. 0.05 cm^3 s^{-1}

12. (a) $\dfrac{3}{4}$ units s^{-1} (b) 10 cm s^{-1}

13. $\dfrac{1}{2}(p^3 + p^2)$; 140 sq. units s^{-1}

14. 9 cm^2 s^{-1}

15. -2.4π cm^3 s^{-1}

16. (a) 15π cm^3 s^{-1} (b) 2.4 cm s^{-1}

Exercise 16.3

1. -0.1, decrease

2. 2, increase

3. -0.005

4. $-\dfrac{1}{60}$

5. $960p$ (increase)

6. 0.12π cm^2

7. 0.002 5 cm

8. $8x + 3$; -4.15 cm^3

9. $4\pi r^2$; $\dfrac{1}{180}$ cm

10. $\dfrac{\pi}{\sqrt{10x}}$; $\dfrac{\pi}{3\sqrt{10}}$ s

11. $\dfrac{6}{(1-2x)^2}$, $\dfrac{2}{3}p$

12. $-\dfrac{18x^2}{(x^3+1)^2}$ (a) $-\dfrac{8}{9}p$ (b) $\dfrac{2}{9}p$

13. 0.36π cm^3; 36.36π cm^3

14. 31 cm^3; 2131 cm^3

15. 10.01

16. 3%

17. -1%

18. -2, $-200p\%$

19. $10x - 4$

20. 6%, 4%

21. (a) $\frac{k\delta x}{2\sqrt{x}}$ (b) $\frac{\delta x}{2x}$; $\frac{1}{2}p\%$

Miscellaneous Exercise 16

1. (a) $\frac{8}{\sqrt{\pi}}$ cm (b) $\frac{1}{4\sqrt{\pi}}$ cm s^{-1}
2. $V = x^3$, $A = 6x^2$, $\frac{dV}{dx} = 3x^2$, $\frac{dA}{dx} = 12x$ (a) 1.5 cm^3 s^{-1} (b) 0.12 cm^2
3. (a) 204.8π cm^3 (b) $\frac{12}{\pi}$ cm s^{-1}
4. (a) $-\frac{600}{p^2}$, 0.075 cu. units (b) decrease at 4.5 units per second
5. (a) $-\frac{4}{x^2}$, p (b) -25.6
6. $\frac{1}{2\sqrt{x}}$ (a) 1.99 (b) 2.02
7. $\frac{4}{25}\pi x^2$ (a) 3.2π cm^3 (b) $3p\%$
8. (a) (i) $\frac{39}{50}p$ (ii) $2p\%$ (b) (i) 4 (ii) 2 cm s^{-1}
9. (a) $\frac{22 + x^2}{2x^{\frac{3}{2}}\sqrt{22 - x^2}}$ (b) $-\frac{13}{6}$ m s^{-1}
10. (a) 4π cm s^{-1} (b) 16π cm^2 s^{-1}
11. (a) $\frac{6}{7}$ m s^{-1} (b) $\frac{20}{7}$ m s^{-1}
12. (a) 15.6 unit^2 s^{-1} (b) 5.36 units s^{-1}
13. $\frac{8}{3}$ m s^{-1}
14. 128π cm^2 s^{-1}

Exercise 17.1

1. (2, 2) min. pt.
2. (b) (0, 0), (1, 1), (2, 0) (c) min, max, min
3. (b) (0, 12), (3, –15) (c) max. pt., min. pt.

Exercise 17.2

1. (a) $\left(2\frac{1}{2}, -5\frac{1}{4}\right)$ min. pt. (b) (3, –4) min. pt. (c) (2, –16) min. pt., (–2, 16) max. pt.
   (d) (0, 2) max. pt., (2, –14) min. pt., (–2, –14) min. pt.
   (e) (2, 32) max. pt., (6, 0) min. pt. (f) (–4, –8) max. pt., (4, 8) min. pt.
2. (a) (1, –2) min. pt. (b) (0, 0) max. pt., (2, –4) min. pt.
   (c) (–3, –12) max. pt., (3, 12) min. pt. (d) (2, 12) min. pt.
   (e) $\left(1\frac{1}{2}, 12\right)$ min. pt., $\left(-1\frac{1}{2}, -12\right)$ max. pt. (f) (0, 0) min. pt., (–2, –4) max. pt.
3. $\left(\frac{1}{2}, 6\right)$ min. pt.
5. (–1, –9) min. pt.
6. (a) (1, 3) (b) $a = 2$, $b = 3$
7. $a = 9$, $b = -8$, max.
8. (a) $a = 2$, $b = 9$ (b) (–1, –5)
9. $\frac{2}{\sqrt{4x - 7}} - \frac{2}{3}$; (4, 3)

Exercise 17.3

1. (a) 2, min. (b) $\frac{2}{5}$, min. (c) 0.144, max. (d) 5.14, max.

2. (a) 3 (b) $-6\frac{1}{4}$ (c) $\frac{1}{3\sqrt{3}}$ (d) $-\frac{1}{2}, \frac{1}{2}$

3. 12.5 4. $x = 1, y = 1$ 5. $x = 2, y = 2$ 6. 4

7. 400 m^2 8. $y = 52 - 8x, x = 5.2, y = 10.4$

9. $\frac{1}{3}x(1.08 - 4x^2), x = 0.3$ 10. (b) 2 11. $x = 0.451, y = 2.55$

12. 3.99 13. 77 cm^3 14. $h = \frac{18}{x^2}, x = 3, h = 2$

15. 40.5 cm^2 16. (a) 60 cm^2 (b) 9 cm

17. (a) $y = \frac{250}{x^2}$ (c) $24x - \frac{3000}{x^2}$ (d) 900 18. $96\sqrt{3}\pi$ cm^3

Miscellaneous Exercise 17

1. $(-2, 8)$ min., $(2, 8)$ min. 2. $\left(1\frac{1}{2}, 27\right)$, min. pt., 27 3. 2

4. (a) $a = -4, b = 10$ (b) $\left(1\frac{1}{3}, 25\frac{11}{27}\right)$ 5. $p = 1\frac{1}{2}, q = 4$

6. $h = 90 - 3x$; 24 000 cm^3 7. $12\frac{1}{2}$, min.

8. $y = \frac{12}{5}(5 - x)$; $2\frac{1}{2}$, 30 cm^2 9. 134, min. 10. 11.2, min.

11. (a) $r = \frac{1}{4}L$, $\theta = 2$ rad.; max. 12. (a) $r = 5, h = 10$ (b) $r = 5.93, h = 7.11$

13. (b) $\frac{10}{3}$ 15. 3, max.

16. (a) (i) 25 cm (ii) 114.6° (b) $\frac{1}{2}$, 2

17. (a) $h = 5r - r^2$, max. $h = \frac{25}{4}$ (b) min. $A = 1600$ m^2

18. $y = \frac{27-4x^3}{\pi x^2}$ (a) $\frac{3}{2}$ cm (b) $A = 54$ cm^2, (min), $y = \frac{6}{\pi}$ cm

19. (a) $\left(\frac{1}{2}, 8\right)$, min. (b) $\frac{60x - 5x^2}{12}$, 15 cm^2

20. (a) $\frac{4}{(2 - x)^2} - \frac{9}{(x - 3)^2}, \frac{8}{(2 - x)^3} + \frac{18}{(x - 3)^3}$ (b) -25, max; -1, min.

21. $\frac{20}{3}$ 22. (1, 1)

18

Exercise 18.1

1. (a) $4 \cos x$ (b) $2x + 5 \sin x$ (c) $2 \cos x - 3 \sin x$ (d) $8x + 3 \sec^2 x$

2. (a) $2x \cos x - x^2 \sin x$ (b) $x \sec^2 x + \tan x$

(c) $2(x + 1) \sin x + (x + 1)^2 \cos x$ (d) $\frac{\sin x - 2}{\cos^2 x}$

4. (a) $3(1 - \cos x)^2 \sin x$ (b) $6 \cos x (3 \sin x + 2)$

(c) $-\frac{\sec^2 x}{2\sqrt{2 - \tan x}}$ (d) $\frac{\cos x - 2 \sin x}{\sqrt{2 \sin x + 2 \cos x}}$

5. (a) $3\cos 3x - 4\sin 4x$ (b) $2\cos\frac{1}{2}x$ (c) $2\cos(2x-5)$ (d) $-2\sin\left(2x+\frac{\pi}{3}\right)$

(e) $6\sec^2 2x$ (f) $3\sec^2\frac{1}{2}x$ (g) $2\sin\left(\frac{\pi}{4}-x\right)$ (h) $6\cos\left(\frac{3x-5}{4}\right)$

6. (a) $\cos x\cos 3x - 3\sin x\sin 3x$ (b) $2x\tan 5x + 5(1+x^2)\sec^2 5x$

(c) $\sec 2x(1+2x\tan 2x)$ (d) $-\frac{\sin x\sin 3x + 3\cos x\cos 3x}{\sin^2 3x}$

7. (a) $6\sin^2 x\cos x$ (b) $2\cos 2x + 12\cos^3 x\sin x$

(c) $-6\sin 3x\cos 3x$ (d) $1 + 30\sin^4 2x\cos 2x$

(e) $40\tan 5x\sec^2 5x$ (f) $(\cos 2x - 14x\sin 2x)\cos^6 2x$

(g) $4(1-x)^2\sin^3 x\cos x - 2(1-x)\sin^4 x$ (h) $-\frac{2x\sin x\cos x + \cos^2 x}{x^2}$

8. $1-\sin x$, $\frac{\pi}{60}$ 9. 3, $\frac{\pi}{44}$

10. 0.012 8; 0.306 unit s^{-1} 11. 0.756% (3 sig. fig.); 0.0192 unit s^{-1}

12. $x+2y = 2+\sqrt{3}+\frac{\pi}{6}$, $y-2x = 1+\frac{\sqrt{3}}{2}-\frac{\pi}{3}$ 13. $\frac{\pi}{3}$

14. 0.983, max. 15. $(0, 0)$, (π, π) 16. 0.927, 4.069

Miscellaneous Exercise 18

1. (a) $3+2\sin x$ (b) $4\cos 4x + 6\sin 2x$

(c) $-4\sec^2 x(1-2\tan x)$ (d) $2\cos 2x\cos 3x - 3\sin 2x\sin 3x$

(e) $6x\sec^2 3x + 2\tan 3x$ (f) $\frac{2x\cos 2x - \sin 2x}{x^2}$

(g) $\frac{2\cos 2x + \sin x}{2\sqrt{\sin 2x - \cos x}}$ (h) $\frac{4\sin 4x}{(1+\cos 4x)^2}$

2. (a) $2\cos 2x + \sin 2x$ (b) $96\sin^3 8x\cos 8x$

(c) $\cos^2 3x - 3x\sin 6x$ (d) $2x - 24\tan 4x\sec^2 4x$

3. (a) $2x\cos(x^2+2)$ (b) $4x\sin(1-x^2)$ (c) $(4x+1)\sec^2(2x^2+x)$

4. (b) $-\frac{1}{40}$ rad. s^{-1} 5. $-\frac{9}{8}$, $\frac{3\pi}{112}$ 6. 0.000 5; -1 unit s^{-1}

7. 4 8. $\sqrt{3}$, $-\sqrt{3}$ 9. (a) $\frac{7}{6}\pi$, $\frac{11}{6}\pi$, $\frac{1}{2}\pi$ (b) $\frac{1}{2}\pi$, $\frac{3}{2}\pi$, 0.253, 2.89

10. (0.615, 0.385), $\left(\frac{\pi}{2}, 0\right)$, (2.526, 0.385) 13. $\left(\frac{\pi}{3}, \frac{3\sqrt{3}}{16}\right)$, $\left(\frac{2\pi}{3}, -\frac{3\sqrt{3}}{16}\right)$, max $\left(\frac{\pi}{3}, \frac{3\sqrt{3}}{16}\right)$

15. 0.848 (max) 16. $a\sec^2\theta(2\sin\theta - 1)$, $\frac{\pi}{6}$, $a\left(2+\sqrt{3}\right)$ 17. $5\sqrt{5}$

9

Exercise 19.1

1. (a) $(0, 1)$ (b) $\left(0, \sqrt{e}\right)$ (c) $(0, e^5)$

2. 0.65 3. -0.59 4. (a) 0.280 (b) -0.053

5. (a) $(0, 1)$ (b) $(0, 1)$ (c) $(0, 3)$ (d) $\left(0, \frac{1}{4}\right)$

6. (a) 2 (b) 2 7. (a) $\frac{1}{4}$ (b) $2\frac{1}{2}$

8. (a) $\frac{1}{2}e^{\frac{1}{2}x}$ (b) $4e^{4x-1}$ (c) $-2e^{-2x}$ (d) $-7e^{2-7x}$

(e) $3e^{3x}$ (f) $10e^{5x}$ (g) $4e^{x}$ (h) $-25e^{3-5x}$

(i) $e^{x} - e^{-x}$ (j) $3e^{3x} + 3e^{-x}$ (k) $2e^{\frac{1}{4}x} + 3e^{6x}$ (l) $3e^{9x}$

9. (a) e^{4x-1}, $4e^{4x-1}$ (b) e^{x+3}, e^{x+3} (c) e^{2-2x}, $-2e^{2-2x}$ (d) $e^{\frac{3}{2}x}$, $\frac{3}{2}e^{\frac{3}{2}x}$

10. (a) $6e^{2x}(e^{2x}+1)^2$ (b) $5(1+e^x)(x+e^x)^4$ (c) $(3+2x)e^{2x}$

(d) $e^x(\sin x + \cos x)$ (e) $(3x^2+2)e^{x^3+2x}$ (f) $-e^{-x}(\cos 2x + 2\sin 2x)$

11. (a) $\dfrac{xe^x}{(1+x)^2}$ (b) $\dfrac{1+e^{-x}+xe^{-x}}{(1+e^{-x})^2}$ (c) $\dfrac{e^x(1-e^{2x})}{(1+e^{2x})^2}$

(d) $\dfrac{e^x(\cos x - \sin x) - \cos x}{(e^x-1)^2}$ (e) $\dfrac{(3\cos x + \sin x)e^{3x}}{\cos^2 x}$ (f) $\dfrac{1-x+e^{2x}}{e^x}$

12. (a) $y = e^3(2x-3)$, $y = -\dfrac{1}{2e^3}x + \dfrac{1}{e^3} + e^3$ (b) $\dfrac{e^3(1+2e^6)}{4}$

13. $y = 2e^3(3x-2)$, $y = -\dfrac{x}{6e^3} + \dfrac{1}{6e^3} + 2e^3$ **14.** $x = \ln 2$, $y = -0.386$

15. (a) 3 (b) $2e$, $-e$ **16.** $-\frac{1}{3}e^{-2x}$, $-\frac{1}{3}e^{-2}p$

17. (a) $\dfrac{12e-8}{e}$ (b) $\dfrac{4}{e}$ (c) 13.5% **18.** -0.33 unit s^{-1}

Exercise 19.2

1. (a) $\left(\frac{1}{3}, 0\right)$ (b) $(0, 0)$ (c) $(2, 0)$ **2.** $(0, 0)$ **3.** 0.829 **4.** 1.16

5. (a) $\dfrac{5}{5x+1}$ (b) $\dfrac{8}{4x-3}$ (c) $\dfrac{3x^2}{x^3-8}$

(d) $\dfrac{2x}{2x-5} + \ln(5-2x)$ (e) $\dfrac{2(x-1)}{(x+1)(x-3)}$ (f) $\dfrac{\cos x - \sin x}{\sin x + \cos x}$

(g) $\dfrac{e^x}{1+e^x}$ (h) $\dfrac{1-\ln x}{x^2}$ (i) $-\dfrac{1}{\sqrt{x^2+1}}$

6. $(3, 0)$, $y = \frac{1}{2}x + \ln 2 - 2$, $(0, \ln 2 - 2)$ **8.** $k = 5$, $3y + x = 2$ **9.** 1.07

10. (a) $x > \frac{5}{2}$, $(3, 0)$ (b) $x < \frac{3}{2}$, $(1, 0)$ (c) $x > 3$, $(4, 0)$ (d) $x > \frac{2}{3}$, $(1, 0)$

11. $\frac{6}{5}p$ **12.** (a) $20e^{-1}$ (b) $-4e^{-1}$ (c) -2% **13.** $-\dfrac{3}{2e+1}$

Miscellaneous Exercise 19

1. (a) $x^2e^x(3+x)$ (b) $-xe^{-x}$ (c) $e^{2x}(\cos x + 2\sin x)$ (d) $3x(1+2\ln x)$

(e) $2x\ln(2x+1) + \dfrac{2(x^2-1)}{2x+1}$ (f) $e^{\frac{1}{2}x}\left(\dfrac{1}{2}\ln(5-4x) - \dfrac{4}{5-4x}\right)$

2. (a) $\dfrac{e^x(2x-1)}{(2x+1)^2}$ (b) $\dfrac{2x\ln 2x - x}{(\ln 2x)^2}$ (c) $\dfrac{e^{\frac{1}{2}x}(\cos 2x + 4\sin 2x)}{2\cos^2 2x}$

(d) $\dfrac{1}{\sqrt{x^2+1}}$ (e) $3\cot x$ (f) $\dfrac{1+\cos x}{x+\sin x}$

3. (a) $e^{x^3-2x}(3x^2-2)$ (b) $e^{1+\sin x}\cos x$ (c) $-2e^{\cos 2x}\sin 2x$ (d) $\dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$

(e) e^{e^x+x} (f) $\dfrac{2x}{x^2-1}$ (g) $\dfrac{\sec^2 x}{3+\tan x}$ (h) $2\cot 2x$

(i) $\dfrac{2\ln x}{x}$ (j) $-\dfrac{1}{x(\ln x)^2}$ (k) $\dfrac{1+xe^x}{2x\sqrt{\ln x+e^x}}$ (l) $\dfrac{2e^{2x}+2x}{e^{2x}+x^2}$

4. (a) $\dfrac{4}{x}$ (b) $\dfrac{3}{(2x-1)(1+x)}$ (c) $\dfrac{12}{5+4x}$ (d) $\tan x$

(e) $\dfrac{1}{x}+2\cot 2x$ (f) $\dfrac{1}{1-x^2}$ (g) $x^2(6\ln(\cos x)-2x\tan x)$

(h) $e^x\left(\dfrac{2x}{x+1}+\ln\left(x^2+1\right)\right)$ (i) $\dfrac{e^{\sin x}\cos x}{e^{\sin x}+1}$

5. (0, 1) 6. (a) $1\frac{1}{2}$ (b) 3 7. (a) 3 (b) 0.549, –2

8. $\left(2\frac{1}{2}, 0\right)$ 9. $\ln x=-\frac{1}{2}x+1.24$, $a=-\frac{1}{2}$, $b\approx 1.24$

10. –0.31

11. $(x+1)e^{x-3}$, 4 12. (a) $\left(\frac{1}{4}\ln 2, 8\right)$, min. (b) (e, –e)

13. $y=e(2x-1)$, $y=-\frac{1}{e}$ 14. $10e^x(x+3)$, (–3, –0.498), (–2, 0), (0, 20)

15. (a) 80 (b) 26.4 (c) 5.81 (d) –1.05°C/min; 8

16. (a) $m=13.2$ (b) $t=34.7$ (c) 0.177 g/day 17. (a) 770 (b) 1841

Revision Exercise 13

1. (a) $a=\frac{7}{3}$, $b=\frac{4}{3}$ (b) (i) $\dfrac{-x}{\sqrt{2-x^2}}$ (ii) $1-\dfrac{1}{\sqrt{x}}$ (c) $3x-4y=2$

2. (a) $5y+4x=13$ (b) $y=-4x-1$, $y=4x-9$ (c) $a=-1$, $b=-3$

3. (a) $9p$ (b) 0.4 cm s^{-1} 4. (a) –1; 0.01 (b) 2 cm s^{-1} 5. 108

6. (a) (i) $\dfrac{(2x+1)e^{2x}}{(x+1)^2}$ (ii) $\dfrac{4}{4-x^2}$ (b) –0.5 7. (1.25, 3.16) max. pt.

Revision Exercise 14

1. (a) (–1, 5), (2, –13) (b) (i) $\dfrac{2x-\sqrt{x}+1}{2\sqrt{x}\sqrt{x+1}}$ (ii) $\dfrac{8-x}{x^3}$ (c) –1

2. (a) (i) $3\sec^2(3x-1)$ (ii) $2x\cos 2x-2(x^2-1)\sin 2x$
(b) $y=-x$ (c) 2.62, 5.76

3. (a) 30 cm^3 s^{-1} (b) $\dfrac{1}{3\left(\sqrt[3]{8+x}\right)^2}$, $\dfrac{p}{12}$ 4. (a) $\dfrac{7p}{50}$ (b) $2p\%$

5. $\dfrac{dy}{dx}=-2$, $y=3-2p$ 6. $y=40-5x-2\pi x$, 3.01, maximum

7. **(a) (i)** $\dfrac{e^x\left(x^2 + x + 1\right)}{(1 + x)^2}$ **(ii)** $\dfrac{\cos 2x}{\sqrt{1 + \sin 2x}}$ **(b)** (0.955, 0.385), (2.186, –0.385)

Revision Exercise 15

1. **(a)** (–1, –2) **(b)** (2, 12), –3, 3
2. **(a)** $y = x + 1$, $y = 8x - 31$, $\left(\dfrac{32}{7}, \dfrac{39}{7}\right)$ **(b)** $\dfrac{-4}{(x - 1)^2}$; $\left(-2, -\dfrac{1}{3}\right)$, $\left(4, \dfrac{7}{3}\right)$
3. **(a) (i)** $y = 20(\sin \theta + \cos \theta)$ **(ii)** $10\left(\sqrt{3} - 1\right)p$ **(b)** $x = 3$, $2p\%$
4. **(b)** 9.71 m^3
5. **(a)** $\dfrac{2\left(1 - x^2\right)}{\left(1 + x^2\right)^2}$ **(b)** $\dfrac{\pi}{2}$
6. **(a) (i)** $(3 + \cos x)e^{3x + \sin x}$ **(ii)** $\dfrac{1}{2\left(\sqrt{x} + x\right)}$ **(b)** 0
7. **(a)** $\left(2, \dfrac{1}{2}\right)$, max. pt. **(b)** (1, 0)

Revision Exercise 16

1. **(a)** $\dfrac{3}{2}$, 3 **(b)** $2y = x + 4$; $\left(-3, \dfrac{1}{2}\right)$
2. **(a)** $\dfrac{dy}{dx} = 1 - \cos x$, $\dfrac{1}{2}p$ **(b)** $2\dfrac{1}{2}$ units s^{-1}
3. **(a) (i)** 0.6 $cm^2\ s^{-1}$ **(ii)** –0.06 cm s^{-1} **(b) (ii)** $3p$ cm
4. **(a)** (1, 3), min. pt.; $\left(\dfrac{1}{3}, 3\dfrac{4}{27}\right)$, max. pt. **(b)** $l = 12x + 2y$, 3
5. $A = \pi\sqrt{\dfrac{144}{h^2} + 12h}$, $h = \sqrt[3]{24}$, min.
6. **(a)** $4 \tan (2x + 1) \sec^2 (2x + 1)$, $4x \cos (2x^2 - 1)$ **(b)** 18 units per second
7. **(a) (i)** (2, 0) **(ii)** 4 **(b)** (4, 1)

20

Exercise 20.1

1. **(a)** $y = \dfrac{1}{2}x^4 + c$ **(b)** $y = -5x + c$ **(c)** $y = \dfrac{2}{3}x^{\frac{3}{2}} + c$
   (d) $y = \dfrac{1}{x} + c$ **(e)** $y = 4\sqrt{x} + c$ **(f)** $y = -\dfrac{1}{4x^2} + c$
2. **(a)** $y = 3x^2 + 3x + c$ **(b)** $y = 4x + c$ **(c)** $y = x^3 + 3x^2 + c$
   (d) $y = \dfrac{1}{3}x^3 + \dfrac{1}{2}x^2 - 2x + c$ **(e)** $y = x^2 + x + c$ **(f)** $y = 2x - \dfrac{3}{x} + c$
3. **(a)** $\dfrac{1}{3}x^3 - \dfrac{1}{x} + c$ **(b)** $\dfrac{1}{2}x - \dfrac{1}{2x} + c$ **(c)** $3x - \dfrac{2}{3}x^{\frac{3}{2}} + c$ **(d)** $\dfrac{1}{2}x^2 + 2x^{\frac{3}{2}} + c$
4. **(a)** $\dfrac{1}{2}ax^2 + bx + c$ **(b)** $ax - \dfrac{1}{3}bx^3 + c$
5. **(a)** $2x + 2x^2 - x^3 + c$ **(b)** $\dfrac{1}{5}x^5 + \dfrac{1}{x} + c$ **(c)** $\dfrac{4}{3}x^3 - \dfrac{8}{5}x^{\frac{5}{2}} + \dfrac{1}{2}x^2 + c$ **(d)** $\dfrac{2}{3}x^{\frac{3}{2}} + 2\sqrt{x} + c$

6. $y = x^3 - \frac{1}{2}x^2 - 2$

7. $y = \frac{1}{4}x^4 - x^3 + 2$

8. $y = x^2 - \frac{3}{x} + 1$

9. $y = x^4 - 3x^2 + 5$

10. 14

11. $y = x^2 - 3x + 4$

12. 19

13. $x = t^3 + 2t + 1$

14. $A = 2t^3 - t^2 + t + 3$

15. (a) $\frac{1}{15}(3x+1)^5 + c$ (b) $-\frac{1}{4}(1-x)^4 + c$ (c) $-\frac{1}{4}(2x+5)^{-2} + c$
(d) $\frac{1}{9}(6x-1)^{\frac{3}{2}} + c$ (e) $-\frac{1}{2x-7} + c$ (f) $-\sqrt{3-2x} + c$
(g) $-\frac{1}{25(3x-1)^5} + c$ (h) $\frac{4}{9}\sqrt{6x-1} + c$ (i) $\frac{8}{1-2x} + c$

16. (a) $-\frac{1}{7}(1-x)^7 + c$ (b) $\frac{1}{2}(2x-5)^3 + c$ (c) $\frac{2}{1-x} + c$ (d) $\frac{1}{6}(4t-1)^{\frac{3}{2}} + c$

17. 1.625

18. $y = \frac{1}{2}(4x-1)^3 + \frac{1}{2}$

19. $A = (x-1)^3 + 2$

20. $s = (2t-1)^3 + t + 2$

Exercise 20.2

1. (a) $31\frac{1}{2}$ (b) $17\frac{1}{3}$ (c) $2\frac{1}{4}$ (d) $\frac{1}{18}$ (e) 2 (f) $12\frac{2}{5}$

2. (a) −8 (b) 7 (c) 31 (d) $\frac{2}{3}$ (e) $\frac{1}{3}$ (f) $28\frac{1}{4}$

3. (a) 0 (b) $-1\frac{1}{6}$ (c) $1\frac{3}{4}$ (d) $-2\frac{2}{3}$ (e) 2 (f) $-\frac{4}{21}$

4. (a) $3\frac{3}{4}$ (b) $-2\frac{1}{2}$ (c) $7\frac{1}{3}$ (d) 0 (e) $-\frac{1}{9}$ (f) $-3\frac{1}{2}$

5. (a) $8\frac{2}{3}$ (b) $2\frac{1}{2}$ (c) $1\frac{1}{3}$ (d) $\frac{5}{6}$

6. −5

7. $1\frac{2}{3}$

8. $\frac{26}{3}\pi$

9. $78\frac{1}{3}$

10. 36

11. $\frac{4}{9}$

12. $\frac{16}{3}$

13. 6

14. (a) 3 (b) 8

15. (a) 24 (b) 0; $\frac{24}{55}$

16. (a) 9 (b) −1, $k = 3$

17. $\frac{6}{25}$

Exercise 20.3

1. (a) $-\cos x + 2x + c$ (b) $x - 3\sin x + c$ (c) $\sin x + \cos x + c$
(d) $\tan x + 4\cos x + c$ (e) $3\sin x + 2\cos x + c$ (f) $4\sin x + 3\tan x + c$

2. (a) $\frac{1}{2}\sin 2x + c$ (b) $-\frac{1}{3}\cos 3x + c$ (c) $\frac{1}{2}\sin 4x + c$
(d) $2\sin\frac{1}{2}x + c$ (e) $-2\cos\frac{1}{4}x + c$ (f) $\frac{1}{3}\sin 3x + c$
(g) $6\cos\frac{1}{2}x + c$ (h) $-2\sin(1-x) + c$ (i) $-2\cos(3x+2) + c$
(j) $-\frac{1}{2}\sin(1-2x) + c$ (k) $\frac{1}{2}\cos(2x+1) + c$ (l) $3\cos(2-x) + c$
(m) $\frac{1}{2}\sin\left(2x + \frac{\pi}{4}\right) + c$ (n) $-4\cos\left(x - \frac{\pi}{4}\right) + c$

3. (a) $\frac{1}{2}$ (b) 1 (c) $2\sqrt{2}$ (d) 1

(e) $\frac{\pi}{2} - 2$ (f) $\frac{3}{2} - \frac{\pi}{3}$ (g) $\frac{1}{2}$ (h) $\frac{1}{3}$

(i) $\frac{1}{2}$ (j) $-\frac{1}{2}$ (k) 2 (l) $\frac{1}{\sqrt{2}} - \frac{1}{2}$

4. 1

Exercise 20.4

1. (a) $e^x + x + c$ (b) $\frac{1}{2}e^{2x} + c$ (c) $\frac{2}{3}e^{3x} + c$ (d) $-e^{-x} - e^x + c$

(e) $-\frac{1}{2}e^{-2x} + c$ (f) $4e^{\frac{1}{2}x} + c$ (g) $\frac{1}{2}e^{2x+1} + c$ (h) $-3e^{1-x} + c$

(i) $\frac{1}{6}e^{3x+2} + c$ (j) $-8e^{\frac{1}{2}(1-x)} + c$

2. (a) $e^2 - 1$ (b) $\frac{1}{2}(e^2 - 1)$ (c) $2\left(1 - \frac{1}{e}\right)$ (d) $1 - \frac{1}{e}$ (e) $\frac{7}{3}$ (f) 2

3. $y = x - 3e^x + 7$ 4. $y = -2e^{-x} + 1$ 5. $y = \frac{1}{2}(e^{2x} + 3)$

6. $e^{\sin x} \cos x$, $e^{\sin x} + c$ 7. $2xe^{x^2}$, $\frac{1}{2}(e - 1)$

Miscellaneous Exercise 20

1. (a) $\frac{1}{4}x^4 - x^3 + c$ (b) $x^2 - 2\sqrt{x} + c$ 2. $8\frac{2}{3}$

3. $41\frac{9}{10}$ 4. $4\frac{7}{15}$ 5. $-1, 4$ 6. 4

7. (a) -12 (b) $y = x^3 - 12x - 10$, $(0, -10)$

8. $a = -1$, $y = x^2 - x + 5$ 9. (a) 3 (b) $y = \frac{3}{2}x^2 - 5x + 6$ 10. 5

11. 15.22, $F(x) = \frac{1}{3}x^3 - \frac{2}{3}x^{\frac{3}{2}} + \frac{10}{3}$ 12. 1 13. 74

14. (a) 0 (b) -6 (c) 16

15. (a) $2\sqrt{2} - 3$ (b) $\frac{1}{4}\left(\sqrt{3} - 1\right)$ 16. $\sqrt{2}$ 17. $\frac{1}{3}$

19. (a) $4x + 4e^x + \frac{1}{2}e^{2x} + c$ (b) $\frac{1}{2}e^{2x} + 2e^{-x} + c$ 20. $\frac{e^{\sqrt{x}}}{2\sqrt{2}}$; $2(e - 1)$

21. $\frac{1}{2}\ln 2$ 22. $\ln 2$ 23. $\ln\frac{3}{2}$ 24. $6\sin^2 2x \cos 2x$ (a) $\frac{1}{6}$ (b) $\frac{1}{3}$

Exercise 21.1

1. (a) $10\frac{2}{3}$ sq. units (b) 6 sq. units (c) $3\frac{1}{3}$ sq. units (d) 0.5 sq. units

(e) $e + 1$ sq. units (f) $2\frac{2}{3}$ sq. units (g) $\frac{1}{2}(e^2 - 1)$ sq. units (h) 0.5 sq. units

2. (a) $4\frac{1}{3}$ sq. units (b) 6 sq. units

3. **(a)** $16\frac{1}{4}$ sq. units **(b)** $2\frac{2}{3}$ sq. units

4. $4\frac{1}{2}$ sq. units

5. $\frac{4}{3}$ sq. units

6. **(a)** $\frac{\pi}{3}$ **(b)** $\frac{3}{4}$ sq. units

7. **(a)** $1 - e^{-a}$ sq. units **(b)** 0.69

8. $a + \frac{a^3}{3}$ sq. units; $a \approx 1.73$

9. $7\frac{1}{3}$ sq. units

10. **(a)** 9 sq. units **(b)** $11\frac{1}{4}$ sq. units **(c)** $2\frac{2}{3}$ sq. units **(d)** $1\frac{1}{3}$ sq. units

Exercise 21.2

1. **(a)** 2, $\frac{4}{3}$ sq. units **(b)** 1, $\frac{17}{6}$ sq. units **(c)** 2, $9\frac{1}{3}$ sq. units

2. $10\frac{2}{3}$ sq. units

3. $\frac{2}{3}$ sq. units

4. 1.05 sq. units

5. **(a)** $A(1, 0)$, $B(4, 3)$ **(b)** 9 sq. units

6. **(a)** $A(2, 12)$, $B(6, 12)$ **(b)** $21\frac{1}{3}$ sq. units

7. $2\frac{1}{3}$ sq. units; 9 sq. units

8. **(a)** $P(7, 0)$, $Q(-1, 0)$, $R(0, 7)$, $S(6, 7)$ **(b)** $4\frac{1}{3}$ sq. units, $3\frac{2}{3}$ sq. units, 36 sq. units

9. $k > 1$, $\frac{k^2}{2} + \frac{1}{k} - \frac{3}{2}$ sq. units

12. $2\sqrt{2} - 2$ sq. units

Miscellaneous Exercise 21

1. **(a)** $P(0, 2)$, $Q(3, 5)$ **(b)** 6 sq. units **(c)** $4\frac{1}{2}$ sq. units

2. **(a)** $8 - \frac{8}{p}$ sq. units **(b)** $\frac{12}{7}$

3. $4\frac{1}{2}$ sq. units

4. 12 sq. units

5. **(a)** $A(4, 8)$, $B(6, 0)$ **(b)** $25\frac{1}{2}$ sq. units

6. **(b)** $\frac{1}{6}$ sq. units

7. $\frac{1}{2}$ sq. units

8. **(a)** $2 - \sqrt{2}$ sq. units **(b)** $\sqrt{2} - 1$ sq. units

9. 10 sq. units

10. 1.2

11. **(a)** $(1, 0)$

12. 3, $y = 2x$, $y = -2x + 8$, $\frac{2}{3}$ sq. units

14. **(a)** $A\left(\frac{\pi}{2}, 0\right)$ **(b)** 0.57 sq. units

15. **(a)** 0.59 sq. units **(b)** 0.32 sq. units

16. $\frac{\sqrt{3}}{2}$ sq. units

22

Exercise 22.1

1. **(a)** 15 m s^{-1} **(b)** 2, 6 **(c)** -6 m s^{-2}

*2. **(a)** -12 m s^{-2} **(b)** -24 m s^{-1} **(c)** 23 m

3. **(a)** 60 m **(b)** 6 m s^{-2}, 6 m s^{-2} **(c)** $10\frac{1}{4}$ m s^{-1}

4. **(a)** 81 **(b)** 25 m **(c)** 40 m s^{-1}

5. **(a)** 7 m s^{-1} **(b)** 3 m **(c)** 4 m s^{-2}

*6. **(a)** 6 m s^{-1} **(b)** 48 m **(c)** 8 s
7. **(a)** 47 m, 15 m **(b)** 1.5, 4.5 **(c)** 6 m s^{-1} **(d)** 46 m
*8. **(a)** 9 m s^{-1} **(b)** 18 m s^{-2} **(c)** 27 m **(d)** 8.5 m/s (ms^{-1})
*9. **(a)** 24 m s^{-1}, –18 m s^{-2} **(b)** 2, 4 **(c)** 24 m
(d) 5.6 ms^{-1}
10. **(a)** 2, $-1\frac{2}{3}$ m **(b)** $1\frac{1}{2}$ m, $7\frac{5}{6}$ m
11. **(a)** $8 - 6t$ m s^{-2} **(b)** $-t^3 + 4t^2 + 5t$ **(c)** 3 **(d)** 5
12. **(a)** 3 **(b)** $3\sqrt{3}$, 18 m s^{-1} **(c)** 36 m
13. **(a)** 6 **(b)** 78 m from 0
14. **(a)** 0 m s^{-2} **(b)** 2 m, 84 m **(c)** 41 m s^{-1}
15. **(a)** 12.5 s **(b)** $78\frac{1}{8}$ m 16. **(a)** $p = -4$, $q = 3$ **(b)** $4\frac{2}{9}$ m s^{-1}
17. $p = 4$, $q = -3$; 4 m 18. **(a)** 4 s **(b)** $21\frac{1}{3}$ m **(c)** 8 m s^{-1}
19. **(a)** 42 m s^{-1} **(b)** 6 s **(c)** 198 m
20. **(a)** 3.6 m s^{-1} **(b)** 2.30 **(c)** –0.4 m s^{-2}
21. **(a)** 1.2 m s^{-2} **(b)** 9.69 m; No

Exercise 22.2

1. **(a)** 0 m s^{-1}, 4 m s^{-1} **(b)** 10 m, 2.5 m s^{-1}
2. **(a)** 18 m **(b)** $v = 6 - 2t$ **(c)** Decelerating; Yes
3. **(a)** – **(e)** **(b)** – **(f)** **(c)** – **(d)**
4. **(b)** area of A + area of B = area of C
5. **(a)** $\frac{1}{3}$ m s^{-2} **(b)** 850 m **(c)** 950 m **(d)** 750 m
6. **(a)** 3 m s^{-2} **(b)** 1850 m **(c)** 37 m s^{-1} **(d)** 2300 m **(e)** 1400 m
7. **(a)** 7.2 m **(b)** 0.6 m s^{-2} **(c)** 3.6 m s^{-1} **(d)** 3.6 m **(e)** 10.9
8. **(a)** 4 **(b)** 1.25 m s^{-2} **(c)** 10 m
9. **(b)** 3 **(c)** 12 **(d)** 4 m s^{-2}
10. 120; 20 11. $\frac{5}{6}$ m s^{-2}; 40 s 12. $v = 4t$; $v = 8$, $t = 2$
13. **(c)** 33 s **(d)** $1\frac{2}{3}$
14. **(b)** 12.5 s **(c)** 125 m **(d)** 80 m; $T = 52$, $AC = 95$ m, $BC = 23$ m
15. **(b)** 30 **(c)** 50

Exercise 22.3

1. 7 m s^{-2}; 80 m 2. 40 m; 50 m 3. 7; 119 m
4. 1200 km h^{-2}, 56 km h^{-1} 5. **(a)** 0.1 **(b)** 595 m
6. **(a)** 1.5 m s^{-2} **(b)** 300 m **(c)** 7.5 m s^{-1}
7. 3 s; 30 m s^{-1} 8. **(a)** 16.2 m **(b)** –18 m s^{-1}
9. **(a)** 2 s **(b)** 10 m s^{-1} **(c)** 8 s
10. 12 m s^{-1}; 22.95 m
11. **(a)** 25 **(b)** 61.25 m, 2.5 s **(c)** –35 m s^{-1} **(d)** 6 s
12. **(a)** 50 m s^{-1} **(b)** 125 m **(c)** 2 s
13. 40 m, 40 s; 20 s; –8 m s^{-1} 14. $S_1 = 5V$; $S_2 = \frac{576 - V^2 8}{2}$; 8, 32; 14 s
15. **(a)** **(i)** $\frac{65}{3}$ m s^{-1} **(ii)** 78 m **(b)** 23 m s^{-1}; 15 m s^{-1}, 44 m

Miscellaneous Exercise 22

1. **(a)** -18 cm s^{-2} **(b)** -54 cm s^{-1} **(c)** 10 cm **(d)** 64 cm

2. **(a)** 3.5 m s^{-1} **(b)** 36 m

3. **(a)** 1 **(b)** $\frac{3}{2}$ **(c)** -9 m s^{-2}, 4 m

4. **(a)** 10.3 m s^{-1} **(b)** 2.26 m

5. **(a)** 9 m **(b)** 7 m **(c)** 10 m **(d)** 8 m s^{-1}; $t = 5$

6. **(a)** $x = \frac{1}{3}t^3 - 4t^2 + 7t$ **(b)** 36 m **(c)** 54 m; Nearer to O

7. $v = 4 - 10t - 3t^2$, $a = -10 - 6t$; 4 m s^{-1}, -10 m s^{-2}; $t = 2\frac{1}{3}$, same direction

8. 6 m s^{-1}, $4\sqrt{3}$ m

9. **(a)** 1.46 s, 4.47 m s^{-2} **(b)** $x = 24 - 24\cos\frac{1}{2}t$

10. **(a)** $v = 24\cos 2t$, $a = -48\sin 2t$ **(b)** $\frac{\pi}{12}$ s **(c)** 18 m **(d)** $6\sqrt{7}$ m s^{-1}

11. **(b)** -0.893 **(c)** $x = 16\left(1 - e^{-\frac{1}{2}t}\right)$

12. **(b)** 1 m s^{-2} **(c)** 36 s **(d)** 288.5 m **(e)** 207.5 m

13. **(a)** 468 m **(b)** 11.7 m s^{-1} **(c)** 1.5 m s^{-2} **(d)** 9 m s^{-1} **(e)** 4, 37

15. **(a)** 20 **(b)** 45.75; 1.5 m s^{-2}

16. **(a)** $s = 120$, $t = 15$ **(b)** **(i)** $T = 50$, $S = 750$
(ii) $v_c = 10$ m s^{-1}; $v_m = 30$ m s^{-1} **(iv)** 25 s

17. **(a)** 235 m, 6 s **(b)** 200 m **(c)** -68.6 m s^{-1}

18. **(a)** 88 m s^{-1} **(b)** 8.8 s **(c)** 387.2 m **(d)** 16.4 s

19. **(a)** 20 s **(b)** -1 m s^{-1}, No

20. **(a)** 4 **(b)** 2.4 m **(c)** 8 **(d)** 40 m
(e) $v_P = 7$ m s^{-1}; $v_Q = 8.2$ m s^{-1}

21. **(a)** 8 m s^{-1}; 2 m s^{-2} **(b)** 16 m s^{-1} **(c)** 20 **(d)** 48 m s^{-1}

22. **(a)** **(i)** 6 **(ii)** 3 **(b)** **(i)** 1.875 **(ii)** 20 s; 15 m s^{-1}, 22.5 m s^{-1}

23. **(a)** **(i)** 12 **(ii)** $\frac{5}{6}$ m s^{-2} **(b)** **(i)** 5.5 m **(ii)** 24 m

24. 30 m s^{-1}; $t = 1$, $h = 40$ **25.** **(ii)** 224 m, 99 m

26. 18 **(c)** 20 m

23 Exercise 23.1

1. **(a)** $\overrightarrow{AP}$ **(b)** $\overrightarrow{AB}$ **(c)** $\overrightarrow{AD}$

2. **(a)** 2**a**, 3**a** **(b)** 3**b** **(c)** 2**a** + 5**b**

4. **(a)** $\frac{1}{2}\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$, $\frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right)$ **(b)** $-\frac{1}{2}\overrightarrow{OA} + \left(m - \frac{1}{2}\right)\overrightarrow{OB}$

5. $\overrightarrow{OM} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$, $\overrightarrow{ON} = \frac{2}{5}\mathbf{a} + \frac{3}{5}\mathbf{b}$ **6.** $\overrightarrow{OE} = -\frac{k}{2}\mathbf{a} + \mathbf{b}$, $\overrightarrow{DE} = -\frac{1}{2}(1 + k)\mathbf{a} + \frac{1}{2}\mathbf{b}$

7. **(a)** If $k = \frac{1}{2}$, P is the midpoint of AB. **(b)** If $k = 1$, P is the point B.
(c) If $k = 2$, P is at a distance $2AB$ from A.

8. **(a)** **p** + **q** **(b)** **p** − **q** **(c)** **q** − $\frac{1}{2}$**p**; $\sqrt{2}$ units

9. (a) $(1-\lambda)\,\mathbf{a}+\frac{1}{2}\lambda\mathbf{b}$ (b) $\frac{2}{3}\mu\,\mathbf{a}+(1+\mu)\mathbf{b}$

10. (a) $-2\lambda\mathbf{a}+(1+\lambda)\mathbf{b}$ (b) $-2\mathbf{a}+(1+\mu)\mathbf{b}$

Exercise 23.2

1. $\frac{1}{2}(\mathbf{a}+\mathbf{b}),\ \mathbf{b}-\frac{1}{2}\mathbf{a},\ \mathbf{b}-\frac{3}{2}\mathbf{a}$ 2. $\mathbf{p}+\mathbf{q},\ 2\mathbf{p}+\mathbf{q},\ \mathbf{p}+2\mathbf{q},\ \mathbf{q}-\mathbf{p}$ 3. $s=\frac{4}{7},\ t=-\frac{5}{7}$

4. -1 5. (a) $\left(1-\frac{1}{2}k\right)\mathbf{a}+k\mathbf{c}$ (b) $n\mathbf{c},\ n=2,\ k=2$

6. (a) $(1-k)\mathbf{a}+k\mathbf{b}$ (b) $n\mathbf{a}+2n\mathbf{b};\ n=\frac{1}{3},\ k=\frac{2}{3}$

7. $\overrightarrow{AB}=\mathbf{b}-\mathbf{a},\ \overrightarrow{PQ}=2\mathbf{b}-\frac{2}{3}\mathbf{a}$ (a) $\frac{2}{3}(1-n)\mathbf{a}+2n\mathbf{b}$ (b) $(1-k)\mathbf{a}+k\mathbf{b};\ n=\frac{1}{4},\ k=\frac{1}{2}$

Exercise 23.3

1. (a) $\frac{1}{3}(2\mathbf{a}+\mathbf{b}),\ \frac{1}{3}(\mathbf{a}+2\mathbf{b})$ (b) $\mathbf{a}+k(\mathbf{b}-\mathbf{a})$

2. $m\mathbf{a}+\mathbf{b}$ 3. $2(\mathbf{a}-\mathbf{b}),\ 5(\mathbf{a}-\mathbf{b})$

4. (a) $\overrightarrow{AB}=2\mathbf{p}-3\mathbf{q},\ \overrightarrow{AC}=5\mathbf{p}+(k-1)\mathbf{q}$ (b) $k=\frac{13}{2},\ \lambda=\frac{2}{5}$

6. $(\lambda-1)\mathbf{p}+(\lambda+1)\mathbf{q},\ (\lambda+1)\mathbf{q},\ \lambda=1$

7. (a) $\frac{1}{2}(\mathbf{p}+\mathbf{q})$ (b) $\mathbf{q}-\mathbf{p}$ (c) $\frac{1}{2}\mathbf{p}-\mathbf{q}$

8. (a) $(1-n)k\mathbf{a}+\frac{1}{2}n\mathbf{b}$ (b) $\frac{n-2}{2(n-1)}$

9. (a) $(m+1)\mathbf{q}-m\mathbf{p}$ (b) $(3+n)\mathbf{q}-9\mathbf{p};\ m=9,\ n=7,\ 10\mathbf{q}-9\mathbf{p}$

10. $\overrightarrow{AQ}=2\mathbf{b}-\mathbf{a},\ \overrightarrow{BP}=3\mathbf{a}-\mathbf{b}$

(a) $(1-\lambda)\mathbf{a}+2\lambda\mathbf{b}$ (b) $3\mu\mathbf{a}+(1-\mu)\mathbf{b};\ \lambda=\frac{2}{5},\ \mu=\frac{1}{5},\ \frac{3}{5}\mathbf{a}+\frac{4}{5}\mathbf{b}$

11. $\frac{2}{5}(1-l)\mathbf{a}+\frac{2}{5}l\mathbf{b},\ \frac{4}{5}(\mathbf{b}-k\mathbf{a});\ k=\frac{1}{4},\ l=\frac{1}{2}$

12. $\overrightarrow{OG}=\frac{1}{2}\mathbf{a}+\frac{1}{2}\mathbf{d},\ \overrightarrow{OC}=-\frac{1}{2}\mathbf{a}+\mathbf{b}+\frac{1}{2}\mathbf{d},\ \overrightarrow{OE}=\mathbf{a}-\mathbf{b}+\mathbf{d},\ \overrightarrow{OF}=\frac{3}{2}\mathbf{a}-\mathbf{b}+\frac{1}{2}\mathbf{d}$

13. $\overrightarrow{OD}=3(\mathbf{p}-2\mathbf{q}),\ \lambda=-4$ 14. $\overrightarrow{OD}=\mathbf{a}+\mathbf{c}-\mathbf{b},\ \overrightarrow{OM}=\frac{1}{2}(\mathbf{a}+\mathbf{c})$

15. (a) $(1-\lambda)\mathbf{a}+\lambda\mathbf{b}$ (b) $2\mu\mathbf{a}+3\mu\mathbf{b};\ \lambda=\frac{3}{5},\ \mu=\frac{1}{5}$.

Exercise 23.4

1. (a) $5\mathbf{i}+\mathbf{j}$ (b) $13\mathbf{i}+4\mathbf{j}$ (c) $-3\mathbf{j}$ (d) $\mathbf{j}$

2. (a) $\begin{pmatrix}3\\4\end{pmatrix}$ (b) $\begin{pmatrix}4\\1\end{pmatrix}$ (c) $\begin{pmatrix}12\\-1\end{pmatrix}$ (d) $\begin{pmatrix}-5\\12\end{pmatrix}$

3. (a) $5,\ \frac{1}{5}(3\mathbf{i}+4\mathbf{j}),\ 13(3\mathbf{i}-4\mathbf{j})$ (b) $13,\ \frac{1}{13}(5\mathbf{i}-12\mathbf{j}),\ 5(5\mathbf{i}-12\mathbf{j})$

4. (a) $m=\frac{2}{5},\ n=\frac{11}{5}$ (b) $m=3,\ n=-2$

5. (a) $\begin{pmatrix}-7\\8\end{pmatrix}$ (b) $\begin{pmatrix}2-3\lambda\\3\lambda-1\end{pmatrix},\ \lambda=\frac{7}{15}$

6. (a) $\sqrt{41}$ (b) $\frac{1}{\sqrt{41}}\begin{pmatrix}5\\4\end{pmatrix}$ (c) $m = 3$, $n = 1$

7. $q = 2p$, $p = \sqrt{5}$, $q = 2\sqrt{5}$ 8. (a) $3q - p = 12$ (b) $p^2 + q^2 - 2p = 3$

9. (a) $\begin{pmatrix}5\\1\end{pmatrix}$ (b) $\begin{pmatrix}-5\\6\end{pmatrix}$ 10. $\begin{pmatrix}3\\6\end{pmatrix}$, $\begin{pmatrix}-1\\2\end{pmatrix}$ (a) $\begin{pmatrix}4p-1\\4p+2\end{pmatrix}$

(b) $\begin{pmatrix}1-4q\\2+4q\end{pmatrix}$; $p = \frac{1}{4}$, $q = \frac{1}{4}$ 11. $\frac{1}{4}$ 12. $\begin{pmatrix}3+\lambda\\5-2\lambda\end{pmatrix}$, 1, $\frac{9}{5}$.

Miscellaneous Exercise 23

1. (a) $k = -\frac{31}{3}$, 3 : 8 (b) $\frac{1}{3}(4\mathbf{a} - 3\mathbf{b})$

2. 1 : 2, $\overrightarrow{OD} = 5\mathbf{q} - 2\mathbf{p}$, $\overrightarrow{OP} = 2(k - 1)\mathbf{p} + (5 - k)\mathbf{q}$, $k = \frac{1}{3}$

3. $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, $\overrightarrow{AC} = 3\mathbf{b} - \mathbf{a}$; $\overrightarrow{OX} = \frac{a+2b}{3}$, $\overrightarrow{OY} = \frac{a+3kb}{1+k}$; $k = \frac{2}{3}$

4. $\frac{1-k}{2k}\mathbf{a} + \frac{1}{2}\mathbf{b}$, $k = \frac{1}{6}$, 1 : 1 5. $\begin{pmatrix}0\\3\end{pmatrix}$

6. $\overrightarrow{OP} = \begin{pmatrix}2\\1\end{pmatrix}$, $\overrightarrow{OY} = \begin{pmatrix}6\\-2\end{pmatrix}$, $\overrightarrow{OR} = \begin{pmatrix}4+5k\\2-5k\end{pmatrix}$, $k = 2$ 7. $\overrightarrow{OM} = \lambda(\mathbf{a} - 2\mathbf{b})$, $\lambda = 3$; 8

8. (a) (i) 1 (ii) $\frac{1}{2}$

(b) (i) $2k\mathbf{p} - 3k\mathbf{q}$ (ii) $(2m + 1)\mathbf{p} + (3 - 5m)\mathbf{q}$; $k = \frac{11}{4}$, $m = \frac{9}{4}$, $\overrightarrow{OX} = \frac{11}{2}\mathbf{p} - \frac{33}{4}\mathbf{q}$

9. (a) $\frac{1}{5}\begin{pmatrix}3\\-4\end{pmatrix}$, 10n (b) $\begin{pmatrix}15\\36\end{pmatrix}$ (c) $\begin{pmatrix}4\\-3\end{pmatrix}$, $\frac{1}{5}\begin{pmatrix}4\\-3\end{pmatrix}$

10. $\frac{4}{3}\mathbf{a}$, $\frac{1}{6}(2\mathbf{a} + \mathbf{b})$, $m = 1$, $n = \frac{1}{6}$

11. $\overrightarrow{OP} = 3\mathbf{a} - \mathbf{b}$, $\overrightarrow{OQ} = 2(\mathbf{a} - \mathbf{b})$, $\overrightarrow{OX} = 3\mathbf{a} + 4\mathbf{b}$, $\overrightarrow{OY} = 4\mathbf{a} + 3\mathbf{b}$

12. $\frac{1}{3}(\mathbf{a} + 2\mathbf{b})$, $\frac{1}{7}(k\mathbf{a} + 6\mathbf{b})$ (a) 3 (b) $\frac{7}{2}$

13. (a) $\begin{pmatrix}18\\1\end{pmatrix}$ (b) $p\mathbf{a} + \mathbf{b}$, $\frac{q}{3}\mathbf{a} + \frac{2q}{3}\mathbf{b}$; $q = \frac{3}{2}$, $p = \frac{1}{2}$

14. (a) $a = 8$, $n = 2$ (b) $b = 8$ 15. $t = 2$, 2 : 1

16. (a) (i) $\mathbf{a} + \mathbf{c} - \mathbf{b}$ (ii) $\mathbf{a} + \mathbf{c} = 2\mathbf{b}$ (b) $3\mathbf{p} - 3\mathbf{q}$

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Exercise 24.1

1. (a) $\mathbf{v}_{B/A} = 2$ m s^{-1} → (b) $\mathbf{v}_{B/A} = 8$ m s^{-1} ← (c) $\mathbf{v}_{B/A} = 8$ m s^{-1} → (d) $\mathbf{v}_{B/A} = 2$ m s^{-1} ←

2. (a) $\mathbf{v}_{P/Q} = 5.5$ m s^{-1} → (b) 20 s, 50 m 3. (a) 1.3 m s^{-1} → (b) 2.5 m s^{-1} →

4. 2 m s^{-1} (upstream); 3 m s^{-1} (upstream) 5. (a) 2 m s^{-1} (b) 0.5 m s^{-1}; 30 m

6. 5.1 m s^{-1}; 33.9 s 7. 4 m s^{-1} ←; 50 s

Exercise 24.2

1. 5 m s^{-1}, in the direction 143.1° 2. 4.36 m s^{-1}, on the bearing 006.6°

3. 6.5 m s^{-1}, makes an angle 67.4° (downstream) with the bank

4. **(a)** 1.79 m s^{-1} makes an angle 39.1° (downstream) with the bank **(b)** 17.7 s
5. **(a)** 45.4° **(b)** 4.33 m s^{-1} **(c)** 7 s; makes an angle 64.3° (upstream) with the bank
6. 250 km h^{-1} on the bearing 053.1° **7.** $\theta = 80.0°$, 45 min
8. **(a) (i)** 50.3° **(ii)** 310 km h^{-1} **(b) (i)** 2.8 m s^{-1} **(ii)** 21.8°
9. **(a)** 73.1° or 73.2° **(b)** 6.70 m s^{-1} **(c)** 29.8 s; 3.82 m s^{-1}

Exercise 24.3

1. **(a)** 6.24 m s^{-1}, in the direction 166° **(b)** 5 s
2. **(a)** 17.1 km h^{-1}, makes an angle 13° with the initial $\overrightarrow{PQ}$ **(b)** 1153 m
3. **(a)** 2.05 m s^{-1}, in the direction 013.1° **(b)** 464 m
4. **(a)** 9.43 m s^{-1} in the direction 122° **(b)** 2.72 km
5. **(a)** 725.5 km h^{-1}, in the direction 281.9° **(b)** 12 04 14
6. **(a)** 23.6° **(b)** 3107 s **7. (a)** $\theta = 300°$ **(b)** 48.1 s
8. **(a) (i)** 17.3 m s^{-1}, 345° **(ii)** 2.99 s **(b) (i)** 37.3 m s^{-1} **(ii)** 1.49 s

Miscellaneous Exercise 24

1. **(a)** 38.5 km h^{-1}, 113.5° **(b)** 3.16 km
2. **(a)** 38.6 km h^{-1}, 165° **(b)** 1.73 km
3. 43.6 km h^{-1}, 203.4°, 48.4 km h^{-1}, 258.1°
4. **(a)** 63° upstream with the bank **(b)** 17 s **(c)** 15 s **(d)** 45 m
5. **(a)** 006° **(b)** 1 h 52 min **(c)** 176.6° **(d)** 1 h 35 min
6. **(a)** 10.6 cm s^{-1} in the direction 289.1° **(b)** 16 s **(c)** 58.2 cm
7. **(a)** 302°, 22 min **(b)** 332° or 028°
8. **(a)** 41.4°, 91 s **(b) (i)** 12 cm s^{-1}, 090° **(ii)** 26.8, 63.4° **(c)** 21.8° or 158.2°
9. **(a) (i)** 175 km h^{-1}, 078.5° **(ii)** 3.5 min **(b)** 50.4 km h^{-1} at 217.5°
10. **(a) (i)** 072.5° **(ii)** 33.5 min **(b)** 15.9 km h^{-1}, 229.1°
11. **(a) (i)** 117° **(ii)** 45 min **(b)** 40.4 km h^{-1}
12. **(a) (i)** 36.9° **(ii)** 30 s **(b) (i)** 076.7° **(ii)** 08 31 h
13. **(a)** 40 m s^{-1}, 4.5 s **(b) (i)** 059.2° **(ii)** 1 h 19 min, 109.2 km h^{-1}
14. **(a) (i)** 17.0 **(ii)** 97.2 m **(iii)** 39.4° **(b)** 83.0 km h^{-1}, 218.9°
15. **(a)** 5.1 m s^{-1}, 33.9 s **(b) (i)** 28.1° **(ii)** 10.5 m s^{-1} **(iii)** 4 s
16. **(a)** 28.6 km h^{-1}, 114.8° **(b)** 16 00 h, 48.0 km **(c)** 65.3 km
17. **(a)** 13.0 km h^{-1}, 067.6° **(b)** 20.6 km, 61 km h^{-1}, 280.4°
18. **(a)** 64 km h^{-1} **(b)** 252 km h^{-1} **(c)** 014.7°, 036.9°, 26 min

Revision Exercise 17

1. **(a)** $y = 2\sqrt{x} - x^2 + 10$ **(b) (i)** $\frac{1}{8}(2x-1)^4 + c$ **(ii)** $-\frac{2}{3}(4-x)^{\frac{3}{2}} + c$

   (c) (i) $\frac{1}{2}e^{2x} + \frac{1}{2}x^2 + c$ **(ii)** $-\frac{1}{2}\cos 2x + \frac{1}{3}\sin 3x + c$
2. **(a) (i)** 39 **(ii)** 8 **(b)** $\frac{2}{\sqrt{5}}$ **3. (a)** $\frac{7}{6}$ sq. units **(b)** $\frac{5}{6}$ sq. units
4. **(a)** $p^2 + q^2 - 2p - 2q - 18 = 0$ **(b)** $p^2 + q^2 - 2p - 2q - 18 = 0$
5. $2\mathbf{b} - \mathbf{a}$, $\frac{3}{2}\mathbf{a} - \mathbf{b}$ **(a)** $(1-\lambda)\mathbf{a} + 2\lambda\mathbf{b}$

   (b) $\frac{3}{2}\mu\mathbf{a} + (1-\mu)\mathbf{b}$; $\lambda = \frac{1}{4}$, $\mu = \frac{1}{2}$, $\frac{3}{4}\mathbf{a} + \frac{1}{2}\mathbf{b}$
6. **(a)** 3 s **(b)** 0, 2 **(c)** –12 m s^{-1} **(d)** 24 m **(e)** 3 m s^{-1}
7. **(a) (ii)** 5s **(iii)** 18 **(iv)** 1.8 m s^{-1} **(b)** 4 s, 40 m s^{-1}
8. 0.7 m s^{-1}; 50 s; 14 m

Revision Exercise 18

1. (a) (i) $\frac{4}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}} + c$ (ii) $\frac{1}{2(1-2t)} + c$ (b) 28
2. (a) (i) –8 (ii) 9 (iii) 3, $m = 3$ (b) $\frac{10}{3}$
3. (a) (i) (1, 5), (2, 5) (ii) $\frac{2}{3}$ sq. units (b) 1 : 1
4. (a) (i) $\frac{1}{2}x + \frac{1}{12}\sin 6x + c$ (ii) $\frac{1}{2}e^{2x-1} + c$ (b) (i) –14 (ii) 84 m
5. (a) $m = 3$, $n = -1$ (b) $4\frac{4}{5}$ (c) 7 (d) –1, 5
6. 1; 1 : 1
7. (a) (i) 5 (ii) 6 m s^{-1} (b) (i) 10 m (ii) 160 m
8. (a) 25 s, 50 m (b) 53.1°; 31.25 s; *P* crosses first.

Revision Exercise 19

1. (a) (i) $k = 2$ (ii) $y = \frac{4}{3}\sqrt{x}(x - 3) + 4\frac{2}{3}$
   (b) (i) $2x - 2\cos\frac{1}{2}x + c$ (ii) $\frac{2}{3}x\sqrt{x} - x + c$
2. (a) $1 - \frac{\pi}{4}$
3. (a) 2.55 sq. units (b) 3
4. (a) $\mu\mathbf{r} - \mathbf{p}$ (b) $\frac{1}{1+\lambda}\mathbf{r} - \mathbf{p}$
5. (a) $2\frac{1}{2}$ m s^{-1} (b) 1.84 m s^{-2} (c) 8.77 m
6. (a) 5 m s^{-1}, 4 m s^{-2} (b) 21 m s^{-1} (c) 6 s (d) 29 m s^{-1}
7. (a) (i) 004.1° (ii) 1 h 55 min (b) 27.8 km h^{-1}, 085.9°

Revision Exercise 20

1. (a) $y = \frac{2}{3}e^{3x} + \frac{4}{3}$ (b) (i) $-\frac{1}{2}\cos\left(2x - \frac{\pi}{3}\right) + c$ (ii) $\frac{1}{2(3-2x)} + c$
2. (a) (i) 1.79 (ii) 2.66
   (b) $-6\cos^2 2x\sin 2x$ (i) $-\frac{1}{6}\cos^3 2x + c$ (ii) $-\frac{1}{2}\cos 2x + \frac{1}{6}\cos^3 2x + c$
3. (a) 6.36 (b) $3\cos 3x - \sin 3x$ (c) $2x\ln x + x - 1$, $2\ln 2 - \frac{3}{4}$
4. (a) $2\frac{5}{6}$ sq. units (b) $17\frac{1}{15}$ sq. units
5. (a) $\frac{2}{3}\lambda\mathbf{a} + \lambda\mathbf{b}$ (b) $(1 - \mu)\mathbf{a} + \mu\mathbf{b}$; $\lambda = \frac{3}{5}$, $\mu = \frac{3}{5}$, $\frac{3}{2}$
6. (a) 2 m s^{-1} (b) 16 m s^{-2} (c) 33 m
7. (a) 16 m s^{-1} (b) 9.8 s (c) 0.8 m s^{-2}, 10 s
8. (a) 16.6 km h^{-1}, 015.7° (b) 13 15 h, 5.64 km (c) 7.84 km

ASSESSMENT PAPERS

Paper 1

1. $x = -\frac{1}{3}$, $y = 6$ or $x = 3$, $y = 1$
2. $M\left(3, \frac{\lambda-2}{2}\right)$, $\lambda = 4, 6$
3. $1 - 21x + 189x^2 - 945x^3$; 504
4. 5040; 15

5. (i) 10.9 cm (ii) 2.02 cm^2
6. (a) $y = x^3 - 3x^2 + 5$ (b) $(-1, -2)$
7. (a) $\{x : x < -1 \text{ or } x > 1, x \in \mathbb{R}\}$ (b) $b = \frac{a^2 + 9a}{3}$
8. (ii) 10 (iii) $x = t^2$
9. (a) $9p$ (b) 0.4 cm s^{-1}
10. (a) (i) $a = 2, b = 1$ (ii) $f^{-1}: x \text{ a } \frac{2x}{x+1}, x \neq -1$ (b) $2 < x < \frac{8}{3}$
11. (b) (i) 10°, 30°, 190°, 210° (ii) 63.4°, 161.6°, 243.4°, 341.6°
12. (a) (i) 81 m s^{-1} (ii) 2.6 m s^{-1} (iii) 139.5 m
(b) (i) $y = \frac{864}{x+4} - 6$ (ii) $x = 20, y = 30$

Paper 2

1. $0 \leqslant g(x) \leqslant 5$
2. (a) $\begin{pmatrix} -7 & 15 \\ 2 & -6 \end{pmatrix}$

No, **A** is not a square matrix; Yes, **AB** is a non-singular square matrix
(b) $x = 2, y = -1$

3. $2\frac{5}{6}$ sq. units 4. (i) -6 (ii) $\pm\frac{1}{2}, -6$
5. (a) $-\frac{1}{3}, 3$ (b) 1.83 6. 1.28, 2.85
7. $A = \{4, 8, 12, 16\}$, $B = \{6, 8, 10\}$ (i) $\{4, 6, 8, 10, 16\}$, $\{8\}$, $\{4, 12, 16\}$ (ii) 6 (iii) 8
8. (i) 6 s (ii) 4 m s^{-1}, 29 m s^{-1}
9. (a) $2x \ln x + x - 1$, $2 \ln 2 - \frac{3}{4}$ (b) (i) 2.32 (ii) $\frac{8}{35}$
10. (a) 3.87, 3.67
11. (a) (i) 25 m s^{-1} on the bearing 97° (ii) 60 s, 184 m (iii) 22°
(b) $\overrightarrow{BC} = 2\mathbf{a} - \mathbf{b}$; $\overrightarrow{DA} = \mathbf{a} - \frac{1}{3}\mathbf{b}$ (i) $\overrightarrow{OP} = (1 + \mu)\mathbf{a} - \frac{1}{3}\mu\mathbf{b}$
(ii) $\overrightarrow{OP} = 2\lambda\mathbf{a} + (1 - \lambda)\mathbf{b}$; $\lambda = 2, \mu = 3$, $\overrightarrow{OP} = 4\mathbf{a} - \mathbf{b}$, $\overrightarrow{OQ} = \mathbf{b} - \mathbf{a}$

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